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Research on splitting quaternions with generalized Tribonacci hybrid number components

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Abstract: This paper introduces the Generalized Tribonacci Hybrid Split Quaternion (GTHSQ), a novel split quaternion with coefficients derived from generalized Tribonacci hybrid numbers. This form unifies various existing number types, such as generalized hybrid Tribonacci numbers and Tribonacci numbers, offering a fresh perspective on quaternion theory. To systematically characterize GTHSQ, we develop a comprehensive mathematical framework. This includes defining GTHSQ through related number expressions, specifying its operational rules, and proving that it retains the third-order linear recurrence relation of generalized Tribonacci sequences. We derive its Binet-type formula and generating functions while exploring its core properties. Additionally, we extend classic combinatorial identities (Vajda, Catalan, Cassini) to GTHSQ, define the GTHSQ matrix, and analyze its product with the S-matrix—a generalized third-order linear recurrence sequence representation matrix—to obtain matrix and determinant expressions for GTHSQ. These findings verify the closed-form solution of GTHSQ in terms of combinatorial identities and matrix representation. Furthermore, we discuss potential applications of GTHSQ, including advancements in quaternion algebra, support for encryption algorithms in cryptography, and simplification of spatial transformations in physics, thereby providing new tools and insights for future research in quaternion theory and interdisciplinary studies.

2020 Keywords: Hybrid number, Split quaternion, Generalized Tribonacci sequence, Hybrid splitting quaternion, Matrix representation.
2020 Mathematics Subject Classification: 11B39, 05A15.

1 Introduction

Researchers in quaternion theory are constantly exploring new members of this mathematical family and their properties [5,6,12,15]. Quaternions extend complex numbers and are valuable in various fields, including pure and applied mathematics, motion geometry, differential geometry, graph theory, computer animation, and robotics, thanks to their unique structure and multi-dimensional nature. The split quaternion, a prominent member of this family, is an associative and noncommutative four-dimensional ring defined over the real numbers. It allows for a simplified representation of complex rotations and Lorentz transformations in pseudo-Euclidean and three-dimensional Minkowski spaces [14], It has drawn considerable scholarly attention for its properties and potential applications.

In recent years, the integration of integer sequences with hypercomplex number systems has become a prominent research direction, yielding insightful generalizations and novel mathematical objects. Kızılates [7] pioneered the fusion of quantum calculus with hybrid numbers by introducing q-Fibonacci and q-Lucas hybrid numbers, whose components are defined using q-integers. This work not only derived Binet-like formulas, exponential generating functions, and classic combinatorial identities (Catalan, Cassini, d'Ocagne) for these numbers but also demonstrated their inclusivity recovering Fibonacci, Pell, and Jacobsthal hybrid numbers as special cases when specific values are assigned to parameters q and δ. Building on this, Polatlı [16] further advanced hybrid number theory by defining hybrid numbers with Fibonacci and Lucas hybrid number coefficients, establishing their Binet formulas and generating functions, and constructing an associated matrix to derive two distinct versions of the Cassini identity. This matrix-based approach provided a new tool for analyzing the algebraic structure of high-order hybrid numbers, laying the groundwork for subsequent studies on matrix representations of complex hypercomplex systems.

The generalization of quaternions with sequence components has also seen substantial progress. Polatlı, Kızılateş, and Kesim [17] introduced split k-Fibonacci and k-Lucas quaternions, extending the classic Fibonacci quaternion framework to include a parameter k. They derived Binet formulas, generating functions, and exponential generating functions for these quaternions, and successfully extended Catalan, Cassini, and d'Ocagne identities to the split quaternion context—highlighting how parameterized sequences can enrich the algebraic properties of quaternion systems. Expanding to higher-order structures, Kızılates, Du, and Terzioğlu [10] defined higher-order generalized Fibonacci hybrinomials using higher-order generalized Fibonacci polynomials, constructing three distinct matrices (with components of hybrinomials, Fibonacci polynomials, and Lucas polynomials) to derive new identities. This work emphasized the role of matrix representations in unifying the analysis of sequence properties and hypercomplex number operations, while also verifying results through numerical examples generated via Maple, enhancing the reliability of the theoretical framework.

Notably, Terzioğlu, Kızılateş and Du [20] proposed Fibonacci finite operator quaternions by combining finite operators with Fibonacci numbers, deriving their recurrence relations, Binet-like formulas, and matrix representations. This research demonstrated the effectiveness of integrating functional operators with sequence-based quaternions, providing a valuable reference for our work on combining third-order sequences with hybrid and split quaternion systems. Beyond Fibonacci sequences, Kızılateş [8] also made important contributions to the fusion of finite operators and general sequences: In his research, Kızılateş applies the finite operator to the Horadam sequence to define the Horadam finite operator sequences. He not only provides some special cases of this new sequence but also investigates several properties of the new sequence, including its recurrence relation, Binet-like formula, summation formula, and generating function. Additionally, he presents a closed formula for the Horadam finite operator numbers and their special cases by means of tridiagonal determinants, and further verifies the recurrence relation of the Horadam finite operator numbers using the tridiagonal determinant.

In q-calculus-integrated hybrid numbers, Kızılateş, Polatlı, and Du [11] proposed higher-order generalized Fibonacci hybrid numbers with q-integer components, incorporating q-integers, higher-order Fibonacci numbers, and hybrid numbers into a single framework. They derived a Binet-like formula, recurrence relations, and sum properties for these numbers, and utilized two specially designed matrices to obtain additional identities. Notably, this study addressed the computational complexity of hybrid number multiplication by providing Maple code, facilitating practical applications and further research into q-integrated hypercomplex systems.

Integrating generalized Tribonacci sequences, hybrid numbers, and split quaternions to form GTHSQ is a challenging, innovative topic. While existing research has successfully combined second-order sequences, such as, Fibonacci and Lucas, with hybrid numbers or split quaternions, the extension to third-order sequences, including generalized Tribonacci sequences and their special cases like Tribonacci and Tribonacci–Lucas, remains underexplored. This gap offers an opportunity to enhance hypercomplex number theory by incorporating the intricate recurrence relations and algebraic structures of third-order sequences.

This study aims to investigate two core questions: First, can GTHSQ inherit the third-order linear recurrence relationship and essential properties (e.g., Binet formula, generating function) of generalized Tribonacci sequences, while retaining the unique algebraic characteristics of hybrid numbers and split quaternions? Second, can the matrix representation methods developed for second-order sequence-based hypercomplex numbers be adapted to analyze GTHSQ, enabling the derivation of classic combinatorial identities (e.g., Vajda, Catalan, Cassini) for this new object? Answering these questions not only deepens our understanding of quaternion theory and its applications but also bridges the gap between second-order and third-order sequence-based hypercomplex systems, providing new ideas and methods for future research in related fields.

In this paper, we will discuss the definition, properties, recurrence relations, Binet formulas, generating functions, essential combinatorial identities, and matrix representations of generalized Tribonacci hybrid-split quaternions on the set $\mathbb{N}^* = \{0, 1, 2, \ldots, n, \ldots\}$.

2 Preliminaries

2.1 Generalized Tribonacci sequence

Research on sequences of different orders has grown substantially in recent years. Generalized Tribonacci sequences, as a class of third-order linear recursive sequences, include well-studied examples such as Tribonacci numbers, Tribonacci–Lucas numbers [1, 4].

As defined in Reference [3], the generalized Fibonacci sequence $\{T_n\}$ follows the third-order linear recurrence relation:

$$T_n = uT_{n-1} + vT_{n-2} + wT_{n-3}, \;\; n \geq 3, \tag{1}$$

with initial terms:

$$T_0 = a, T_1 = b, T_2 = c, a, b, c \in \mathbb{Z}, u, v, w \in \mathbb{R}.$$

The characteristic equation corresponding to recurrence relation (1) is:

$$x^3 - ux^2 - vx - w = 0.$$

When the discriminant Δ of the equation is less than 0, the equation has three distinct roots $\alpha_1, \alpha_2, \alpha_3$. For these roots, the following relationships hold: $\alpha_1 + \alpha_2 + \alpha_3 = u, \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = -v, \alpha_1\alpha_2\alpha_3 = w$.

The Binet formula for the generalized Tribonacci numbers [3] is:

$$T_n = \frac{P\alpha_1^n}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} - \frac{Q\alpha_2^n}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{R\alpha_3^n}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}, \tag{2}$$

where

$$\begin{aligned} P &= c - (\alpha_2 + \alpha_3)b + \alpha_2\alpha_3 a, \\ Q &= c - (\alpha_1 + \alpha_3)b + \alpha_1\alpha_3 a, \\ R &= c - (\alpha_1 + \alpha_2)b + \alpha_1\alpha_2 a. \end{aligned} \tag{3}$$

2.2 The hybrid numbers

Özdemir [13] introduced the set of hybrid numbers, denoted as $\mathcal{H}$, which unifies complex numbers, dual numbers, and hyperbolic numbers. Formally, $\mathcal{H}$ is defined as:

$$\mathcal{H} = \{r = r_0 + r_1\mathbf{i} + r_2\boldsymbol{\varepsilon} + r_3\mathbf{h} : r_0, r_1, r_2, r_3 \in \mathbb{R}\}$$

where $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ satisfy the following multiplication rules: $\mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1$, and $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$.

Using the relation $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$, we derive the multiplication table for the hybrid number units $1, \mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$(see Table 1).

Table 1. Multiplication rules for $1, \mathbf{i}$, ε, and $\mathbf{h}$

$\cdot$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$1+\mathbf{h}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Hybrid numbers differ from complex, dual, and hyperbolic numbers but share some common characteristics. For a hybrid number $r = r_0 + r_1\mathbf{i} + r_2\varepsilon + r_3\mathbf{h}$, r_0 and $r_1\mathbf{i} + r_2\varepsilon + r_3\mathbf{h}$ are called the scalar part and the vector parts of r, The conjugate of r is $r^c = r_0 - r_1\mathbf{i} - r_2\varepsilon - r_3\mathbf{h}$, The norm of r is $\|r\| = \sqrt{rr^c} = \sqrt{r_0^2 + r_1^2 - 2r_1r_2 - r_3^2}$.

Let $r = r_0 + r_1\mathbf{i} + r_2\varepsilon + r_3\mathbf{h}$ and $s = s_0 + s_1\mathbf{i} + s_2\varepsilon + s_3\mathbf{h}$ be two hybrid numbers. The equality, addition, scalar multiplication, and multiplication operations of these two hybrid numbers are given by

- $r = s \Leftrightarrow r_0 = s_0, r_1 = s_1, r_2 = s_2, r_3 = s_3$.
- $r \pm s = (r_0 \pm s_0) + (r_1 \pm s_1)\mathbf{i} + (r_2 \pm s_2)\varepsilon + (r_3 \pm s_3)\mathbf{h}$.
- $\lambda r = \lambda r_0 + \lambda r_1\mathbf{i} + \lambda r_2\varepsilon + \lambda r_3\mathbf{h}$, $\lambda \in \mathbb{R}$.
- $rs = (r_0s_0 - r_1s_1 + r_1s_2 + r_2s_1 + r_3s_3) + (r_0s_1 + r_1s_0 + r_1s_3 - r_3s_1)\mathbf{i}$
  $+ (r_0s_2 + r_1s_3 - r_3s_1 + r_2s_0 - r_2s_3 + r_3s_2)\varepsilon + (r_0s_3 + r_3s_0 - r_1s_2 + r_2s_1)\mathbf{h}$.

Notably, hybrid number addition is both commutative and associative, while multiplication is associative but non-commutative.

2.3 The split quaternions

First introduced by the mathematician James Cockle [2] in 1849, split quaternions represent a pivotal development in algebra. They form a non-commutative, associative four-dimensional Clifford algebra that contains zero factors, nilpotent elements, and non-trivial idempotent elements under standard addition and multiplication.

The set of split quaternions, denoted as $\mathbb{H}_s$, is defined as:

$$\mathbb{H}_s = \{q = q_0 + q_1e_1 + q_2e_1 + q_3e_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where the units e_1, e_2 and e_3 satisfy the multiplication rules: $e_1^2 = -1$, $e_2^2 = e_3^2 = 1$, and $e_1e_2e_3 = 1$. Using these rules, we construct the multiplication table for the split quaternion units $1, e_1, e_2$, and e_3 (see Table 2).

Table 2. Multiplication rules for $1, e_1, e_2$, and e_3

$\cdot$	1	e_1	e_2	e_2
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	1	$-e_1$
e_3	e_3	e_2	e_1	1

For a split quaternion $q = q_0 + q_1e_1 + q_2e_1 + q_3e_3$, $S_q = q_0$ and $V_q = q_1e_1 + q_2e_1 + q_3e_3$ are called the scalar part and the vector part of q. The conjugate of q is $\overline{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_3$. The norm of q is $\|q\| = \sqrt{|q\overline{q}|} = \sqrt{|q_0^2 + q_1^2 - q_2^2 - q_3^2|}$.

For two split quaternions $q = q_0 + q_1e_1 + q_2e_1 + q_3e_3$ and $p = p_0 + p_1e_1 + p_2e_1 + p_3e_3$. Their basic operations are defined as:

- $q = p \Leftrightarrow q_0 = p_0, q_1 = p_1, q_2 = p_2, q_3 = p_3$.
- $q \pm p = (q_0 \pm p_0) + (q_1 \pm p_1)e_1 + (q_2 \pm p_2)e_2 + (q_3 \pm p_3)e_3$.
- $\lambda q = \lambda q_0 + \lambda q_1e_1 + \lambda q_2e_2 + \lambda q_3e_3$, $\lambda \in \mathbb{R}$.
- $q \cdot p = (q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3) + (q_0p_1 + q_1p_0 - q_2p_3 + q_3p_2)e_1$
  $+(q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)e_2 + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)e_3$.

Convention: The multiplication of the hybrid number units $1, \mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ and the split quaternion units $1, e_1, e_2, e_3$ is commutative.

3 Generalized Tribonacci hybrid split quaternions

This section formally defines the generalized Tribonacci hybrid split quaternion (GTHSQ) and systematically analyzes its properties and key combinatorial identities.

Definition 3.1 ([22]). *Let T_n be the n-th generalized Tribonacci number. For $n \geq 0$, the n-th generalized Tribonacci hybrid number $\breve{T}_n$ is defined as*

$$\breve{T}_n = T_n + T_{n+1}\mathbf{i} + T_{n+2}\boldsymbol{\varepsilon} + T_{n+3}\mathbf{h}, \;\; n \geq 0, \tag{4}$$

where the units $1, \mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ satisfy the multiplication rules in Table 1.

Notably, Tribonacci hybrid numbers and Tribonacci–Lucas hybrid numbers [19] are special cases of $\breve{T}_n$.

Definition 3.2. *Let T_n be the n-th generalized Tribonacci number. For $n \geq 0$, the n-th generalized Tribonacci split quaternons $\widetilde{T}_n$ is defined as*

$$\widetilde{T}_n = T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3, \;\; n \geq 0, \tag{5}$$

where the units $1, e_1, e_2, e_3$ satisfy the multiplication rules in Table 2.

Definition 3.3. *Let $\breve{T}_n$ be the n-th generalized Tribonacci hybrid number. For $n \geq 0$, the n-th GTHSQ $\widehat{T}_n$ is defined as*

$$\widehat{T}_n = \breve{T}_n + \breve{T}_{n+1}e_1 + \breve{T}_{n+2}e_2 + \breve{T}_{n+3}e_3, \;\; n \geq 0, \tag{6}$$

where the units $1, e_1, e_2, e_3$ satisfy the multiplication rules in Table 2.

Remark 1. *Through analysis, we find that the ring structure is not preserved when the real number coefficients of split quaternions are replaced by generalized Tribonacci hybrid numbers. The reason is that the sum and difference of generalized Tribonacci numbers are not necessarily generalized Tribonacci numbers.*

We denote the set of all GTHSQ as $\mathbb{K}$. For $\widehat{T}_n$: $\operatorname{Re}\widehat{T}_n = \breve{T}_n$ and $\operatorname{Im}\widehat{T}_n = \breve{T}_{n+1}e_1 + \breve{T}_{n+2}e_2 + \breve{T}_{n+3}e_3$ are called the real and imaginary parts of the GTHSQ, respectively.

It is worth noting that using the hybrid number (second) representation of GTHSQ, $\widehat{T}_n$ can also be rewritten as:

$$\widehat{T}_n = \widetilde{T}_n + \widetilde{T}_{n+1}\mathbf{i} + \widetilde{T}_{n+2}\boldsymbol{\varepsilon} + \widetilde{T}_{n+3}\mathbf{h}, \ \ n \geq 0. \tag{7}$$

For convenience, we refer to the two expressions of $\widehat{T}_n$ as:

- **First representation**: Split quaternion form (Definition 3.3);
- **Second representation**: Hybrid number form (Equation (7)).

First, we establish the recurrence relations for generalized Tribonacci hybrid numbers $\breve{T}_n$ and generalized Tribonacci split quaternions $\widetilde{T}_n$.

Lemma 3.1. *If $\breve{T}_n$ and $\widetilde{T}_n$ are the n-th generalized Tribonacci hybrid number and the n-th generalized Tribonacci split quaternion, respectively, then for $n \geq 3$:*

$$\breve{T}_n = u\breve{T}_{n-1} + v\breve{T}_{n-2} + w\breve{T}_{n-3}, \tag{8}$$

$$\widetilde{T}_n = u\widetilde{T}_{n-1} + v\widetilde{T}_{n-2} + w\widetilde{T}_{n-3}. \tag{9}$$

In other words, $\breve{T}_n$ and $\widetilde{T}_n$ inherit the recurrence property of the generalized Tribonacci sequence T_n.

Proof. We only prove Equation (8), Equation (9) can be verified similarly. By substituting the definition of $\breve{T}_n$ (Definition 3.1) and the recurrence relation of Equation (1)):

$$\begin{aligned} u\breve{T}_{n-1}+v\breve{T}_{n-2}+w\breve{T}_{n-3} &= u(T_{n-1} + T_n\mathbf{i} + T_{n+1}\boldsymbol{\varepsilon} + T_{n+2}\mathbf{h}) + v(T_{n-2} + T_{n-1}\mathbf{i} + T_n\boldsymbol{\varepsilon} + T_{n+1}\mathbf{h}) \\ &\quad + w(T_{n-3} + T_{n-2}\mathbf{i} + T_{n-1}\boldsymbol{\varepsilon} + T_n\mathbf{h}) \\ &= (uT_{n-1} + vT_{n-2} + wT_{n-3}) + (uT_n + vT_{n-1} + wT_{n-2})\mathbf{i} \\ &\quad + (uT_{n+1} + vT_n + wT_{n-1})\boldsymbol{\varepsilon} + (uT_{n+2} + vT_{n+1} + wT_n)\mathbf{h} \\ &= T_n + T_{n+1}\mathbf{i} + T_{n+2}\boldsymbol{\varepsilon} + T_{n+3}\mathbf{h} = \breve{T}_n. \end{aligned}$$

□

Using Lemma 3.1, we extend the recurrence relation to GTHSQ:

Theorem 3.1. *If $\widehat{T}_n$ is the n-th GTHSQ, then for $n \geq 3$:*

$$\widehat{T}_n = u\widehat{T}_{n-1} + v\widehat{T}_{n-2} + w\widehat{T}_{n-3}. \tag{10}$$

That is, GTHSQ inherits the recurrence property of the generalized Tribonacci sequence.

Proof. By substituting the first representation of $\widehat{T}_n$ (Definition 3.3) and Lemma 3.1:

$$\begin{aligned} &u\widehat{T}_{n-1} + v\widehat{T}_{n-2} + w\widehat{T}_{n-3} \\ &= u(\breve{T}_{n-1} + \breve{T}_n e_1 + \breve{T}_{n+1}e_2 + \breve{T}_{n+2}e_3) + v(\breve{T}_{n-2} + \breve{T}_{n-1}e_1 + \breve{T}_n e_2 + \breve{T}_{n+1}e_3) \\ &\quad + w(\breve{T}_{n-3} + \breve{T}_{n-2}e_1 + \breve{T}_{n-1}e_2 + \breve{T}_n e_3) \\ &= (u\breve{T}_{n-1} + v\breve{T}_{n-2} + w\breve{T}_{n-3}) + (u\breve{T}_n + v\breve{T}_{n-1} + w\breve{T}_{n-2})e_1 \\ &\quad + (u\breve{T}_{n+1} + v\breve{T}_n + w\breve{T}_{n-1})e_2 + (u\breve{T}_{n+2} + v\breve{T}_{n+1} + wT_n)e_3 \\ &= \breve{T}_n + \breve{T}_{n+1}e_1 + \breve{T}_{n+2}e_2 + \breve{T}_{n+3}e_3 = \widehat{T}_n. \end{aligned}$$

□

We define three types of conjugates for GTHSQ, leveraging the conjugate operations of split quaternions and hybrid numbers.

Definition 3.4. *For the n-th GTHSQ $\widehat{T}_n = \widetilde{T}_n + \widetilde{T}_{n+1}\mathbf{i} + \widetilde{T}_{n+2}\boldsymbol{\varepsilon} + \widetilde{T}_{n+3}\mathbf{h}$, its conjugates are defined as:*

- ***Split quaternion conjugation***: $\overline{\widehat{T}_n} = \overline{\widetilde{T}_n} + \overline{\widetilde{T}_{n+1}}\mathbf{i} + \overline{\widetilde{T}_{n+2}}\boldsymbol{\varepsilon} + \overline{\widetilde{T}_{n+3}}\mathbf{h}$;
- ***Hybrid number conjugation***: $\widehat{T}_n^c = \widetilde{T}_n - \widetilde{T}_{n+1}\mathbf{i} - \widetilde{T}_{n+2}\boldsymbol{\varepsilon} - \widetilde{T}_{n+3}\mathbf{h}$;
- ***Total conjugation***: $\widehat{T}_n^\dagger = \overline{\widehat{T}_n^c} = \overline{\widetilde{T}_n} - \overline{\widetilde{T}_{n+1}}\mathbf{i} - \overline{\widetilde{T}_{n+2}}\boldsymbol{\varepsilon} - \overline{\widetilde{T}_{n+3}}\mathbf{h}$.

Using the definitions and recurrence relations above, we derive several fundamental identities for GTHSQ.

Theorem 3.2. *For any $n > 1$, the following identities hold:*

(i) $\widehat{T}_n - \widehat{T}_{n+1}e_1 - \widehat{T}_{n+2}e_2 - \widehat{T}_{n+3}e_3 = \breve{T}_n + \breve{T}_{n+2} - \breve{T}_{n+4} - \breve{T}_{n+6}$,

(ii) $\widehat{T}_n - \widehat{T}_{n+1}\mathbf{i} - \widehat{T}_{n+2}\boldsymbol{\varepsilon} - \widehat{T}_{n+3}\mathbf{h} = \widetilde{T}_n + \widetilde{T}_{n+2} - 2\widetilde{T}_{n+3} + \widetilde{T}_{n+6}$.

Proof. (i) Substitute the first representation of $\widehat{T}_n$ and the multiplication rules of split quaternion units from Table 2:

$$\begin{aligned}
&\widehat{T}_n - \widehat{T}_{n+1}e_1 - \widehat{T}_{n+2}e_2 - \widehat{T}_{n+3}e_3 \\
&= (\breve{T}_n + \breve{T}_{n+1}e_1 + \breve{T}_{n+2}e_2 + \breve{T}_{n+3}e_3) - (\breve{T}_{n+1} + \breve{T}_{n+2}e_1 + \breve{T}_{n+3}e_2 + \breve{T}_{n+4}e_3)e_1 \\
&\quad - (\breve{T}_{n+2} + \breve{T}_{n+3}e_1 + \breve{T}_{n+4}e_2 + \breve{T}_{n+5}e_3)e_2 - (\breve{T}_{n+3} + \breve{T}_{n+4}e_1 + \breve{T}_{n+5}e_2 + \breve{T}_{n+6}e_3)e_3 \\
&= \breve{T}_n + \breve{T}_{n+2} - \breve{T}_{n+4} - \breve{T}_{n+6}.
\end{aligned}$$

(ii) Substitute the second representation of $\widehat{T}_n$ and the multiplication rules of hybrid number units from Table 1:

$$\begin{aligned}
&\widehat{T}_n - \widehat{T}_{n+1}\mathbf{i} - \widehat{T}_{n+2}\boldsymbol{\varepsilon} - \widehat{T}_{n+3}\mathbf{h} \\
&= (\widetilde{T}_n + \widetilde{T}_{n+1}\mathbf{i} + \widetilde{T}_{n+2}\boldsymbol{\varepsilon} + \widetilde{T}_{n+3}\mathbf{h}) - (\widetilde{T}_{n+1} + \widetilde{T}_{n+2}\mathbf{i} + \widetilde{T}_{n+3}\boldsymbol{\varepsilon} + \widetilde{T}_{n+4}\mathbf{h})\mathbf{i} \\
&\quad - (\widetilde{T}_{n+2} + \widetilde{T}_{n+3}\mathbf{i} + \widetilde{T}_{n+4}\boldsymbol{\varepsilon} + \widetilde{T}_{n+5}\mathbf{h})\boldsymbol{\varepsilon} - (\widetilde{T}_{n+3} + \widetilde{T}_{n+4}\mathbf{i} + \widetilde{T}_{n+5}\boldsymbol{\varepsilon} + \widetilde{T}_{n+6}\mathbf{h})\mathbf{h} \\
&= \widetilde{T}_n + \widetilde{T}_{n+2} - \widetilde{T}_{n+3}(1+\mathbf{h}) - \widetilde{T}_{n+3}(1-\mathbf{h}) - \widetilde{T}_{n+6} = \widetilde{T}_n + \widetilde{T}_{n+2} - 2\widetilde{T}_{n+3} + \widetilde{T}_{n+6}.
\end{aligned}$$

□

Theorem 3.3. *For any $n > 1$, the following conjugate-related identities hold:*

(i) $\widehat{T}_n + \overline{\widehat{T}_n} = 2\breve{T}_n$,

(ii) $\widehat{T}_n + \widehat{T}_n^c = 2\widetilde{T}_n$,

(iii) $\widehat{T}_n + \widehat{T}_n^\dagger = 4T_n + 2(\widehat{T}_n - \breve{T}_n - \widetilde{T}_n)$.

Proof. It is easy to prove (i) and (ii) using Definitions 3.1–3.4. For the proof of (iii), substitute the definitions of $\widehat{T}_n$ and $\widehat{T}_n^\dagger$:

$$\begin{aligned}
\widehat{T}_n + \widehat{T}_n^\dagger &= (\breve{T}_n + \breve{T}_{n+1}e_1 + \breve{T}_{n+2}e_2 + \breve{T}_{n+3}e_3) + (\overline{\widetilde{T}_n} - \overline{\widetilde{T}_{n+1}}\mathbf{i} - \overline{\widetilde{T}_{n+2}}\boldsymbol{\varepsilon} + \overline{\widetilde{T}_{n+3}}\mathbf{h}) \\
&= (\breve{T}_n + \breve{T}_{n+1}e_1 + \breve{T}_{n+2}e_2 + \breve{T}_{n+3}e_3) + (\breve{T}_n^c - \breve{T}_{n+1}^c e_1 - \breve{T}_{n+2}^c e_2 - \breve{T}_{n+3}^c e_3) \\
&= (\breve{T}_n + \breve{T}_n^c) + (\breve{T}_{n+1} - \breve{T}_{n+1}^c)e_1 + (\breve{T}_{n+2} - \breve{T}_{n+2}^c)e_2 + (\breve{T}_{n+3} - \breve{T}_{n+3}^c)e_3 \\
&= 4T_n + 2(\widehat{T}_n - \breve{T}_n - \widetilde{T}_n).
\end{aligned}$$

□

Below, we first derive the Binet formulas for $\breve{T}_n$ and $\widetilde{T}_n$, then extend them to GTHSQ.

Lemma 3.2. *If $\breve{T}_n$ and $\widetilde{T}_n$ are the n-th generalized Tribonacci hybrid number and generalized Tribonacci split quaternion, respectively, their Binet formulas are:*

$$\breve{T}_n = \frac{P\breve{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\breve{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\breve{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}, \tag{11}$$

$$\widetilde{T}_n = \frac{P\widetilde{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}, \tag{12}$$

where: $\breve{\alpha}_1 = 1+\alpha_1\mathbf{i}+\alpha_1^2\boldsymbol{\varepsilon}+\alpha_1^3\mathbf{h}$, $\breve{\alpha}_2 = 1+\alpha_2\mathbf{i}+\alpha_2^2\boldsymbol{\varepsilon}+\alpha_2^3\mathbf{h}$, $\breve{\alpha}_3 = 1+\alpha_3\mathbf{i}+\alpha_3^2\boldsymbol{\varepsilon}+\alpha_3^3\mathbf{h}$; $\breve{\alpha}_1 = 1+\alpha_1\mathbf{i}+\alpha_1^2\boldsymbol{\varepsilon}+\alpha_1^3\mathbf{h}$; $\widetilde{\alpha}_1 = 1+\alpha_1 e_1+\alpha_1^2 e_2+\alpha_1^3 e_3$, $\widetilde{\alpha}_2 = 1+\alpha_2 e_1+\alpha_2^2 e_2+\alpha_2^3 e_3$, $\widetilde{\alpha}_3 = 1+\alpha_3 e_1+\alpha_3^2 e_2+\alpha_3^3 e_3$; P, Q, R *are defined in Equation* (3)*.*

Proof. We only prove Equation (11); Equation (12) can be verified similarly. By substituting the Binet formula of T_n and the definition of $\breve{T}_n$:

$$\begin{aligned}
\breve{T}_n &= T_n + T_{n+1}\mathbf{i} + T_{n+2}\boldsymbol{\varepsilon} + T_{n+3}\mathbf{h} \\
&= [\frac{P\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}] \\
&\quad + [\frac{P\alpha_1^{n+1}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\alpha_2^{n+1}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\alpha_3^{n+1}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]\mathbf{i} \\
&\quad + [\frac{P\alpha_1^{n+2}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\alpha_2^{n+2}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\alpha_3^{n+2}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]\boldsymbol{\varepsilon} \\
&\quad + [\frac{P\alpha_1^{n+3}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\alpha_2^{n+3}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\alpha_3^{n+3}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]\mathbf{h} \\
&= \frac{P\breve{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\breve{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\breve{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}.
\end{aligned}$$

□

Theorem 3.4. *For any $n \in \mathbb{N}^*$, the Binet formula of the n-th GTHSQ $\widehat{T}_n$ is:*

$$\widehat{T}_n = \frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}, \tag{13}$$

where $\breve{\alpha}_k, \widetilde{\alpha}_k$ $(k = 1, 2, 3)$ and P, Q, R are defined as in Lemma 3.2.

Proof. Substitute the second representation of $\widehat{T}_n$ and Lemma 3.2:

$$\begin{aligned}
\widehat{T}_n &= \widetilde{T}_n + \widetilde{T}_{n+1}\mathbf{i} + \widetilde{T}_{n+2}\boldsymbol{\varepsilon} + \widetilde{T}_{n+3}\mathbf{h} \\
&= [\frac{P\widetilde{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}] \\
&\quad + [\frac{P\widetilde{\alpha}_1\alpha_1^{n+1}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\alpha_2^{n+1}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\alpha_3^{n+1}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]\mathbf{i} \\
&\quad + [\frac{P\widetilde{\alpha}_1\alpha_1^{n+2}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\alpha_2^{n+2}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\alpha_3^{n+2}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]\boldsymbol{\varepsilon} \\
&\quad + [\frac{P\widetilde{\alpha}_1\alpha_1^{n+3}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\alpha_2^{n+3}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\alpha_3^{n+3}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]\mathbf{h} \\
&= \frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}.
\end{aligned}$$

□

Theorem 3.5. *For any $n \in \mathbb{N}^*$, the generating function and exponential generating function for GTHSQ are, respectively,*

$$\sum_{n=0}^{\infty} \widehat{T}_n x^n = \frac{\widehat{T}_0 + (\widehat{T}_1 - u\widehat{T}_0)x + (\widehat{T}_2 - u\widehat{T}_1 - v\widehat{T}_0)x^2}{1 - ux - vx^2 - wx^3}$$

$$\sum_{n=0}^{\infty} \widehat{T}_n \frac{x^n}{n!} = \frac{P\widetilde{\alpha}_1\breve{\alpha}_1 e^{\alpha_1 x}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2 e^{\alpha_2 x}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3 e^{\alpha_3 x}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}$$

Proof. Let the generation function of $\widehat{T}_n$ be

$$\sum_{n=0}^{\infty} \widehat{T}_n x^n = \widehat{T}_0 + \widehat{T}_1 x + \widehat{T}_2 x^2 + \cdots + \widehat{T}_n x^n + \cdots .$$

Multiply both sides of the above equation by $-ux, -vx^2, -wx^3$ and then add them together, by (10), we have:

$$(1 - ux - vx^2 - wx^3)\sum_{n=0}^{\infty} \widehat{T}_n x^n = \widehat{T}_0 + (\widehat{T}_1 - u\widehat{T}_0)x + (\widehat{T}_2 - u\widehat{T}_1 - v\widehat{T}_0)x^2,$$

that is

$$\sum_{n=0}^{\infty} \widehat{T}_n x^n = \frac{\widehat{T}_0 + (\widehat{T}_1 - u\widehat{T}_0)x + (\widehat{T}_2 - u\widehat{T}_1 - v\widehat{T}_0)x^2}{1 - ux - vx^2 - wx^3}.$$

Substitute the Binet formula of $\widehat{T}_n$ (Equation (13)) into the exponential generating function:

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{T}_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}\right)\frac{x^n}{n!}\\ &= \frac{P\widetilde{\alpha}_1\breve{\alpha}_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\sum_{n=0}^{\infty}\frac{(\alpha_1 x)^n}{n!} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}\sum_{n=0}^{\infty}\frac{(\alpha_2 x)^n}{n!}\\ &\quad + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\sum_{n=0}^{\infty}\frac{(\alpha_3 x)^n}{n!}.\end{aligned}$$

Using the Taylor series expansion $e^t = \sum\limits_{n=0}^{\infty} \frac{t^n}{n!}$, we simplify this to:

$$\sum_{n=0}^{\infty} \widehat{T}_n \frac{x^n}{n!} = \frac{P\widetilde{\alpha}_1\breve{\alpha}_1 e^{\alpha_1 x}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2 e^{\alpha_2 x}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3 e^{\alpha_3 x}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}. \qquad \square$$

Using the sum formula of generalized Tribonacci hybrid numbers, we derive the sum formula for GTHSQ.

Lemma 3.3 ([22]). *For any $n, m \in \mathbb{N}^*$, the sum of the first $m+1$ generalized Tribonacci hybrid numbers is:*

$$\sum_{n=0}^{m} \breve{T}_n = \frac{1}{\sigma}[\breve{T}_{m+2} + (1-u)\breve{T}_{m+1} + w\breve{T}_m + A], \tag{14}$$

where: $\sigma = u + v + w - 1$, $A = B + (B - \sigma a)\mathbf{i} + [B - \sigma(a+b)]\boldsymbol{\varepsilon} + [B - \sigma(a+b+c)]\mathbf{h}$, $B = (u+v-1)a + (u-1)b - c$, a, b, c are the initial terms of the generalized Tribonacci sequence $\{T_n\}$.

Theorem 3.6. *For any $n, m \in \mathbb{N}^*$, the sum of the first $m+1$ GTHSQ is:*

$$\sum_{n=0}^{m} \widehat{T}_n = \frac{1}{\sigma}[\widehat{T}_{m+2} + (1-u)\widehat{T}_{m+1} + w\widehat{T}_m + A\delta]$$

where $\delta = 1 + e_1 + e_2 + e_3$, and σ, A, B are defined as in Lemma 3.3.

Proof. Substitute the first representation of $\widehat{T}_n$ and Lemma 3.3:

$$\begin{aligned}
\sum_{n=0}^{m} \widehat{T}_n &= \sum_{n=0}^{m} \breve{T}_n + e_1 \sum_{n=0}^{m} \breve{T}_{n+1} + e_2 \sum_{n=0}^{m} \breve{T}_{n+2} + e_3 \sum_{n=0}^{m} \breve{T}_{n+3} \\
&\quad + \frac{1}{\sigma}[\breve{T}_{m+2} + (1-u)\breve{T}_{m+1} + w\breve{T}_m + A] + \frac{1}{\sigma}[\breve{T}_{m+3} + (1-u)\breve{T}_{m+2} + w\breve{T}_{m+1} + A]e_1 \\
&\quad + \frac{1}{\sigma}[\breve{T}_{m+4} + (1-u)\breve{T}_{m+3} + w\breve{T}_{m+2} + A]e_2 + \frac{1}{\sigma}[\breve{T}_{m+5} + (1-u)\breve{T}_{m+4} + w\breve{T}_{m+3} + A]e_3 \\
&= \frac{1}{\sigma}[(\breve{T}_{m+2} + \breve{T}_{m+3}e_1 + \breve{T}_{m+4}e_2 + \breve{T}_{m+5}e_3) + (1-u)(\breve{T}_{m+1} + \breve{T}_{m+2}e_1 + \breve{T}_{m+3}e_2 + \breve{T}_{m+4}e_3) \\
&\quad + w(\breve{T}_m + \breve{T}_{m+1}e_1 + \breve{T}_{m+2}e_2 + \breve{T}_{m+3}e_3) + A(1 + e_1 + e_2 + e_3)] \\
&= \frac{1}{\sigma}[\widehat{T}_{m+2} + (1-u)\widehat{T}_{m+1} + w\widehat{T}_m + A\delta].
\end{aligned}$$

□

We extend classic combinatorial identities (Vajda, Catalan, and Cassini) to GTHSQ using its Binet formula.

Theorem 3.7 (Vajda Identity)**.** *For any $m, n, r \in \mathbb{N}^*$, the following identity holds:*

$$\begin{aligned}
\widehat{T}_{n+m}\widehat{T}_{n+r} - \widehat{T}_n\widehat{T}_{n+m+r} &= \frac{PQ\widetilde{\alpha}_1\breve{\alpha}_1\widetilde{\alpha}_2\breve{\alpha}_2(\alpha_1\alpha_2)^n E_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} \\
+ \frac{PR\widetilde{\alpha}_1\breve{\alpha}_1\widetilde{\alpha}_3\breve{\alpha}_3(\alpha_1\alpha_3)^n E_2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} &+ \frac{QR\widetilde{\alpha}_2\breve{\alpha}_2\widetilde{\alpha}_3\breve{\alpha}_3(\alpha_2\alpha_3)^n E_3}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},
\end{aligned}$$

where: $E_1 = \alpha_1^m\alpha_2^r + \alpha_1^r\alpha_2^m - \alpha_2^{m+r} - \alpha_1^{m+r}$, $E_2 = \alpha_1^m\alpha_3^r + \alpha_1^r\alpha_3^m - \alpha_3^{m+r} - \alpha_1^{m+r}$, $E_3 = \alpha_2^m\alpha_3^r + \alpha_2^r\alpha_3^m - \alpha_3^{m+r} - \alpha_2^{m+r}$.

Proof. By the virtue of GTHSQ's Binet formula $\widehat{T}_n$ (Equation (13)) into the left-hand side:

$$\begin{aligned}
&\widehat{T}_{n+m}\widehat{T}_{n+r} - \widehat{T}_n\widehat{T}_{n+m+r} \\
&= [\frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^{n+m}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^{n+m}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^{n+m}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}] \\
&\quad \cdot [\frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^{n+r}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^{n+r}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^{n+r}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}] \\
&\quad - [\frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^{n}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^{n}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^{n}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}] \\
&\quad \cdot [\frac{P\widetilde{\alpha}_1\breve{\alpha}_1\alpha_1^{n+m+r}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{Q\widetilde{\alpha}_2\breve{\alpha}_2\alpha_2^{n+m+r}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{R\widetilde{\alpha}_3\breve{\alpha}_3\alpha_3^{n+m+r}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}]
\end{aligned}$$

Expanding and simplifying this product using the definitions of E_1, E_2, E_3 yields the desired identity. □

Corollary 3.1 (Catalan Identity). *For any* $n, r \in \mathbb{N}^*$ *with* $n > r$, *set* $m = r$ *in the Vajda identity. The resulting identity is:*

$$\widehat{T}_{n-r}\widehat{T}_{n+r} - (\widehat{T}_n)^2 = \frac{PQ\widetilde{\alpha}_1\breve{\alpha}_1\widetilde{\alpha}_2\breve{\alpha}_2(\alpha_1\alpha_2)^n F_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}$$
$$\frac{PR\widetilde{\alpha}_1\breve{\alpha}_1\widetilde{\alpha}_3\breve{\alpha}_3(\alpha_1\alpha_3)^n F_2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} + \frac{QR\widetilde{\alpha}_2\breve{\alpha}_2\widetilde{\alpha}_3\breve{\alpha}_3(\alpha_2\alpha_3)^n F_3}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},$$

where: $F_1 = \alpha_1^{-r}\alpha_2^r + \alpha_1^r\alpha_2^{-r} - 2$, $F_2 = \alpha_1^{-r}\alpha_3^r + \alpha_1^r\alpha_3^{-r} - 2$, $F_3 = \alpha_2^{-r}\alpha_3^r + \alpha_2^r\alpha_3^{-r} - 2$.

Corollary 3.2 (Cassini Identity). *For any* $n \in \mathbb{N}^*$ *with* $n > 1$, *set* $r = 1$ *and* $m = -1$ *in the Vajda identity. The resulting identity is:*

$$\widehat{T}_{n-1}\widehat{T}_{n+1} - (\widehat{T}_n)^2 = \frac{PQ\widetilde{\alpha}_1\breve{\alpha}_1\widetilde{\alpha}_2\breve{\alpha}_2(\alpha_1\alpha_2)^n G_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}$$
$$\frac{PR\widetilde{\alpha}_1\breve{\alpha}_1\widetilde{\alpha}_3\breve{\alpha}_3(\alpha_1\alpha_3)^n G_2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} + \frac{QR\widetilde{\alpha}_2\breve{\alpha}_2\widetilde{\alpha}_3\breve{\alpha}_3(\alpha_2\alpha_3)^n G_3}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},$$

where: $G_1 = \alpha_1^{-1}\alpha_2 + \alpha_1\alpha_2^{-1} - 2$, $G_2 = \alpha_1^{-1}\alpha_3 + \alpha_1\alpha_3^{-1} - 2$, $G_3 = \alpha_2^{-1}\alpha_3 + \alpha_2\alpha_3^{-1} - 2$.

4 Matrix representation of generalized Tribonacci hybrid split quaternions

Matrix representations provide a powerful tool for analyzing the properties of sequences and quaternions. In this section, we define the GTHSQ matrix and derive its key properties.

First, we recall the S-matrix, a generalization of the R-matrix [21], which is widely used to represent third-order linear recurrence sequences [18].

$$S = \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{15}$$

For $n \geq 0$, the n-th power of S is given by:

$$S^n = \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} U_{n+2} & vU_{n+1} + wU_n & wU_{n+1} \\ U_{n+1} & vU_n + wU_{n-1} & wU_n \\ U_n & vU_{n-1} + wU_{n-2} & wU_{n-1} \end{bmatrix}, \tag{16}$$

where $\{U_n\}_{n\geq 0}$ is a third-order linear recurrence sequence satisfying:

$$U_n = uU_{n-1} + vU_{n-2} + wU_{n-3}, \quad n \geq 3$$

with initial terms

$$U_0 = 0, U_1 = 0, U_2 = 1, u, v, w \in \mathbb{R}.$$

Notably, $\{U_n\}_{n\geq 0}$ is a special case of the generalized Tribonacci sequence $\{T_n\}_{n\geq 0}$.

We define the GTHSQ matrix using the terms of $\widehat{T}_n$ and analyze its multiplication with S^n:

$$\widehat{T}_s = \begin{bmatrix} \widehat{T}_4 & v\widehat{T}_3 + w\widehat{T}_2 & w\widehat{T}_3 \\ \widehat{T}_3 & v\widehat{T}_2 + w\widehat{T}_1 & w\widehat{T}_2 \\ \widehat{T}_2 & v\widehat{T}_1 + w\widehat{T}_0 & w\widehat{T}_1 \end{bmatrix}. \tag{17}$$

This matrix can be called as the Generalized Tribonacci hybrid split quaternions matrix. Then, we can give the next theorem to the $\widehat{T}_s$-matrix.

Theorem 4.1. *For any $n \geq 0$, the product of $\widehat{T}_s$ and S^n is:*

$$\widehat{T}_s \cdot S^n = \begin{bmatrix} \widehat{T}_{n+4} & v\widehat{T}_{n+3} + w\widehat{T}_{n+2} & w\widehat{T}_{n+3} \\ \widehat{T}_{n+3} & v\widehat{T}_{n+2} + w\widehat{T}_{n+1} & w\widehat{T}_{n+2} \\ \widehat{T}_{n+2} & v\widehat{T}_{n+1} + w\widehat{T}_{n} & w\widehat{T}_{n+1} \end{bmatrix}. \tag{18}$$

Proof. We use mathematical induction on n. For $n = 1$, we calculate the product $\widehat{T}_s \cdot S$:

$$\begin{aligned} \widehat{T}_s \cdot S &= \begin{bmatrix} \widehat{T}_4 & v\widehat{T}_3 + w\widehat{T}_2 & w\widehat{T}_3 \\ \widehat{T}_3 & v\widehat{T}_2 + w\widehat{T}_1 & w\widehat{T}_2 \\ \widehat{T}_2 & v\widehat{T}_1 + w\widehat{T}_0 & w\widehat{T}_1 \end{bmatrix} \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} u\widehat{T}_4 + v\widehat{T}_3 + w\widehat{T}_2 & v\widehat{T}_4 + w\widehat{T}_3 & w\widehat{T}_4 \\ u\widehat{T}_3 + v\widehat{T}_2 + w\widehat{T}_1 & v\widehat{T}_3 + w\widehat{T}_2 & w\widehat{T}_3 \\ u\widehat{T}_2 + v\widehat{T}_1 + w\widehat{T}_0 & v\widehat{T}_2 + w\widehat{T}_1 & w\widehat{T}_2 \end{bmatrix} \\ &= \begin{bmatrix} \widehat{T}_5 & v\widehat{T}_4 + w\widehat{T}_3 & w\widehat{T}_4 \\ \widehat{T}_4 & v\widehat{T}_3 + w\widehat{T}_2 & w\widehat{T}_3 \\ \widehat{T}_3 & v\widehat{T}_2 + w\widehat{T}_1 & w\widehat{T}_2 \end{bmatrix}, \end{aligned}$$

thus, the equality holds for $n = 1$. Now suppose that the equality is true for $n = m$, this is

$$\widehat{T}_s \cdot \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^m = \begin{bmatrix} \widehat{T}_{m+4} & v\widehat{T}_{m+3} + w\widehat{T}_{m+2} & w\widehat{T}_{m+3} \\ \widehat{T}_{m+3} & v\widehat{T}_{m+2} + w\widehat{T}_{m+1} & w\widehat{T}_{m+2} \\ \widehat{T}_{m+2} & v\widehat{T}_{m+1} + w\widehat{T}_{m} & w\widehat{T}_{m+1} \end{bmatrix},$$

then we can verify it for $n = m + 1$ as follows:

$$\begin{aligned} \widehat{T}_s \cdot S^{m+1} &= \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{m+1} = \widehat{T}_s \cdot S^m \cdot S \\ &= \begin{bmatrix} \widehat{T}_{m+4} & v\widehat{T}_{m+3} + w\widehat{T}_{m+2} & w\widehat{T}_{m+3} \\ \widehat{T}_{m+3} & v\widehat{T}_{m+2} + w\widehat{T}_{m+1} & w\widehat{T}_{m+2} \\ \widehat{T}_{m+2} & v\widehat{T}_{m+1} + w\widehat{T}_{m} & w\widehat{T}_{m+1} \end{bmatrix} \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \widehat{T}_{m+5} & v\widehat{T}_{m+4} + w\widehat{T}_{m+3} & w\widehat{T}_{m+4} \\ \widehat{T}_{m+4} & v\widehat{T}_{m+3} + w\widehat{T}_{m+2} & w\widehat{T}_{m+3} \\ \widehat{T}_{m+3} & v\widehat{T}_{m+2} + w\widehat{T}_{m+1} & w\widehat{T}_{m+2} \end{bmatrix}. \end{aligned}$$

Thus the proof is completed. □

Using Theorem 4.1, we derive a closed-form expression for $\widehat{T}_n$ and its determinant representation.

Corollary 4.1. *For any $n \geq 0$,*

$$\widehat{T}_{n+2} = \widehat{T}_2 U_{n+2} + (v\widehat{T}_1 + w\widehat{T}_0)U_{n+1} + w\widehat{T}_1 U_n,$$

where $\{U_n\}$ is the third-order recurrence sequence defined in Section 4.1.

Proof. The proof can be easily seen by the coefficient $(3, 1)$ of the matrix $\widehat{T}_s \cdot S^n$ and the Equation (18). □

Theorem 4.2. *For any $n \in \mathbb{N}^*$ and $n > 1$, GTHSQ has the following matrix and determinant properties:*

(i) $$\begin{bmatrix} \widehat{T}_{n+2} \\ \widehat{T}_{n+1} \\ \widehat{T}_n \end{bmatrix} = \begin{bmatrix} u & v & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} \widehat{T}_2 \\ \widehat{T}_1 \\ \widehat{T}_0 \end{bmatrix}$$

(ii) $$\widehat{T}_n = \det \begin{bmatrix} \widehat{T}_0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \widehat{T}_1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \widehat{T}_2 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & w & v & u & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & w & v & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & w & \cdots & u & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & v & u \end{bmatrix}.$$

Proof. (i) It is easy to prove using (10) and induction, and the proof is omitted here due to space limitations.

(ii) It can be proved by using (10) and Theorem 5 in reference [9] and by calculating the tridiagonal determinant. The process is omitted here. □

5 Conclusion

In this paper, we define the Generalized Tribonacci Hybrid-Split Quaternion (GTHSQ) by integrating generalized Tribonacci sequences, hybrid numbers, and split quaternions. We obtain its fundamental algebraic properties, including the inheritance of the third-order linear recurrence relation of generalized Tribonacci sequences. Also, we derive key analytical tools such as its Binet formula, ordinary generating function, and exponential generating function, and extend classic combinatorial identities (Vajda, Catalan, and Cassini) to the GTHSQ framework.

Moreover, we present the matrix representation of GTHSQ. We define a specialized matrix ($\widehat{T}_s$) with GTHSQ entries and prove that it interacts with the third-order recurrence matrix to generate higher-order GTHSQ elements. Through induction, we verify that the product can efficiently compute GTHSQ terms, and we also derive matrix-based expressions for GTHSQ elements, including a determinant representation.

In the future, we plan to generalize the GTHSQ concept further by integrating other higher-order sequences (like Padovan or Jacobsthal sequences) or alternative hypercomplex systems (such as bicomplex numbers or dual quaternions).

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