Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 4, 839–845
DOI: 10.7546/nntdm.2025.31.4.839-845

Inequalities between some arithmetic functions, II

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Received: 13 August 2025 **Revised:** 8 November 2025
Accepted: 11 November 2025 **Online First:** 12 November 2025

Abstract: As a continuation of Part I (see [1]), we offer new inequalities for classical arithmetic functions such as the Euler's totient function, the Dedekind's psi function, the sum of the positive divisors function, the number of divisors function, extended Jordan's totient function, generalized Dedekind's psi function.

Keywords: Arithmetic functions, Inequalities.
2020 Mathematics Subject Classification: 11A25, 26D99.

1 Introduction

Let $\varphi(n)$ and $\psi(n)$ denote the Euler's totient and Dedekind's psi functions. Their generalizations are Jordan's totient $J_s(n)$ and $\psi_s(n)$, defined by

$$J_s(1)=1, J_s(n)=n^s \cdot \prod_{p|n}\Big(1-\frac{1}{p^s}\Big)$$

for $n > 1$, s – arbitrary complex number, and

$$\psi_s(1)=1, \psi_s(n)=n^s \cdot \prod_{p|n}\Big(1+\frac{1}{p^s}\Big)$$

for $n > 1$, repsectively, where p denotes a prime number (see [3]). Clearly, $J_1(n) = \varphi(n)$ and $\psi_1(n) = \psi(n)$.

The sum of the positive divisors function is denoted by $\sigma(n)$, while the number of divisors is denoted by $d(n)$.

Let $\omega(n)$ denote the number of distinct prime factors of n and $\gamma(n)$ – the product of distinct prime divisors of n.

There are many important inequalities for the above functions, for a survey of results, see [2]. In a recent paper [1], certain new inequalities have been provided. The aim of this paper is to offer more new inequalities, as well as some generalizations.

2 Main results

Theorem 1. *For all* $n \neq 2, 3, 4, 6, 8, 10, 18, 30$, *one has*

$$(\varphi(n))^2 > \sigma(n). \tag{1}$$

Proof. By the classical inequality

$$\varphi(n)\sigma(n) < n^2$$

(see, e.g., [2]), one has

$$\sigma(n) < \frac{n^2}{\varphi(n)}.$$

On the other hand, by a result by D. G. Kendall and R. Osborn (see [2]), we have

$$\varphi(n) > n^{\frac{2}{3}} \quad \text{for } n > 30. \tag{2}$$

Then we get from the above that

$$\sigma(n) < \frac{n^2}{\varphi(n)} < (\varphi(n))^2 \quad \text{for } n \geq 31. \tag{3}$$

An easy computation shows that (1) is true also for the specified values. □

Corollary 1. *For* $n \geq 7$, *and* $n \neq 10, 12, 18, 30$, *one has*

$$(\varphi(n))^2 > \psi(n). \tag{4}$$

Proof. As $\sigma(n) \geq \psi(n)$, relation (4) follows by (1) for the values given there. An easy verification shows that (4) is also true for the above stated values. □

Corollary 2. *For each* $n \geq 5$, *one has*

$$(\varphi(n))^{d(n)} > (\psi(n))^{\omega(n)}. \tag{5}$$

Proof. Since $d(n) \geq 2\omega(n)$, we get that

$$(\varphi(n))^{d(n)} \geq (\varphi(n))^{2\omega(n)} > (\psi(n))^{\omega(n)},$$

by Corollary 1. A verification shows that the inequality is also valid for $n = 6, 10, 12, 18, 30$. □

Theorem 2. *For all* $n \geq 31$, *one has*

$$2^{\varphi(n)} > \psi(n). \tag{6}$$

Proof. Since $\varphi(n)\psi(n) < n^2$, it will be sufficient to prove that

$$\varphi(n) \cdot 2^{\varphi(n)} > n^2 \quad \text{for } n \geq 31. \tag{7}$$

By inequality (2) it will be sufficient to show that

$$2^x > x^2 \quad \text{for } n \geq 4, \tag{8}$$

where $x = n^{\frac{2}{3}}$.

Now by taking logarithms in (8), and considering the function

$$f(x) = x \ln 2 - 2 \ln x,$$

since

$$f'(x) = \frac{x \ln 2 - 2}{x},$$

f will be strictly increasing for

$$x \geq \frac{2}{\ln 2} = 2.8\ldots .$$

As $f(4) = 0$, Inequality (8) follows. Now remark that $n^{\frac{2}{3}} \geq 4$ for $n \geq 8$. Thus (6) holds true for any $n \geq 31$. □

Corollary 3. *For each* $n \geq 5$, *one has*

$$(d(n))^{\varphi(n)} > (\psi(n))^{\omega(n)}. \tag{9}$$

Proof. Indeed, as $d(n) \geq 2^{\omega(n)}$, one has

$$(d(n))^{\varphi(n)} \geq 2^{\omega(n)\varphi(n)} > (\psi(n))^{\omega(n)}$$

by Theorem 2, for $n \geq 31$. A direct verification shows that (9) holds true also for $5 \leq n \leq 30$. □

Theorem 3. *For each natural number* n *one has*

$$\frac{\psi(n)}{\varphi(n)} \leq \frac{3^{\omega(n)}}{\omega(n)}. \tag{10}$$

Proof. Let $p_1, \ldots, p_r$ be the distinct prime factors of n, with $p_1 < \cdots < p_r$, with $r = \omega(n)$. We will prove by induction that

$$\frac{\psi(n)}{\varphi(n)} = \frac{(p_1 + 1) \cdots (p_r + 1)}{(p_1 - 1) \cdots (p_r - 1)} \leq \frac{3^r}{r}. \tag{11}$$

For $r = 1$, it is true, as this is equivalent to $p_1 \geq 2$.

Now, if (11) is true for r, in order to prove it for $r+1$, it is immediate that we have to prove that $\frac{p+1}{p-1} \leq \frac{3r}{r+1}$, where $p = p_{r+1}$. This inequality however can be rewritten as $p(2r-1) \geq 4r+1$. Now as $r \geq 2$ and $p = p_{r+1} \geq 3$, this will follow by $3 \cdot (2r-1) \geq 4r+1$. Thus relation (11) follows. □

Corollary 4. *For each natural number n one has*

$$\varphi(n)(d(n))^2 \geq \psi(n)\omega(n). \tag{12}$$

Proof. Remark that (12) can be rewritten as

$$\frac{\psi(n)}{\varphi(n)} \leq \frac{(d(n))^2}{\omega(n)}. \tag{13}$$

Clearly,

$$\frac{(d(n))^2}{\omega(n)} \geq \frac{4^{\omega(n)}}{\omega(n)} > \frac{3^{\omega(n)}}{\omega(n)},$$

so (13) follows by (10) for $n \geq 2$. For $n = 1$, there is equality. □

Now, we will obtain a result similar to one in [1].

Theorem 4. *One has*

$$\frac{\psi(n)}{\varphi(n)} \leq (\gamma(n))^\lambda \qquad \text{for } n \geq 3, \text{odd}, \tag{14}$$

$$\frac{\psi(n)}{\varphi(n)} \leq \left(\frac{3}{2^\lambda}\right)(\gamma(n))^\lambda \quad \text{for } n \text{ even}, \tag{15}$$

where $\lambda = \frac{\ln 2}{\ln 3}$.

Proof. Let us consider the function

$$g(x) = \frac{x^\lambda \cdot (x-1)}{(x+1)}$$

for $x > 0$. An easy computation shows that

$$g'(x) \cdot (x+1)^2 = 2x^\lambda > 0.$$

Thus, g is a strictly increasing function, implying that for $p \geq 3$ one has $g(p) \geq g(3)$, where $\frac{3^\lambda}{2} = 1$. Thus we get the inequality:

$$\frac{p+1}{p-1} \leq p^\lambda \quad (p \geq 3). \tag{16}$$

Now, relation (14) follows from the fact that $\frac{\psi(n)}{\varphi(n)}$ is the product of terms of type $\frac{p+1}{p-1}$ for primes $p \geq 3$. Relation (15) follows again by (16), by writing $n = 2^k \cdot N$, with N odd, $N \geq 3$. For $N = 1$, there is an equality. □

Corollary 5. *For $n \geq 31$ and odd, one has*

$$\frac{\psi(n)}{\varphi(n)} \leq (\gamma(n))^{\lambda} < (\gamma(n))^{\frac{2}{3}} < \varphi(n). \tag{17}$$

Proof. This follows by the (14) and $\gamma(n) \leq n$, combined with relation (2). □

Theorem 5. *Let $\Omega(n)$ denote the total number of prime factors of n. If $n \geq 3$ is odd, and*

$$\frac{\Omega(n)}{\omega(n)} \geq \lambda + 1, \tag{18}$$

then

$$\psi(n) \cdot \omega(n) \leq (\varphi(n))^2. \tag{19}$$

Proof. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorization of n. Then $\Omega(n) = a_1 + \cdots + a_r$. By (14) one has

$$\frac{\psi(n)}{\varphi(n)} \leq (\gamma(n))^{\lambda} = (p_1 \cdots p_r)^{\lambda}.$$

We want to see when this last term is less than or equal to $\frac{\varphi(n)}{r}$, where $r = \omega(n)$. This is equivalent to

$$p_1^{a_1-\lambda-1} \cdots p_r^{a_r-\lambda-1} \cdot (p_1 - 1) \cdots (p_r - 1) \geq r. \tag{20}$$

Clearly, $(p_1 - 1) \cdots (p_r - 1) \geq r$ as $p_1 - 1 \geq 1, \ldots, p_r - 1 \geq r$. Now,

$$p_1^{a_1-\lambda-1} \cdots p_r^{a_r-\lambda-1} \geq 2^{\Omega(n)-(\lambda+1)\omega(n)} \geq 1,$$

by (18). Thus, inequality (19) follows. □

Corollary 6. *When n is odd and squarefull, then* (19) *holds true.*

Proof. If n is squarefull, then all $a_i \geq 2$ $(i = 1, 2, \ldots, r)$, so

$$\frac{\Omega(n)}{\omega(n)} \geq 2,$$

and (18) holds true, as $\lambda + 1 = 1.63092\ldots$. □

As a generalization of Corollary 1, we now state the following theorem.

Theorem 6. *For each real number $s \geq 2$ and positive integer $n \geq 2$, the inequality*

$$(J_s(n))^{s+1} > (\psi_s(n))^s \tag{21}$$

holds true.

Proof. The functions J_s and ψ_s being multiplicative (see [2,3]), it will be sufficient to prove (21) for prime powers $n = p^{\alpha}$ $(\alpha \geq 1, p \text{ prime})$.

Inequality (21) can be rewritten in this case as

$$(p^s)^{\alpha-1} \cdot (p^s - 1)^{s+1} > (p^s + 1)^s. \tag{22}$$

Since $(p^s)^{\alpha-1} \geq 1$, for the proof of (22) it is enough the prove the inequality

$$(p^s-1)^{\frac{p^s}{s}} > \Big(1+\frac{2}{p^s-1}\Big)^{p^s} \tag{23}$$

for any real numbers $s \geq 2, k \geq 2$.

To prove (23), we shall use the function

$$h(x) = \Big(1+\frac{2}{x}\Big)^{x+1},$$

which is strictly decreasing for $x > 0$. Indeed, for $x > 0$, we have $h'(x) = h(x) \cdot t(y)$, where $y = \frac{2}{x}$ and

$$t(y) = \ln(1+y) - \frac{y(y+2)}{2(1+y)}.$$

We will prove that $t(y) < 0$ by using the classical logarithmic inequality $L > G$, where

$$L(a,b) = \frac{b-a}{\ln b - \ln a}$$

for $a \neq b$ is the logarithmic mean, and

$$G = \sqrt{ab}$$

is the geometric mean. One has particularly that

$$L(1,1+y) > G(1,1+y),$$

so

$$\frac{\ln(1+y)}{y} < \frac{1}{\sqrt{1+y}}.$$

This is less than $\frac{y+2}{2(1+y)}$, as

$$\sqrt{1+y} < \frac{y+2}{2} = 1+\frac{y}{2}.$$

Thus $t(y) < 0$, and the result follows.

Therefore, $h'(x) < 0$ for $x > 0$. Hence h is a strictly decreasing function for $x > 0$, and therefore for $x \geq 3$, too. So for $x = p^s - 1$, we obtain

$$\frac{625}{81} = \Big(1+\frac{2}{2^2-1}\Big)^{2^2} > \Big(1+\frac{2}{p^s-1}\Big)^{p^s}$$

for $p \geq 2$ and $s \geq 2$. On the other hand, we have

$$(p^s-1)^{\frac{p^s}{s}} \geq (2^s-1)^{\frac{2^s}{s}} \geq (2^2-1)^{\frac{2^2}{2}} = 9.$$

Since $9 > \frac{625}{81}$, inequality (22) follows. □

Theorem 7. *For each real number $s \geq 2$ and positive integer $n \geq 2$, the inequality*

$$(J_s(n))^{s+\theta} > (\psi_s(n))^s. \tag{24}$$

holds, where θ is an arbitrary real number satisfying

$$1 \geq \theta > 2\frac{\ln 5}{\ln 3} - 2 = 0.929947\ldots.$$

Proof. As in the proof of Theorem 6, it will be sufficient to consider prime powers $n = p^\alpha$. Thus, we have to prove that

$$(p^{s\theta})^{\alpha-1} \cdot (p^s - 1)^{s+\theta} > (p^s + 1)^s. \tag{25}$$

Since $\alpha - 1 \geq 0$, we have $(p^{s\theta})^{\alpha-1} \geq 1$. Therefore, (25) will be proved if the inequality

$$(p^s - 1)^{s+\theta} > (p^s + 1)^s \tag{26}$$

holds for all real numbers $p \geq 2$ and $s \geq 2$. Relation (26) can be rewritten as

$$\left((p^s - 1)^{\frac{p^s}{s}}\right)^\theta > \left(1 + \frac{2}{p^s - 1}\right)^{p^s}. \tag{27}$$

Let LHS be the left-hand side of (27) and RHS be the right-hand side, respectively. From the proof of Theorem 6, we know that it is fulfilled:

$$LHS \leq (3^2)^\theta, \tag{28}$$

$$\frac{625}{81} \geq RHS. \tag{29}$$

Therefore, to prove (27), it remains to verify the inequality

$$3^{2\theta} > \frac{625}{81}. \tag{30}$$

Let $\mu = 2\,\frac{\ln 5}{\ln 3} - 2$. Then it is easy to see that

$$\mu = 2\log_3 5 - 2 = \log_3\left(\frac{5^2}{3^2}\right) = \log_3\left(\frac{25}{9}\right).$$

Hence,

$$3^{2\mu} = 3^{2\log_3\left(\frac{25}{9}\right)} = \frac{625}{81}.$$

But since $\theta > \mu$, we obtain

$$3^{2\theta} > 3^{2\mu} = \frac{625}{81},$$

and (30) is proved. This finishes the proof of Theorem 7. $\square$

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