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Coefficients of symmetric power L-functions on integers under digital constraints

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Abstract: Let $\lambda_{\mathrm{sym}^r f}(n)$ be the n-th coefficient in the Dirichlet series representing the symmetric power L-function attached to a primitive form f of weight k and level N. In this paper, we give asymptotic formulas for the arithmetic mean of $\lambda_{\mathrm{sym}^r f}(n)$ on integers under digital constraints related to the sum of digits function. Our results throw the light on the behavior of the classical function $\lambda_{\mathrm{sym}^r f}(n)$ on integers in arithmetic progression related to the sum of digits function.
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1 Introduction

Let $k \geq 2$ an even integer and $N \geq 1$ a squarefree integer. Denote $\mathcal{H}_k^*(N)$ the set of holomorphic primitive forms of weight k for the congruence group $\Gamma_0(N)$. Each $f \in \mathcal{H}_k^*(N)$ has a Fourier expansion in ∞ of the form

$$f(z) = \sum_{n=1}^{+\infty} \lambda_f(n) n^{\frac{k-1}{2}} \exp(2i\pi nz), \qquad (\Im z > 0),$$

where $\lambda_f(n)$ is the n-th normalized Fourier coefficient of f and verify the Hecke relation

$$\lambda_f(n)\lambda_f(m) = \sum_{\substack{d|(m,n)\\(d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all positive integers m and n. A result by Deligne [5, 6] implies that for each prime number p there exist two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\begin{cases} |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1, & \text{if } (p,N)=1 \\ \alpha_f(p) = \varepsilon_f(p)p^{-1/2},\ \beta_f(p)=0, & \text{if } p|N \end{cases} \tag{1}$$

with $|\varepsilon_f(p)| = 1$, and

$$\lambda_f(p^\nu) = \begin{cases} \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu, & \text{if } (p,N)=1 \\ \left(\varepsilon_f(p)p^{-1/2}\right)^\nu, & \text{if } p|N \end{cases} \tag{2}$$

for all integer $\nu \geq 1$. The Ramanujan conjecture states that for all prime numbers p, we have

$$|\alpha_f(p)| = |\beta_f(p)| = 1. \tag{3}$$

As a consequence, we get

$$|\lambda_f(n)| \leq d(n) \qquad (n \geq 1), \tag{4}$$

where $d(n)$ is the number of positive divisors of n. In addition, for all $(p,N)=1$, there exists a unique $\theta_f(p) \in [0,\pi]$ such that

$$\lambda_f(p) = 2\cos\theta_f(p). \tag{5}$$

If r is a positive integer and $f \in \mathcal{H}_k^*(N)$, then we construct the r-th symmetric power of f denoted by $\mathrm{sym}^r f$. When $r = 1,2,3,4$, it is known that $\mathrm{sym}^r f$ is an automorphic form. In the general case, it is predicted by the conjecture of Langlands–Serre [7]. It is easy to define it via its L-function. We define then for $\Re e\, s > 1$

$$L(s,\mathrm{sym}^r f) = \prod_p \prod_{0\leq j\leq r} \left(1-\alpha_f(p)^{r-j}\beta_f(p)^j p^{-s}\right)^{-1} = \sum_{n\geq 1} \lambda_{\mathrm{sym}^r f}(n) n^{-s}.$$

It is not difficult to see that the function $n \mapsto \lambda_{\mathrm{sym}^r f}(n)$ is real multiplicative and satisfy

$$|\lambda_{\mathrm{sym}^r f}(n)| \leq d_{r+1}(n) \qquad (n \geq 1), \tag{6}$$

where $d_2(n) = d(n)$ and $d_{r+1}(n) = \sum_{(d,n)=1} d_r(d)$. Let $U_r(u)$ be a Tchebychef polynomial of second kind and of degree r. It is well-known that (see for example [10])

$$\lambda_{\mathrm{sym}^r f}(p) = \lambda_f(p^r) = U_r(\cos\theta_f(p)) = \frac{\sin((r+1)\theta_f(p))}{\sin\theta_f(p)} \qquad (p,N)=1. \tag{7}$$

Therefore, if n is a squarefree integer coprime with N, we have

$$\lambda_{\mathrm{sym}^r f}(n) = \lambda_f(n^r). \tag{8}$$

Mazhouda, Mbarki and Wu established an explicit formula for $\lambda_{\mathrm{sym}^r f}(n)$ in [9, Proposition 2.1] given in the following proposition.

Proposition 1.1. *Let $k \geq 2$ be an even integer, $N \geq 1$ be a squarefree integer and $f \in \mathcal{H}_k^*(N)$. For all integers $r \geq 1$ and $\nu \geq 0$, we have*

$$\lambda_{\mathrm{sym}^r f}(p^\nu) = \prod_{j=1}^{r} \frac{\sin((\nu+j)\theta_f(p))}{\sin(j\theta_f(p))} \qquad (p,N)=1, \tag{9}$$

where $\theta_f(p)$ is defined as in (5).

Now, given a real number x, $[x]$ denotes the greatest integer $\leq x$ and for an arbitrary fixed $q \geq 2$ denote by $\log_q(x) = \frac{\log x}{\log q}$ the logarithm to the base q. The analysis includes the q-adic numbers $\mathbb{Q}_q$ and the ring of integers $\mathbb{Z}_q$. Recall that each $n \in \mathbb{Z}_q$ can be expressed in the form

$$n = \sum_{k=k_0}^{\infty} a_k(n) q^k \tag{10}$$

with $0 \leq a_k(n) \leq q-1$ and $a_{k_0}(n) \neq 0$. The q-adic integers $\mathbb{Z}_q$ correspond to the case $k_0 \geq 0$. We set

$$S(n) = S_q(n) = \sum_{k=k_0}^{\infty} a_k(n).$$

The function S is the sum of digits function in basis q and it is a completely q-additive function which means that $S(0)=0$ and $S(n) = \sum_{k=k_0}^{\infty} S(a_k(n))$. This implies that S is completely determined by its values on the set $\{0, 1, \ldots, q-1\}$. There has been interest in various properties, and in particular asymptotic properties, of this function for a long time. For example, one of the earliest results [1] is that the average of the values of this function over the integers from 1 to n behaves asymptotically as $\frac{q-1}{2}\log_q(n)$. Such functions were introduced by Gelfond [8] and further studied by Delange [4], Coquet [3], and many related results, as well as new developments (see [11, Chapter 4, paragraph 3]). The distribution of the sum of digits function in residue classes has been studied by Gelfond [8], the crucial part in the proof is the following result that we shall need in the sequel.

Theorem A. *Let $q \geq 2, m \geq 2$ be integers such that $(m, q-1) = 1$ and $\delta \in \mathbb{R}$. Then, we have for $1 \leq h \leq m-1$,*

$$\Big|\sum_{n=1}^{N} e\big(\delta n + \frac{h}{m} S(n)\big)\Big| = O_q(N^\lambda) \quad \text{as } N \longrightarrow +\infty,$$

where $\lambda = \frac{1}{2\log q}\left(\frac{q\sin(\pi/2m)}{\sin(\pi/2mq)}\right) < 1$.

Note that the hypothesis $(m, q-1) = 1$ is not necessary but it allows to avoid some degenerated cases which lead to a couple of congruences on n. Indeed, if we put $(m, q-1) = l > 1$, then the congruence $q \equiv 1 (\mathrm{mod}\, l)$ implies the congruence $n \equiv S(n) \equiv a (\mathrm{mod}\, l)$ since $(q-1)|(n - S(n))$. Since $\lambda_{\mathrm{sym}^r f} = \mathbf{1} * g$ where $g = \lambda_{\mathrm{sym}^r f} * \mu$ and μ is the Möbius function. So we write

$$\lambda_{\mathrm{sym}^r f}(n) = \sum_{d|n} g(d) = \sum_{d|n}\sum_{h|d} \mu(h)\lambda_{\mathrm{sym}^r f}\left(\frac{d}{h}\right) = \sum_{d|n} \mu(d) \times [\lambda_{\mathrm{sym}^r f} * \mathbf{1}]\left(\frac{n}{d}\right).$$

Let $a \in \mathbb{Z}$ and let q and m be integers ≥ 2 such that $(m, q-1) = 1$, we define the following function that depends on n, q, a and m by

$$\widetilde{\lambda_{\mathrm{sym}^r f}}(n) = \sum_{\substack{d|n \\ S(d) \equiv a \pmod m}} \mu(d) \times [\lambda_{\mathrm{sym}^r f} * \mathbf{1}]\left(\frac{n}{d}\right),$$

where $S(n) = S_q(n)$ is the sum of digits in base q of a positive integer n.

Then, Section 2 of this paper is devoted to finding estimates of average of the function $\widetilde{\lambda_{\mathrm{sym}^r f}}(n)$ which is the function $\lambda_{\mathrm{sym}^r f}(n)$ defined on numbers under digital constraints.

1.1 Notations

In what follows, p always denotes a prime number. For any real number x, we define $e(x) = e(2\pi i x)$. The greatest common divisor of two integers a and b will be denoted by (a, b). The implied constants in the symbols "O", $\ll$ " are absolute (recall that the notation $U \ll V$ is equivalent to the statement that $U = O(V)$ for positive functions U and V). We also use the symbol "o" with its usual meaning: the statement $U = o(V)$ is equivalent to $U/V \longrightarrow 0$.

Finally, in order to detect the congruences, we shall use the classic orthogonality relation

$$\frac{1}{m}\sum_{j=0}^{m-1} e\left(\frac{j(a-b)}{m}\right) = \begin{cases} 1, & \text{if } a \equiv b \pmod m \\ 0, & \text{else.} \end{cases} \tag{11}$$

2 Auxiliary estimates

Let f be a Hecke eigencusp form of even integral weight k. By the Cauchy–Schwarz inequality we get

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^r f}(n) \ll \sum_{n \leq x} |\lambda_{\mathrm{sym}^r f}(n)| \ll x^{1+\varepsilon}. \tag{12}$$

Tang and Wu [12, Theorem 3] establish some bounds for the sum $\sum_{n \leq x} |\lambda_{\mathrm{sym}^r f}(n)|$. They obtained the following:

$$\sum_{n \leq x} |\lambda_{\mathrm{sym}^r f}(n)| \sim C_r(f) x (\log x)^{-\delta_r} \tag{13}$$

unconditionally for $x \to \infty$, where $\delta_r = 1 - \dfrac{4(r+1)}{\pi r(r+2)} \cot \dfrac{\pi}{2(r+1)}$ and $C_r(f)$ is a positive constant depending on f and r.

Guohua and Xiaoguang [2, Theorem 1] improved the result of Tang and Wu [12] for the $\sum_{n \leq x} \lambda_{\mathrm{sym}^r f}(n)$ and proved that:

Lemma 2.1. *Let f be a primitive cusp form for the full modular group $SL_2(\mathbb{Z})$ of an even weight $k \geq 2$ and let $r \geq 2$ be an integer. Then, for any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^r f}(n) \ll k^{\frac{r(r+1)}{2(r+2)}} x^{\frac{1}{(r+2)}+\varepsilon} + x^{\frac{r}{(r+2)}+\varepsilon},$$

where the implied constant depends on r and ε.

The previous uniform upper bound in Lemma 2.1 is nontrivial in the range $k \ll x^{2/r}$ and in fact it is the current best upper bound up to x^{ε} with respect to x.

3 Asymptotic formulas

The average of the function $\widetilde{\lambda_{\mathrm{sym}^r f}}(n)$ is given in the following theorem.

Theorem 3.1. *Let f be a primitive cusp form for the full modular group $SL_2(\mathbb{Z})$ of an even weight $k \geq 2$ and let $q \geq 2$, $m \geq 2$ be integers such that $(m, q-1) = 1$ and $a \in \mathbb{Z}$. Then, for $r \geq 2$ and for any $0 < \varepsilon < \frac{2}{r+2}$, we have*

$$\sum_{n \leq x} \widetilde{\lambda_{\mathrm{sym}^r f}}(n) = \frac{[\alpha_r + \alpha_{m,r}]}{m} x + O_{r,\varepsilon}\left(k^{\frac{r(r+1)}{2(r+2)}} x\right) + O\left(\frac{x}{(\log x)^{\delta_r}}\right),$$

unconditionally for $x \to \infty$ and uniformly in the range $k \ll x^{2/r}$, where

$$\alpha_r = \int_1^{+\infty} \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{h \leq u} \lambda_{\mathrm{sym}^r f}(h) \frac{du}{u^2},$$

$$\alpha_{m,r} = \sum_{j=1}^{m-1} \int_1^{+\infty} \sum_{d \leq x} \frac{\mu(d)}{d} e\left(\frac{-j}{m} a + \frac{j}{m} S(d)\right) \sum_{h \leq u} \lambda_{\mathrm{sym}^r f}(h) \frac{du}{u^2}$$

and $\delta_r = 1 - \dfrac{4(r+1)}{\pi r(r+2)} \cot \dfrac{\pi}{2(r+1)}$.

Proof. We have

$$\begin{aligned}
\sum_{n \leq x} \widetilde{\lambda_{\mathrm{sym}^r f}}(n) &= \sum_{\substack{d \leq x \\ S(d) \equiv a (\bmod m)}} \mu(d) \sum_{\substack{n \leq x \\ n \equiv 0 (\bmod d)}} [\lambda_{\mathrm{sym}^r f} * \mathbf{1}]\left(\frac{n}{d}\right) \\
&= \sum_{\substack{d \leq x \\ S(d) \equiv a \ (\bmod m)}} \mu(d) \sum_{h \leq \frac{x}{d}} [\lambda_{\mathrm{sym}^r f} * \mathbf{1}](h) \\
&= \sum_{\substack{d \leq x \\ S(d) \equiv a \ (\bmod m)}} \mu(d) \sum_{h \leq \frac{x}{d}} \lambda_{\mathrm{sym}^r f}(h) \sum_{l \leq \frac{x}{dh}} 1 \\
&= \sum_{\substack{d \leq x \\ S(d) \equiv a \ (\bmod m)}} \mu(d) \sum_{h \leq \frac{x}{d}} \lambda_{\mathrm{sym}^r f}(h) \left[\frac{x}{dh}\right] \\
&= \sum_{\substack{d \leq x \\ S(d) \equiv a \ (\bmod m)}} \mu(d) \sum_{h \leq \frac{x}{d}} \lambda_{\mathrm{sym}^r f}(h) \left(\frac{x}{dh} + O(1)\right) \\
&= x \sum_{\substack{d \leq x \\ S(d) \equiv a \ (\bmod m)}} \frac{\mu(d)}{d} \sum_{h \leq \frac{x}{d}} \frac{\lambda_{\mathrm{sym}^r f}(h)}{h} + O\left(x \sum_{d \leq x} \frac{|\mu(d)|}{d} \sum_{h \leq \frac{x}{d}} |\lambda_{\mathrm{sym}^r f}(h)|\right).
\end{aligned}$$

For the error term, we use the result of equation (13) to get

$$\begin{aligned} x\sum_{d\le x}\frac{|\mu(d)|}{d}\sum_{h\le \frac{x}{d}}|\lambda_{\mathrm{sym}^r f}(h)| &\sim C_r(f)x^2\sum_{d\le x}\frac{|\mu(d)|}{d^2}\left[\log\left(\frac{x}{d}\right)\right]^{-\delta_r} \\ &\ll x^2\sum_{d\le x}\frac{1}{d^2\left[\log\left(\frac{x}{d}\right)\right]^{\delta_r}} \\ &\ll \frac{x^2}{(\log x)^{\delta_r}}\sum_{d\le x}\frac{1}{d^2} \\ &\ll \frac{x}{(\log x)^{\delta_r}}. \end{aligned} \tag{14}$$

Now, let us compute the sum $\sum_{h\le \frac{x}{d}}\frac{\lambda_{\mathrm{sym}^r f}(h)}{h}$. By applying Abel's summation formula, we get

$$\sum_{h\le \frac{x}{d}}\frac{\lambda_{\mathrm{sym}^r f}(h)}{h} = \frac{d}{x}\sum_{h\le \frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) + \int_1^{\frac{x}{d}}\sum_{h\le u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}.$$

The last formula combined with the orthogonality relation (11) and equation (14) yield

$$\begin{aligned} \sum_{n\le x}\widetilde{\lambda_{\mathrm{sym}^r f}}(n) = &\sum_{\substack{d\le x\\ S(d)\equiv a(\bmod m)}}\mu(d)\sum_{h\le \frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) \\ &+ x\sum_{\substack{d\le x\\ S(d)\equiv a(\bmod m)}}\frac{\mu(d)}{d}\int_1^{\frac{x}{d}}\sum_{h\le u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} + O\left(\frac{x}{(\log x)^{\delta_r}}\right) \\ = &\frac{1}{m}\sum_{j=0}^{m-1}\sum_{d\le x}\mu(d)e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\sum_{h\le \frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) \\ &+ \frac{x}{m}\sum_{j=0}^{m-1}\sum_{d\le x}\frac{\mu(d)}{d}e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\int_1^{\frac{x}{d}}\sum_{h\le u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} + O\left(\frac{x}{(\log x)^{\delta_r}}\right) \\ = &\frac{1}{m}\sum_{d\le x}\mu(d)\sum_{h\le \frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) + \frac{1}{m}\sum_{j=1}^{m-1}\sum_{d\le x}\mu(d)e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\sum_{h\le \frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) \\ &+ \frac{x}{m}\sum_{d\le x}\frac{\mu(d)}{d}\int_1^{\frac{x}{d}}\sum_{h\le u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} \\ &+ \frac{x}{m}\sum_{j=1}^{m-1}\sum_{d\le x}\frac{\mu(d)}{d}e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\int_1^{\frac{x}{d}}\sum_{h\le u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} + O\left(\frac{x}{(\log x)^{\delta_r}}\right). \end{aligned}$$

So, we get four sums with an error term. Let us investigate them. According to the result of Lemma 2.1, we have

$$\frac{1}{m}\sum_{d\le x}\mu(d)\sum_{h\le \frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) \ll k^{\frac{r(r+1)}{2(r+2)}}x^{\frac{1}{(r+2)}+\varepsilon}\sum_{d\le x}\frac{1}{d^{\frac{1}{(r+2)}+\varepsilon}} + x^{\frac{r}{(r+2)}+\varepsilon}\sum_{d\le x}\frac{1}{d^{\frac{r}{(r+2)}+\varepsilon}}.$$

It is well-known that

$$\sum_{h\leq x}\frac{1}{h^v}=\begin{cases}\frac{x^{1-v}}{1-v}+\zeta(v)+O(x^{-v}), & \text{if } v\neq 1\\ \log(x)+\gamma+O\left(\frac{1}{x}\right), & \text{if } v=1,\end{cases} \tag{15}$$

where ζ is the Riemann zeta-function and γ is the Euler–Mascheroni's constant defined by the equation

$$\gamma=\lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n}-\log(x)\right).$$

So, we get

$$\begin{aligned}\frac{1}{m}\sum_{d\leq x}\mu(d)\sum_{h\leq\frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h) &\ll k^{\frac{r(r+1)}{2(r+2)}}x^{\frac{1}{(r+2)}+\varepsilon}\times x^{1-\left(\frac{1}{(r+2)}+\varepsilon\right)}+x^{\frac{r}{(r+2)}+\varepsilon}\times x^{1-\left(\frac{r}{(r+2)}+\varepsilon\right)}\\ &\ll k^{\frac{r(r+1)}{2(r+2)}}x,\end{aligned} \tag{16}$$

where the implied constant depends on r and ε. Now, for the second one, we use the result of Lemma 2.1 to obtain

$$\frac{1}{m}\sum_{j=1}^{m-1}\sum_{d\leq x}\mu(d)e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\sum_{h\leq\frac{x}{d}}\lambda_{\mathrm{sym}^r f}(h)=O_{r,\varepsilon}\left(k^{\frac{r(r+1)}{2(r+2)}}x\right). \tag{17}$$

For the third sum, since $0<\varepsilon<\frac{2}{r+2}$, then according to Lemma 2.1:

$$\begin{aligned}\int_1^{+\infty}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} &\ll \int_1^{+\infty}\left[k^{\frac{r(r+1)}{2(r+2)}}\frac{u^{\frac{1}{(r+2)}+\varepsilon}}{u^2}+\frac{u^{\frac{r}{(r+2)}+\varepsilon}}{u^2}\right]du\\ &\ll k^{\frac{r(r+1)}{2(r+2)}}\int_1^{+\infty}\frac{1}{u^{\frac{2r+3}{(r+2)}-\varepsilon}}du+\int_1^{+\infty}\frac{1}{u^{\frac{r+4}{(r+2)}-\varepsilon}}du<\infty.\end{aligned}$$

So, we can write

$$\begin{aligned}\frac{x}{m}\sum_{d\leq x}\frac{\mu(d)}{d}\int_1^{\frac{x}{d}}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} &= \frac{x}{m}\int_1^{+\infty}\sum_{d\leq x}\frac{\mu(d)}{d}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}\\ &\quad-\frac{x}{m}\sum_{d\leq x}\frac{\mu(d)}{d}\int_{\frac{x}{d}}^{+\infty}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}.\end{aligned}$$

Notice that if we use equation (15), we get

$$\begin{aligned}\frac{x}{m}\sum_{d\leq x}\frac{\mu(d)}{d}\int_{\frac{x}{d}}^{+\infty}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2} &\ll x\sum_{d\leq x}\frac{1}{d}\times\left[k^{\frac{r(r+1)}{2(r+2)}}u^{1-\frac{2r+3}{(r+2)}+\varepsilon}+u^{1-\frac{r+4}{(r+2)}+\varepsilon}\right]_{\frac{x}{d}}^{+\infty}\\ &\ll k^{\frac{r(r+1)}{2(r+2)}}x\sum_{d\leq x}\frac{\left(\frac{x}{d}\right)^{-\frac{(r+1)}{(r+2)}+\varepsilon}}{d}+x\sum_{d\leq x}\frac{\left(\frac{x}{d}\right)^{-\frac{2}{(r+2)}+\varepsilon}}{d}\\ &\ll x^{1-\frac{2}{(r+2)}+\varepsilon}\sum_{d\leq x}\frac{1}{d^{1+\frac{2}{(r+2)}-\varepsilon}}+k^{\frac{r(r+1)}{2(r+2)}}x^{1-\frac{(r+1)}{(r+2)}+\varepsilon}\sum_{d\leq x}\frac{1}{d^{1+\frac{(r+1)}{(r+2)}-\varepsilon}}\\ &\ll x^{1-\frac{4}{(r+2)}+2\varepsilon}+k^{\frac{r(r+1)}{2(r+2)}}x^{1-\frac{2(r+1)}{(r+2)}+2\varepsilon}\\ &\ll x^{\frac{r-2}{(r+2)}+2\varepsilon}+k^{\frac{r(r+1)}{2(r+2)}}x^{-\frac{r}{(r+2)}+2\varepsilon}.\end{aligned}$$

Thus, we deduce that

$$\frac{x}{m}\sum_{d\leq x}\frac{\mu(d)}{d}\int_1^{\frac{x}{d}}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}=\frac{x}{m}\alpha_r+O_{r,\varepsilon}\left(x^{\frac{r-2}{(r+2)}+2\varepsilon}+k^{\frac{r(r+1)}{2(r+2)}}x^{-\frac{r}{(r+2)}+2\varepsilon}\right), \tag{18}$$

where

$$\alpha_r=\int_1^{+\infty}\sum_{d\leq x}\frac{\mu(d)}{d}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}.$$

We finish with the last sum, if we follow the same steps as above and thanks to **Theorem A** and Lemma 2.1 which are used essentially to check the convergence of our integrals, we obtain

$$\begin{aligned}&\frac{x}{m}\sum_{j=1}^{m-1}\sum_{d\leq x}\frac{\mu(d)}{d}e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\int_1^{\frac{x}{d}}\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}\\&=\frac{x}{m}\times\alpha_{m,r}+O_{r,\varepsilon}\left(x^{\frac{r-2}{(r+2)}+2\varepsilon}+k^{\frac{r(r+1)}{2(r+2)}}x^{-\frac{r}{(r+2)}+2\varepsilon}\right),\end{aligned} \tag{19}$$

where

$$\alpha_{m,r}=\sum_{j=1}^{m-1}\int_1^{+\infty}\sum_{d\leq x}\frac{\mu(d)}{d}e\left(\frac{-j}{m}a+\frac{j}{m}S(d)\right)\sum_{h\leq u}\lambda_{\mathrm{sym}^r f}(h)\frac{du}{u^2}.$$

Combining equations (16), (17), (18) and (19), we find that

$$\begin{aligned}\sum_{n\leq x}\widetilde{\lambda_{\mathrm{sym}^r f}}(n)&=\frac{[\alpha_r+\alpha_{m,r}]}{m}x+O_{r,\varepsilon}\left(x^{\frac{r-2}{(r+2)}+2\varepsilon}+k^{\frac{r(r+1)}{2(r+2)}}x^{-\frac{r}{(r+2)}+2\varepsilon}\right)\\&\quad+O_{r,\varepsilon}\left(k^{\frac{r(r+1)}{2(r+2)}}x\right)+O\left(\frac{x}{(\log x)^{\delta_r}}\right)\\&=\frac{[\alpha_r+\alpha_{m,r}]}{m}x+O_{r,\varepsilon}\left(k^{\frac{r(r+1)}{2(r+2)}}x\right)+O\left(\frac{x}{(\log x)^{\delta_r}}\right).\end{aligned}$$

□

4 Conclusion

The method used in this paper for the case of the function $\lambda_{\mathrm{sym}^r f}(n)$ works for any arithmetic function since any arithmetic function f satisfies: $f=\mathbf{1}*\mu*\mathbf{f}$. It suffices to know the average order of f and for the error term of $|f|$ and to use Abel's summation formula and the orthogonality relation (11) and to check the convergence of the integrals using **Theorem A**.

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