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Alternating generalized Fibonacci sequences

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Abstract: In a recent paper, K. T. Atanassov and A. G. Shannon introduced a Fibonacci-like sequence derived from the generalized Fibonacci sequence by incorporating alternating signs into the recurrence relation. They also proposed explicit formulas for this sequence. In this work, we present the generating function for the sequence using a matrix-based approach. Furthermore, we explore additional variations of the original definition.

Keywords: Fibonacci numbers, Generating function.
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1 Preliminaries

The study of number sequences defined by higher-order homogeneous linear recurrence relations with constant coefficients has somewhat diffuse origins, likely tracing back to the late 19th century.

To the best of our knowledge, while the systematic study of generalized Fibonacci numbers of arbitrary order k was formalized by E. P. Miles, Jr. [16] in 1960, the seeds of this idea, particularly for the Tribonacci case ($k = 3$), were planted much earlier.

The sequence defined by the third-order recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ has a rich and independent history. The earliest known mathematical treatment of this sequence, that is, its study as an object of mathematical interest in its own right, currently appears to be a 1914 note by Nikolai Aleksandrovich Agronomof [2], as recently brought to light in the fascinating and captivating masterpiece of mathematical historiography by Tuenter [19]. Further pre-Miles occurrences include its appearance in a problem on sheep reproduction posed by Martin in 1892 [14] and its connection to a population model for Charles Darwin's elephants [19]. The sequence was also examined by Jarden in his precursor work [12] in 1953.

The modern terminology and a surge of interest in these sequences are often attributed to Mark Feinberg. In 1963, the then fourteen-year-old student, in his award-winning project, coined the terms "Tribonacci" and "Tetranacci" for the cases $k = 3$ and $k = 4$ [7]. In both cases, the first zero terms were ignored, starting the Tribonacci sequence with $1, 1, 2, 4, \ldots$.

It was Miles [16] who provided the comprehensive framework, defining the *generalized Fibonacci numbers* for any $k \geqslant 2$ by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for } n \geqslant k, \tag{1}$$

with initial conditions

$$F_0^{(k)} = F_1^{(k)} = \cdots = F_{k-2}^{(k)} = 0 \quad \text{and} \quad F_{k-1}^{(k)} = 1. \tag{2}$$

The connection with matrix methods is evident from the outset of this general study. Around the same time as Miles, M. E. Waddill, in his 1962 dissertation [22] and later with Sacks [23], established the foundational matrix-theoretic framework for the Tribonacci sequence.

Between the works of Feinberg and Miles, Narayana and Pettigrew [17] studied in 1962 a related recurrence similar to (1), but with the last term subtracted rather than added, illustrating the variety of generalizations being explored at the time.

Since then, the generalized Fibonacci sequence has inspired a wide range of generalizations and extensions. The most recent is due to K. T. Atanassov and A. G. Shannon [4] who considered the alternating generalized Fibonacci sequence defined by

$$f_n^{(k)} = f_{n-1}^{(k)} - f_{n-2}^{(k)} + \cdots + (-1)^{k-1} f_{n-k}^{(k)}, \quad \text{for } n \geqslant k, \tag{3}$$

with generic initial conditions $f_i^{(k)} = a_i$, for $i = 0, \ldots, k-1$. Setting $\bar{n} = n \bmod (k+1)$ and $\dot{n} = n - \bar{n}$, briefly they showed that

$$f_n^{(k)} = (-1)^{\dot{n}} \times \begin{cases} a_{\bar{n}}\,, & \text{if } (k+1) \nmid (n+1) \\ \displaystyle\sum_{i=0}^{k-1} (-1)^{k-i-1} a_i\,, & \text{otherwise.} \end{cases} \tag{4}$$

Encouraged by these results, K. Gryszka proposed in [9] six novel variants with recurrence terms progressing in steps of two.

Our first aim in this note is to determine the generating function for the sequence defined by (3). To achieve this, we will use its matrix representation, a method that, in a sense, brings us back to the foundational approach used in studying numerical sequences defined by recurrence relations. We will then reverse the signs on the right-hand side of (3), replacing each '$+$' with '$-$' and vice versa, resulting in more intricate expressions. These results will be closely related to the generalized Fibonacci sequence defined by (1).

2 The generating function

In this section, we employ a less known linear algebraic approach to derive the generating function of the alternating generalized Fibonacci sequence defined by (3). Let us recall that a sequence (s_n) defined by the identities

$$s_n = p_{n,n-1}\, s_{n-1} + \cdots + p_{n,n-r}\, s_{n-r}\,, \tag{5}$$

for $n > r$, with the given initial conditions

$$s_1 = a_1\,, \ldots, s_r = a_r\,, \tag{6}$$

can be written explicitly as the determinant of a Hessenberg matrix, namely,

$$s_n = \det \begin{pmatrix} a_1 & a_2 & \cdots & a_r & & & & \\ -1 & 0 & \cdots & 0 & p_{r+1,1} & & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & & \\ & & \ddots & 0 & \vdots & & \ddots & \\ & & & -1 & p_{r+1,r} & & & p_{n,n-r} \\ & & & & -1 & \ddots & & \vdots \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & -1 & p_{n,n-1} \end{pmatrix}. \tag{7}$$

For details and interesting applications, the reader is referred to [11, 15, 21]. A general result can be found, for example, in [20, Theorem 4.20]. In case we replace the -1's of the subdiagonal of the Hessenberg matrix defined in (7) by 1's, then s_n is the permanent of such matrix (cf. [5]).

Using standard matrix theory, it is possible to find in many instances explicit formulas or the generating function for (7). Here, we make use of the following result, available in [8, 10, 13].

Theorem 2.1. *The generating function of the sequence (d_n) defined by the determinants*

$$d_n = \det \begin{pmatrix} b_0 & b_1 & \cdots & b_k & & & \\ -1 & c_1 & c_2 & \cdots & c_\ell & & \\ & -1 & c_1 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c_\ell \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 \\ & & & & & -1 & c_1 \end{pmatrix}_{(n+1)\times(n+1)}$$

is

$$\sum_{n=0}^{\infty} d_n z^n = \frac{b_0 + b_1 z + \cdots + b_k z^k}{1 - c_1 z - \cdots - c_\ell z^\ell}\,.$$

This means that the alternating generalized Fibonacci sequence $(f_n^{(k)})$ can be written as

$$f_n^{(k)} = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} & & & & \\ -1 & 0 & \cdots & 0 & (-1)^{k-1} & & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & & \\ & & \ddots & 0 & -1 & & \ddots & \\ & & & -1 & 1 & \ddots & & (-1)^{k-1} \\ & & & & -1 & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & -1 \\ & & & & & & -1 & 1 \end{pmatrix}_{(n+1)\times(n+1)} \tag{8}$$

or, equivalently,

$$f_n^{(k)} = \det \begin{pmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_{k-1} & & & & \\ -1 & 1 & -1 & \cdots & (-1)^{k-2} & (-1)^{k-1} & & & \\ & -1 & 1 & \ddots & \vdots & \vdots & \ddots & & \\ & & -1 & \ddots & -1 & \vdots & & \ddots & \\ & & & \ddots & 1 & -1 & & & (-1)^{k-1} \\ & & & & -1 & 1 & \ddots & & \vdots \\ & & & & & -1 & \ddots & \ddots & \vdots \\ & & & & & & \ddots & \ddots & -1 \\ & & & & & & & -1 & 1 \end{pmatrix}, \tag{9}$$

where

$$\tilde{a}_\ell = a_\ell - \tilde{a}_{\ell-1} = a_\ell - a_{\ell-1} + a_{\ell-2} - \cdots + (-1)^\ell a_0\,,$$

for $\ell = 1, \ldots, k-1$, with $\tilde{a}_0 = a_0$. Therefore, from Theorem 2.1,

$$\sum_{n=0}^{\infty} f_n^{(k)} z^n = \frac{a_0 + \tilde{a}_1 z + \cdots + \tilde{a}_{k-1} z^{k-1}}{1 - z + z^2 - \cdots - (-1)^{k-1} z^k}\,.$$

We observe that using elementary properties of the determinants on the last k rows of the matrix defined in (8) (or in (9)) we clearly find that

$$f_n^{(k)} = (-1)^{k-1} f_{n-k-1}^{(k)}\,.$$

Together with a simple inductive argument we reach (4).

3 A new alternating generalized Fibonacci sequence

Taking into account (7) and Theorem 2.1, we can readily find that

$$
\begin{aligned}
F_n^{(k)} &= \det \begin{pmatrix}
0 & \cdots & 0 & 1 & & & \\
-1 & 0 & \cdots & 0 & 1 & & \\
 & -1 & \ddots & \vdots & \vdots & \ddots & \\
 & & \ddots & 0 & \vdots & & 1 \\
 & & & -1 & 1 & & \vdots \\
 & & & & \ddots & \ddots & \vdots \\
 & & & & & -1 & 1
\end{pmatrix} \\
&= \det \begin{pmatrix}
0 & \cdots & 0 & 1 & & & \\
-1 & 1 & \cdots & 1 & 1 & & \\
 & -1 & \ddots & \vdots & \vdots & \ddots & \\
 & & \ddots & 1 & \vdots & & 1 \\
 & & & -1 & 1 & & \vdots \\
 & & & & \ddots & \ddots & \vdots \\
 & & & & & -1 & 1
\end{pmatrix} .
\end{aligned}
$$

This means that the generating function of $F_n^{(k)}$ is given by

$$\sum_{n=0}^{\infty} F_n^{(k)} z^n = \frac{z^{k-1}}{1 - z - z^2 - \cdots - z^k} .$$

We will next consider the sequence $(G_n^{(k)})$, satisfying the recurrence relation (1) with generic initial conditions

$$G_0^{(k)} = a_0, \quad G_1^{(k)} = a_1, \quad \ldots, \quad G_{k-1}^{(k)} = a_{k-1},$$

and find a formula for its terms, as well as the corresponding generating function. For that purpose, we will first consider the sequences $(G_n^{(k,i)})$, for $i = 0, 1, \ldots, k-1$, also satisfying (1), with initial conditions

$$G_i^{(k,i)} = 1 \quad \text{and} \quad G_j^{(k,i)} = 0 \,, \quad \text{for } j = 0, 1, \ldots, k-1 \quad \text{and} \quad j \neq i.$$

We observe that $G_n^{(k,0)} = G_{n-1}^{(k,k-1)}$. This implies that the generating function of $(G_n^{(k)})$ can be expressed in terms of the generating functions of the sequences $(G_n^{(k,i)})$, as we will see in what follows.

Table 1. First few terms of $(G_n^{(4,i)})$, for $i = 0, 1, 2, 3$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**OEIS**
$k = 4\,, i = 0$	1	0	0	0	1	1	2	4	8	15	29	56	108	208	401	A000078
$k = 4\,, i = 1$	0	1	0	0	1	2	3	6	12	23	44	85	164	316	609	A001630
$k = 4\,, i = 2$	0	0	1	0	1	2	4	7	14	27	52	100	193	372	717	A001631
$k = 4\,, i = 3$	0	0	0	1	1	2	4	8	15	29	56	108	208	401	773	A000078

As before, we have

$$G_n^{(k,i)} = \det \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & & & \\ -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & & \\ & -1 & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \\ & & \ddots & 0 & \vdots & & \vdots & \vdots & & 1 \\ & & & -1 & 0 & & \vdots & \vdots & & \vdots \\ & & & & -1 & \ddots & \vdots & \vdots & & \vdots \\ & & & & & \ddots & 0 & \vdots & & \vdots \\ & & & & & & -1 & 1 & & \vdots \\ & & & & & & & \ddots & \ddots & \vdots \\ & & & & & & & & -1 & 1 \end{pmatrix},$$

where the first row consists of a single 1 in the position $i+1$ and all remaining entries are assumed to be zero. The 1 in the second row is in the position $k+1$. After some elementary operations on the first k columns, we achieve

$$G_n^{(k,i)} = \det \begin{pmatrix} 0 & \cdots & 0 & 1 & -1 & \cdots & -1 & & & \\ -1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & & \\ & -1 & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \\ & & \ddots & 1 & \vdots & & \vdots & \vdots & & 1 \\ & & & -1 & 1 & & \vdots & \vdots & & \vdots \\ & & & & -1 & \ddots & \vdots & \vdots & & \vdots \\ & & & & & \ddots & 1 & \vdots & & \vdots \\ & & & & & & -1 & 1 & & \vdots \\ & & & & & & & \ddots & \ddots & \vdots \\ & & & & & & & & -1 & 1 \end{pmatrix}.$$

We arrive to the generating function for $G_n^{(k,i)}$:

$$\sum_{n=0}^{\infty} G_n^{(k,i)} z^n = \frac{z^i(1 - z^1 - \cdots - z^{k-i-1})}{1 - z - z^2 - \cdots - z^k}.$$

For the general sequence $(G_n^{(k)})$ we have

$$G_n^{(k)} \quad = \quad \det \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} & & & \\ -1 & 0 & \cdots & 0 & 1 & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & \\ & & \ddots & 0 & \vdots & & 1 \\ & & & -1 & 1 & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & -1 & 1 \end{pmatrix} \qquad (10)$$

$$= \det \begin{pmatrix} a_0 & \bar{a}_1 & \cdots & \bar{a}_{k-1} & & & \\ -1 & 1 & \cdots & 1 & 1 & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & \\ & & \ddots & 1 & \vdots & & 1 \\ & & & -1 & 1 & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & -1 & 1 \end{pmatrix},$$

with

$$\bar{a}_\ell = a_\ell - a_{\ell-1} - \cdots - a_0\,, \quad \text{for } \ell = 1, \ldots, k-1.$$

Therefore, from Theorem 2.1,

$$\sum_{n=0}^{\infty} G_n^{(k)} z^n = \frac{a_0 + \bar{a}_1 z + \cdots + \bar{a}_{k-1} z^{k-1}}{1 - z - z^2 - \cdots - z^k}\,.$$

The expansion of the determinant along the first row in (10) yields

$$G_n^{(k)} = G_n^{(k,0)} a_0 + G_n^{(k,1)} a_1 + \cdots + G_n^{(k,k-1)} a_{k-1} = \sum_{i=0}^{k-1} G_n^{(k,i)} a_i\,. \tag{11}$$

For example, the terms of $G_4^{(4)}, \ldots, G_{10}^{(4)}$ are:

$$\begin{aligned} G_4^{(4)} &= a_0 + a_1 + a_2 + a_3, \\ G_5^{(4)} &= a_0 + 2a_1 + 2a_2 + 2a_3, \\ G_6^{(4)} &= 2a_0 + 3a_1 + 4a_2 + 4a_3, \\ G_7^{(4)} &= 4a_0 + 6a_1 + 7a_2 + 8a_3, \\ G_8^{(4)} &= 8a_0 + 12a_1 + 14a_2 + 15a_3, \\ G_9^{(4)} &= 15a_0 + 23a_1 + 27a_2 + 29a_3, \\ G_{10}^{(4)} &= 29a_0 + 44a_1 + 52a_2 + 56a_3\,. \end{aligned}$$

An alternative approach is presented, for example, in [1].

We now reach the main aim of this section. In the spirit of [4], let us consider a change of signs in (3) in the following way

$$g_n^{(k)} = -g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + (-1)^k g_{n-k}^{(k)}\,, \quad \text{for } n \geqslant k, \tag{12}$$

with similar generic initial conditions $g_i^{(k)} = a_i$, for $i = 0, \ldots, k-1$. As before, we have

$$g_n^{(k)} = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} & & & & \\ -1 & 0 & \cdots & 0 & (-1)^k & & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & & \\ & & \ddots & 0 & 1 & & \ddots & \\ & & & -1 & -1 & \ddots & & (-1)^k \\ & & & & -1 & \ddots & \ddots & \vdots \\ & & & & & \ddots & \ddots & 1 \\ & & & & & & -1 & -1 \end{pmatrix}.$$

If we multiply each even column and each odd row (with exception of the first) by -1, we obtain

$$g_n^{(k)} = (-1)^n \det \begin{pmatrix} a_0 & -a_1 & \cdots & (-1)^{k-1}a_{k-1} & & & \\ -1 & 0 & \cdots & 0 & 1 & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & \\ & & \ddots & 0 & \vdots & & 1 \\ & & & -1 & 1 & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & -1 & 1 \end{pmatrix}.$$

From (11), we readily obtain

$$g_n^{(k)} = (-1)^n \sum_{i=0}^{k-1} (-1)^i G_n^{(k,i)} a_i \,. \tag{13}$$

4 New avenues

In recent years, sequences satisfying the second-order recurrence relations

$$x_{n,\ell} = x_{n-1,\ell} - (-1)^{\lfloor (n-2)/\ell \rfloor} x_{n-2,\ell},$$

$$y_{n,\ell} = (-1)^{\lfloor (n-1)/\ell \rfloor} y_{n-1,\ell} - y_{n-2,\ell}\,,$$

and

$$z_{n,\ell} = (-1)^{\lfloor (n-1)/\ell \rfloor} z_{n-1,\ell} - (-1)^{\lfloor (n-2)/\ell \rfloor} z_{n-2,\ell}$$

have been thoroughly investigated in the literature [3,6,18]. Considering these recurrences along with those examined in the previous sections, a potential extension worth considering is, for instance,

$$z_{n,\ell}^{(k)} = (-1)^{\lfloor (n-1)/\ell \rfloor} z_{n-1,\ell}^{(k)} - (-1)^{\lfloor (n-2)/\ell \rfloor} z_{n-2,\ell}^{(k)} - \cdots - (-1)^{\lfloor (n-k)/\ell \rfloor} z_{n-k,\ell}^{(k)}\,.$$

For instance, setting $k = 3$, $\ell = 4$, and the initial conditions

$$z_{0,4}^{(3)} = z_{1,4}^{(3)} = 0 \quad \text{and} \quad z_{2,4}^{(3)} = 1\,,$$

the first few terms are

$$0,\ 0,\ 1,\ 1,\ 0,\ -2,\ 1,\ -3,\ 2,\ 0,\ -5,\ -7,\ -2,\ 14,\ -9,\ 21\,,$$

which, to the best of our knowledge, has not been previously documented. This simple example shows that countless sequences can be generated, offering ample opportunities for exploration by the interested reader.

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