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Power series expansions of arbitrary order functional difference equations

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Abstract: This paper looks at some real and complex generalizations of power series associated with some arbitrary order functional difference equations considered as generalizations and extensions of Fibonacci and Lucas numbers. It does this by drawing on, and interconnecting, some classic number theoretic results of Carlitz, Fasenmyer and Horadam.

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1 Introduction

Functional difference equations have received spasmodic, but serious, attention over the years [3, 7, 12, 17, 22]. Here we propose to consider some real and complex properties related to their power-series [4]. Following Carlitz [5], we define the functional difference equation of arbitrary order, *r*, by.

$$w_n^*(x,\lambda)=\sum_{k=0}^{\infty} w_{n+k}^{(r)}\binom{x}{k}\lambda^k, \tag{1.1}$$

in which λ is an arbitrary integer; and from Jarden [14], $w_n^{(r)}$ is an element of an arbitrary order, *r*, linear recursive sequence such that

$$w_n^{(r)}=\sum_{i=1}^{r}\alpha_{r,i}^n\frac{d_i}{d},$$

where $\alpha_{r,i},\ i=1,2,\ldots,r,$ are the distinct roots of the auxiliary equation of $w_n^{(r)}$, and d_i is the determinant of order *r* formed from

$$d=\begin{vmatrix}1 & \alpha_{r,1} & \cdots & \alpha_{r,1}^{r-1}\\ 1 & \alpha_{r,2} & \cdots & \alpha_{r,2}^{r-1}\\ & & \cdots & \\ 1 & \alpha_{r,r} & \cdots & \alpha_{r,r}^{r-1}\end{vmatrix}$$

on replacing the *i*-th row by the *r* initial terms of $\{w_n^{(r)}\}$.

2 Power series

From (1.1), it follows that

$$w_n^*(0,1)=w_n^{(r)}$$

and, with arbitrary integers $P_{r,j}$,

$$\begin{aligned}w_n^*(x,\lambda)&=\sum_{k=0}^{\infty} w_{n+k}^{(r)}\binom{x}{k}\lambda^k\\ &=\sum_{k=0}^{\infty}\sum_{j=1}^{r}(-1)^{j+1}P_{r,j}w_{n+k-j}^{(r)}\binom{x}{k}\lambda^k\\ &=\sum_{j=1}^{r}(-1)^{j+1}P_{r,j}\sum_{k=0}^{\infty}w_{n+k-j}^{(r)}\binom{x}{k}\lambda^k\\ &=\sum_{j=1}^{r}(-1)^{j+1}P_{r,j}w_{n-j}^*(x,\lambda)\end{aligned}$$

which is a recurrence relation of order *r*. When *r* = 2 and the $P_{r,j}$, are unity, this reduces to an ordinary second order recurrence relation. The second line of this proof comes from [19, Chapters 14, 18].

We are now in a position to show that the power series in (1.1) converges for a sufficiently small λ:

$$\begin{aligned} w_n^*(x+1,\lambda) - w_n^*(x,\lambda) &= \sum_{k=0}^{\infty} w_{n+k}^{(r)} \left(\binom{x+1}{k} - \binom{x}{k} \right) \lambda^k \\ &= \lambda \sum_{k=1}^{\infty} w_{n+k}^{(r)} \binom{x}{k-1} \lambda^{k-1} \\ &= \lambda \sum_{k=0}^{\infty} w_{n+k+1}^{(r)} \binom{x}{k} \lambda^k \\ &= \lambda w_{n+1}^*(x,\lambda), \end{aligned}$$

which is effectively a functional difference equation for $w_n^*(x,\lambda)$. If we set the difference operator

$$\Delta w_n^*(x,\lambda) = w_n^*(x+1,\lambda) - w_n^*(x,\lambda),$$

then we have the rather neat difference calculus result that

$$\Delta w_n^*(x,\lambda) = \lambda w_{n+1}^*(x,\lambda).$$

Furthermore, if

$$w_n^*(x,\lambda) = \sum_{j=1}^{r} A_{r,j} \alpha_{r,j}^n,$$

where $A_{r,j}$ depends on the initial values of $\{ w_n^{(r)} \}$, then from (1.1):

$$\begin{aligned} w_n^*(x,\lambda) &= \sum_{k=0}^{\infty} w_{n+k}^{(r)} \binom{x}{k} \lambda^k \\ &= \sum_{k=0}^{\infty} \sum_{j=1}^{r} A_{r,j} \alpha_{r,j}^{n+k} \binom{x}{k} \lambda^k \\ &= \sum_{j=1}^{r} A_{r,j} \alpha_{r,j}^n \sum_{k=0}^{\infty} \binom{x}{k} \left(\lambda \alpha_{r,j} \right)^k \\ &= \sum_{j=1}^{r} A_{r,j} \alpha_{r,j}^n \left(1 + \lambda \alpha_{r,j} \right)^k \end{aligned} \tag{2.1}$$

For example, from (1.1):

$$w_n^*(1,\lambda) = \sum_{k=0}^{\infty} w_{n+k}^{(r)} \binom{1}{k} \lambda^k = w_n^{(r)} + \lambda w_{n+1}^{(r)}$$

and from (2.1):

$$w_n^*(x,\lambda) = \sum_{j=1}^{r} A_{r,j} \alpha_{r,j}^n + \lambda \sum_{j=1}^{r} A_{r,j} \alpha_{r,j}^{n+1} = w_n^{(r)} + \lambda w_{n+1}^{(r)},$$

as required. It also follows from (2.1) that

$$w_n^*\left(\left(x+y\right),\lambda\right)=\sum_{j=1}^{r}A_{r,j}\alpha_{r,j}^{n}\left(1+\lambda\alpha_{r,j}\right)^{x+y}$$

$$=\sum_{k=0}^{\infty}\sum_{j=1}^{r}A_{r,j}\alpha_{r,j}^{n+k}\left(1+\lambda\alpha_{r,j}\right)^{x}\binom{y}{k}\lambda^{k}$$

$$=\sum_{k=0}^{\infty}w_{n+k}^*\left(x+y\right)\binom{y}{k}\lambda^{k},$$

which agrees with Section 3 of [20]. We can thus also obtain from (2.1) that

$$w_n^*\left(\left(x+y\right),\lambda\right)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}w_{n+j+k}^{(r)}\binom{x}{j}\binom{y}{k}\lambda^{j+k}.$$

Similarly,

$$w_n^*\left(\left(x+y+z\right),\lambda\right)=\sum_{k=0}^{\infty}w_{n+k}^*\left(x+y\right)\binom{z}{k}\lambda^{k}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}w_{n+i+j+k}^{(r)}\binom{x}{i}\binom{y}{j}\binom{z}{k}\lambda^{i+j+k}.$$

It can then be proved by induction that for i = 1, 2, …, k,

$$w_n^*\left(\left(\sum_{i=1}^{k}x_i\right),\lambda\right)=\sum_{j_i}^{\infty}w_{n+\Sigma j_i}^{(r)}\prod_{i=1}^{k}\binom{x_i}{j_i}\lambda^{\Sigma j_i},\ i=1,2,\ldots,k. \tag{2.2}$$

Proof: We have established (2.2) for i = 1,2,3; assume it is also true for i = 4, 5, …, k – 1. Then

$$w_n^*\left(\left(\sum_{i=1}^{k}x_i\right),\lambda\right)=\sum_{j_k}^{\infty}w_{n+j_k}^*\left(\sum_{i=1}^{k-1}x_i\right)\binom{x_j}{j_k}\lambda^{j_k}$$

and the result follows since

$$w_{n+j_k}^*\left(\sum_{i=1}^{k-1}x_i\right)=\sum_{j_i=0}^{\infty}w_{n+\Sigma j_i}^{(r)}\prod_{i=1}^{k-1}\binom{x_i}{j_i}\lambda^{\Sigma j_i}$$

from the inductive hypothesis. □

For further related convergence properties, see [18].

3 Complex plane

We can also use these functional difference equations to investigate analytic extensions of Lucas' fundamental basic and primordial sequences, $\{u_n^{(r)}\}$ and $\{v_n^{(r)}\}$, from r = 2 to arbitrary order, r, to the complex plane as Horadam did for complex Fibonacci numbers [11].

In the terminology of Macmahon [16], $u_n^{(r)}$ is the homogeneous product sum of weight n of the quantities $\alpha_{r,j}$; that is, it is the sum of symmetric functions formed from a partition of n. $v_n^{(r)}$ is then the sum of the n-th powers of the quantities $\alpha_{r,j}$ [13]. Now, if $\alpha_{r,j}<0$, let

$$\mathbb{Z}\alpha_{r,j} = e^{i\pi}\left(-\alpha_{r,j}\right), j=1,2,\ldots,r,$$

and let

$$v^{(r)}\left(\mathbb{Z}\right)=\sum_{j=1}^{r}\alpha_{r,j,}^{\mathbb{Z}}\ \mathbb{Z}\in\mathbb{C},$$

so that when $n \in \mathbb{R}$, $v^{(r)}\left(\mathbb{Z}\right)=v_n^{(r)}$. We look now at the periodic properties of $v^{(r)}\left(\mathbb{Z}\right)$. Let

$$\alpha_{r,j}^{z} = \alpha_{r,j}^{z}e^{2\pi i} = \alpha_{r,j}^{z+p_j}$$

so that p_j is the period of $\alpha_{r,j}^{z}$. Clearly $v^{(r)}\left(z\right)$ is not periodic if $\alpha_{r,j}$ are distinct, since

$$v^{(r)}\left(z+q\right)=\sum_{j=1}^{r}\alpha_{r,j}^{z+q}\neq\sum_{j=1}^{r}\alpha_{r,j}^{z+p_j} = v^{(r)}\left(z\right).$$

Many of the identities of $v_n^{(r)}$ carry over to $v^{(r)}\left(z\right)$. One of these is

$$v^{(r)}\left(z\right)=\sum_{k=0}^{\infty}\sum_{j=1}^{r}\left(\ln\alpha_{r,j}\right)^{k}\alpha_{r,j}^{a}\frac{\left(z-a\right)^{k}}{k!}. \tag{3.1}$$

Proof: Suppose $v^{(r)}(z)$ is analytic inside, and on, a circle with centre at $z = a$. Then, for all complex points z in the circle, the Taylor series representation of $v^{(r)}(z)$ is given by formula (3.1). When $a = 0$ and $z = n$, (3.1) becomes

$$v_n^{(r)}=\sum_{k=0}^{\infty}\sum_{j=1}^{r}\left(\ln\alpha_{r,j}\right)^{k}\frac{n^k}{k!},$$

which agrees with a formula for $v_n^{(2)}$ discovered by Whitney [24]. We can reconcile this representation of $v_n^{(r)}$ with the usual form, as follows

$$\begin{aligned} v_n^{(r)} &= \sum_{k=0}^{\infty}\sum_{j=1}^{r}\left(\ln\alpha_{r,j}\right)^{k}\frac{n^k}{k!} \\ &= \sum_{j=1}^{r}\sum_{k=0}^{\infty}\left(\ln\alpha_{r,j}\right)^{k}\frac{n^k}{k!} \\ &= \sum_{j=1}^{r}\exp\left(\ln\alpha_{r,j}^{n}\right) \\ &= \sum_{1}^{r}\alpha_{r,j}^{n}, \end{aligned}$$

as required. Studies of other analytic properties of $v^{(r)}(z)$, such as the genus, canonical product, and Hadamard's factorization theorem would seem to yield little specific knowledge about $v^{(r)}(z)$ without information about its zeros. However, for the generalized Fibonacci polynomials $u_n^{(r)}$, defined by

$$\sum_{n=0}^{\infty}u_n^{(r)}\left(x\right)\frac{t^n}{n!}=\exp\left(xt+\sum_{m=1}^{\infty}v_m^{(r)}\frac{t^m}{m}\right),$$

we have that

$$u_n^{(r)}(x)=\frac{1}{2\pi i}\oint\frac{\exp\left(zw+v_m^{(r)}\frac{w^m}{m}\right)}{w^{n+1}}\,dw,$$

where the contour encloses the origin. The result is a direct application of Cauchy's formula [6]. Readers who wish to extend these ideas could delve into the work of Fasenmyer, who under the supervision of Earl Rainville [21] extended the work of Bateman's $Z_n(t)$ and J_n^u [2] into pioneering results with hypergeometric polynomials in particular, and in combinatorics in general [1]. Sister Fasenmyer's work [8, 9] also appears in higher degrees by research, as does that of Herb Wilf [25] who championed the role of Fasenmyer: "Wilf and Zeilberger then worked to push Sister Celine's methods even further to produce what today is called 'WZ theory'… It allows an extremely elegant proof of certain classes of combinatorial identities and also provides an algorithm to generate new identities from old ones" [15].

4 Concluding comments for further development

The extension of Jackson's calculus [14] into the realm of discrete analytic functions offers scope for further related research. For instance, Harman [10] extended some of Jackson's calculus of q-functions into the complex plane of lattice points of the form $(tq^m x, tq^n y)$ where $q,x,y\in\mathbb{R}$ and $m,n\in\mathbb{Z}$. Harman also developed Cauchy integral formulas and found an analogue $z^{\underline{a}}$ of the function z^a for monodiffric functions [23], which are discrete analytic functions defined on the set of Gaussian integers to satisfy the forward difference equation

$$i\left(f(z+1)-f(z)\right)=f(z+i)-f(z).$$

Finally, the monodiffric function $z^{\underline{n}}$ which corresponds to z^n has some notational analogies with (1.1) and can be expressed as:

$$z^{\underline{n}}=\sum_{j=0}^{n}\binom{n}{j}x^{\underline{n-j}}i^j y^{\underline{j}},\quad z=(x,y),$$

which suggests some further comparative generalizations.

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