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# On an infinite family of unipotent Sylvester–Kac-like matrices

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**Abstract:** Classical Sylvester–Kac matrices are tridiagonal integral matrices with positive off-diagonal entries and fully integral spectra. Here, by relaxing the positivity requirement and using a lower Pascal triangle framework, we define, for each positive integer $n$, a unipotent Sylvester–Kac-like matrix in which $n$ is the only eigenvalue. This construction highlights the connection to the original Sylvester–Kac matrices while introducing a new family of unipotent matrices with distinctive properties.
**Keywords:** Sylvester–Kac matrix, Tridiagonal matrix, Unipotent matrix, Eigenvalues.
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# 1 Introduction

In 1854, the eclectic English mathematician James Joseph Sylvester, founder of the American Journal of Mathematics, in a half-page note ([16]), conjectured the characteristic polynomial of the matrix

$$A_{n+1} = \begin{pmatrix} 0 & 1 & & & & \\ n & 0 & 2 & & & \\ & n-1 & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & 0 & n \\ & & & & 1 & 0 \end{pmatrix}. \tag{1}$$

The conjectured factorization of these polynomials would immediately imply that the eigenvalues are

$$-n, -(n-2), \ldots, n-2, n\,. \tag{2}$$

Various attempts to prove this seemingly simple result appeared soon after its formulation, and to this day there remains some controversy over who first established it rigorously.

Marc Kac is credited with providing the first methodical proof of the eigenvalues and corresponding eigenvectors of this matrix, presented in his 1950 essay, which was awarded the Chauvenet Prize for excellence in mathematical exposition [11]. However, as Kac himself later admitted, there were previous proofs. For a fairly comprehensive list of references, extensions, proofs, and discussions on this topic, the reader is referred to [1, 5, 6, 8, 10, 12, 15]. Since then, tridiagonal integral matrices with integral spectra (or satisfying certain algebraic relations) with positive sub- and super-diagonal entries have become commonly known as Sylvester–Kac matrices.

A square matrix is called unipotent if its characteristic polynomial is a power of $x-1$, i.e., if all its eigenvalues are equal to $1$. In this work, we relax this condition to allow the eigenvalue to be an arbitrary positive integer. Matrices of this type have been widely studied in various areas of mathematics (cf., e.g., [4, 14, 17]). Recently in [7], it was proved that the Schwarz matrix

$$J_n = \begin{pmatrix} n & \frac{n-1}{1} & & & \\ -\frac{n+1}{3} & 0 & \frac{n-2}{3} & & \\ & -\frac{n+2}{5} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{2n-3} \\ & & & -1 & 0 \end{pmatrix}$$

is unipotent. As in [5], a certain matrix with entries defined by the Pascal triangle plays a crucial role in [7]. In this note, we define the $n \times n$ Pascal matrix as the lower triangular matrix $P_n = (p_{ij})$ with the entries

$$p_{ij} = \begin{cases} \binom{j-1}{i-1}, & \text{if } i \leqslant j\,, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix and its extensions are well-known in the literature (cf., e.g., [2, 3, 9, 13]). Moreover, it is elementary to prove that the inverse of $P$ is the matrix $P_n^{-1} = (q_{ij})$ whose entries are

$$q_{ij} = \begin{cases} (-)^{i+j}\binom{j-1}{i-1}, & \text{if } i \leqslant j\,, \\ 0, & \text{otherwise.} \end{cases}$$

In this brief note, for each positive integer $n$, we construct a unipotent Sylvester–Kac-like matrix whose only eigenvalue is $n$. The main tool in our construction is the Pascal matrix $P_n$ defined above, which allows us to define this new family of matrices based on the structure of $A_{n+1}$ (1).

## 2 Unipotent Sylvester–Kac-like matrices

In this main section, for each positive integer $n$, we present a unipotent Sylvester–Kac matrix where the subdiagonal is all negative, where $n$ is the only eigenvalue.

**Theorem 2.1.** *Let $n$ be a nonnegative integer. Then $n$ is the only eigenvalue of the matrix*

$$U_{n+1} = \begin{pmatrix} 0 & 1 & & & & \\ -n & 2 & 2 & & & \\ & -(n-1) & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & -2 & 2(n-1) & n \\ & & & & -1 & 2n \end{pmatrix}. \tag{3}$$

*Proof.* We claim that

$$U_{n+1}P_{n+1} = P_{n+1}\begin{pmatrix} n & & & & \\ -n & n & & & \\ & \ddots & \ddots & & \\ & & -2 & n & \\ & & & -1 & n \end{pmatrix}, \tag{4}$$

or, equivalently,

$$P_{n+1}^{-1}U_{n+1}P_{n+1} = \begin{pmatrix} n & & & & \\ -n & n & & & \\ & \ddots & \ddots & & \\ & & -2 & n & \\ & & & -1 & n \end{pmatrix}, \tag{5}$$

in which we can deduce that $n$ is the only eigenvalue of $U_{n+1}$.

On the one hand, we can easily see that the $(i, j)$-entry of $U_{n+1}P_{n+1}$ is equal to

$$-(n-i+2)\binom{j-1}{i-2} + 2(i-1)\binom{j-1}{i-1} + i\binom{j-1}{i}.$$

On the other hand, the $(i,j)$-entry of

$$P_{n+1}\begin{pmatrix} n & & & & \\ -n & n & & & \\ & \ddots & \ddots & & \\ & & -2 & n & \\ & & & -1 & n \end{pmatrix}$$

is equal to

$$\binom{j-1}{i-1}n - \binom{j}{i-1}(n-j+1)\,.$$

It is standard (although somewhat tedious) to verify that

$$-(n-i+2)\binom{j-1}{i-2} + 2(i-1)\binom{j-1}{i-1} + i\binom{j-1}{i} = \binom{j-1}{i-1}n - \binom{j}{i-1}(n-j+1)$$

holds for any $i, j$, completing the proof of the claim. $\square$


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## References


[1] Anđelić, M., da Fonseca, C. M., Kılıç, E., & Stanić, Z. (2022). A Sylvester–Kac matrix type and the Laplacian controllability of half graphs. *Electronic Journal of Linear Algebra*, 38, 559–571.

[2] Brawer, R., & Pirovino, M. (1992). The linear algebra of the Pascal matrix. *Linear Algebra and Its Applications*, 174, 13–23.

[3] Call, G. S., & Velleman, D. J. (1993). Pascal's matrices. *The American Mathematical Monthly*, 100, 372–376.

[4] Can, M. B., Howe, R., & Joyce, M. (2013). Unipotent invariant matrices. *Linear Algebra and Its Applications*, 439, 196–210.

[5] Du, Z., & da Fonseca, C. M. (2024). Sylvester–Kac matrices with quadratic spectra: A comprehensive note. *The Ramanujan Journal*, 65, 1313–1322.

[6] Du, Z., & da Fonseca, C. M. (2025). A note on the eigenvalues of a Sylvester–Kac matrix with off-diagonal biperiodic perturbations. *Journal of Computational and Applied Mathematics*, 461, 116429.

[7] Dyachenko, A., da Fonseca, C. M., & Tyaglov, M. (2025). $a$-potent Schwarz matrices and Bessel-like Jacobi polynomials. Submitted.

[8] Edelman, A., & Kostlan, E. (1994). The road from Kac's matrix to Kac's random polynomials. In: *Lewis, J. (Ed.). Proceedings of the Fifth SIAM Conference on Applied Linear Algebra* (pp. 503–507). Philadelphia: SIAM.

[9] Ernst, T. (2015). Factorizations for $q$-Pascal matrices of two variables. *Special Matrices*, 3, 207–213.

[10] da Fonseca, C. M., Mazilu, D. A., Mazilu, I., & Williams, H. T. (2013). The eigenpairs of a Sylvester–Kac type matrix associated with a simple model for one-dimensional deposition and evaporation. *Applied Mathematics Letters*, 26, 1206–1211.

[11] Kac, M. (1947). Random walk and the theory of Brownian motion. *The American Mathematical Monthly*, 54, 369–391.

[12] Kovačec, A. (2021). Schrödinger's tridiagonal matrix. *Special Matrices*, 9, 149–165.

[13] Lawden, G. H. (1972). Pascal matrices. *The Mathematical Gazette*, 56, 325–327.

[14] Mason, A. W. (1998). Unipotent matrices, modulo elementary matrices, in $SL_2$ over a coordinate ring. *Journal of Algebra*, 203, 134–155.

[15] Mersin, E. Ö., & Bahşi, M. (2025). A new approach to tridiagonal matrices related to the Sylvester–Kac matrix. *Notes on Number Theory and Discrete Mathematics*, 31(2), 211–227.

[16] Sylvester, J. J. (1854). Théorème sur les déterminants. *Nouvelles Annales de Mathématiques*, 13, 305.

[17] Zassenhaus, H. (1969). Characterization of unipotent matrices. *Journal of Number Theory*, 1, 222–230.