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A note on normal ordering of degenerate integral powers of number operator

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Abstract: This study derives the normal ordering expansion of degenerate integral powers of the number operator, $(a^{\dagger}a)_{n,\lambda}$, using recurrence relations for the coefficients and an operator action on number states. Here $a^{\dagger}$ and a are respectively the boson creation and annihilation operators. We also determine the inverse of this normal ordering expansion. By analyzing diagonal coherent state elements of the degenerate integral powers of the number operator, we establish a combinatorial identity which yields a Dobinski-like formula for the degenerate Bell numbers at a specific value, connecting degenerate quantum operator calculus with combinatorics.
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1 Introduction

For any nonzero $\lambda \in \mathbb{R}$, the *degenerate integral powers of x* are defined by

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1), \tag{1}$$

(see [5, 6]). The degenerate exponentials are given by

$$e_\lambda^x(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, \qquad e_\lambda(t) = e_\lambda^1(t), \tag{2}$$

(see [4–6]). It is known that the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)\,(x)_{k,\lambda}, \quad (n \geq 0), \tag{3}$$

(see [4–7]), where

$$(x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1).$$

The unsigned degenerate Stirling numbers of the first kind are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_\lambda = (-1)^{n-k} S_{1,\lambda}(n,k), \quad (n \geq k \geq 0), \tag{4}$$

(see [4–7]). Thus, by (3) and (4), we get

$$(x)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda (-1)^{n-k} (x)_{k,\lambda}, \quad (n \geq 0), \quad (\text{see } [4,5]). \tag{5}$$

As the inversion formula of (3), the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda (x)_k, \quad (n \geq 0), \tag{6}$$

(see [4–7]), where $\begin{Bmatrix} n \\ k \end{Bmatrix} = \lim_{\lambda \to 0} \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda$ are the ordinary Stirling numbers of the second kind given by (see [2])

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k, \quad (n \geq 0).$$

From (6), we note that (see [7])

$$\begin{aligned} &\begin{Bmatrix} n+1 \\ k \end{Bmatrix}_\lambda = \begin{Bmatrix} n \\ k-1 \end{Bmatrix}_\lambda + (k - n\lambda) \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda, \quad (n \geq k \geq 1), \\ &\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1, \quad \begin{Bmatrix} n \\ 0 \end{Bmatrix} = 0, \quad (n \geq 1), \quad \begin{Bmatrix} n \\ k \end{Bmatrix} = 0, \quad (0 \leq n < k). \end{aligned} \tag{7}$$

In addition, from (6), we have

$$\frac{1}{k!}\left(e_\lambda(t)-1\right)^k=\sum_{n=k}^{\infty}\left\{\begin{matrix}n\\k\end{matrix}\right\}_\lambda\frac{t^n}{n!},\quad (k\geq 0) \tag{8}$$

(see [4–7]).

The boson annihilation and creation operators a and $a^\dagger$ satisfy the commutation relation:

$$1=[a,a^\dagger]=aa^\dagger-a^\dagger a. \tag{9}$$

The normal ordering of integral powers of the number operator $a^\dagger a$ is written as

$$(a^\dagger a)^k=\sum_{l=0}^{k}\left\{\begin{matrix}k\\l\end{matrix}\right\}(a^\dagger)^l a^l,\quad (k\geq 0) \tag{10}$$

(see [1,3,7–9]).

This paper is inspired by theoretical quantum mechanics and mathematical physics, specifically focusing on operator ordering and quantum statistics. We investigate the normal ordering expansion of degenerate integral powers $(a^\dagger a)_{n,\lambda}$ of the number operator, $a^\dagger a$. We derive the normal ordering expansion for the operator using two distinct methods:

- Determining a recurrence relation for the coefficients $A_\lambda(n,l)$ in the expansion:

$$(a^\dagger a)_{n,\lambda}=\sum_{l=0}^{n}A_\lambda(n,l)(a^\dagger)^l a^l.$$

- Computing the action of the operator $e_\lambda^m(t)$ on a number state, $e_\lambda^m(t)|m\rangle$, in two different ways.

Furthermore, we deduce the inverse of this normal ordering expansion. Finally, by computing the diagonal coherent state elements $\langle\alpha|(a^\dagger a)_{k,\lambda}|\alpha\rangle$ in two ways, we establish an identity that yields a Dobinski-like formula for the degenerate Bell numbers when $\alpha=1$.

2 A note on normal ordering of degenerate integral powers of number operator

The number states $|m\rangle$ satisfy

$$a|m\rangle=\sqrt{m}|m-1\rangle\quad\text{and}\quad a^\dagger|m\rangle=\sqrt{m+1}|m+1\rangle. \tag{11}$$

We now consider normal ordering of degenerate integral powers of the number operator $a^\dagger a$: (see (1))

$$(a^\dagger a)_{n,\lambda}=\sum_{l=0}^{n}A_\lambda(n,l)(a^\dagger)^l a^l,\quad (n\geq 0). \tag{12}$$

From (12), we note that

$$A_\lambda(0,0)=1,\quad A_\lambda(n,0)=0,\quad (n\geq 1),\quad A_\lambda(n,l)=0,\quad (l>n\geq 0). \tag{13}$$

For $l \in \mathbb{N}$, we have (see (9))

$$\begin{aligned}
[a,(a^\dagger)^l] &= a(a^\dagger)^l - (a^\dagger)^l a = a(a^\dagger)^l - (a^\dagger)^{l-1}aa^\dagger + (a^\dagger)^{l-1}aa^\dagger - (a^\dagger)^l a \\
&= \left(a(a^\dagger)^{l-1} - (a^\dagger)^{l-1}a\right)a^\dagger + (a^\dagger)^{l-1}(aa^\dagger - a^\dagger a) \\
&= [a,(a^\dagger)^{l-1}]a^+ + (a^\dagger)^{l-1} = [a,(a^\dagger)^{l-2}](a^\dagger)^2 + 2(a^\dagger)^{l-1} \\
&= [a,(a^\dagger)^{l-3}](a^\dagger)^3 + 3(a^\dagger)^{l-1} = \cdots \\
&= [a,a^\dagger](a^\dagger)^{l-1} + (l-1)(a^\dagger)^{l-1} = l(a^\dagger)^{l-1}.
\end{aligned} \tag{14}$$

Thus, by (14), we get

$$\begin{aligned}
(a^\dagger a)(a^\dagger)^l &= a^\dagger(a(a^\dagger)^l) = a^\dagger(l(a^\dagger)^{l-1} + (a^\dagger)^l a) \\
&= (a^\dagger)^l(l + a^\dagger a).
\end{aligned} \tag{15}$$

Hence, by (9), we get

$$\begin{aligned}
a^l(a^\dagger a) &= a^{l-1}(aa^\dagger)a = a^{l-1}(a^\dagger a + 1)a = a^l + a^{l-1}(a^\dagger a)a \\
&= 2a^l + a^{l-2}(a^\dagger a)a^2 = 3a^l + a^{l-3}(a^\dagger a)a^3 = \cdots \\
&= la^l + (a^\dagger a)a^l = la^l + a^\dagger a^{l+1}.
\end{aligned} \tag{16}$$

From (16), we note that

$$\begin{aligned}
\sum_{l=0}^{n+1} A_\lambda(n+1,l)(a^\dagger)^l a^l &= (a^\dagger a)_{n+1,\lambda} = (a^\dagger a)_{n,\lambda}(a^\dagger a - n\lambda) \\
&= \sum_{l=0}^{n} A_\lambda(n,l)(a^\dagger)^l a^l (a^\dagger a - n\lambda) \\
&= \sum_{l=0}^{n} A_\lambda(n,l)(a^\dagger)^{l+1} a^{l+1} + \sum_{l=0}^{n}(l - n\lambda)A_\lambda(n,l)(a^\dagger)^l a^l \\
&= \sum_{l=0}^{n+1} A_\lambda(n,l-1)(a^\dagger)^l a^l + \sum_{l=0}^{n+1}(l - n\lambda)A_\lambda(n,l)(a^\dagger)^l a^l.
\end{aligned} \tag{17}$$

By (17), we get

$$A_\lambda(n+1,l) = A_\lambda(n,l-1) + (l - n\lambda)A_\lambda(n,l), \tag{18}$$

where $n \geq l \geq 0$. From (7), (13), and (18), we have

$$A_\lambda(n,l) = \begin{Bmatrix} n \\ l \end{Bmatrix}_\lambda, \quad (n,l \geq 0). \tag{19}$$

Thus, from (12) and (19), we obtain:

$$(a^\dagger a)_{n,\lambda} = \sum_{l=0}^{n} \begin{Bmatrix} n \\ l \end{Bmatrix}_\lambda (a^\dagger)^l a^l, \quad (n \geq 0). \tag{20}$$

From (11), we have

$$(a^\dagger)^n a^n |m\rangle = m(m-1)\cdots(m-n+1)|m\rangle = (m)_n|m\rangle. \tag{21}$$

Note that

$$\begin{aligned}(a^\dagger a)_n|m\rangle &= (a^\dagger a)(a^\dagger a-1)\cdots(a^\dagger a-n+1)|m\rangle \\ &= m(m-1)\cdots(m-n+1)|m\rangle \\ &= (m)_n|m\rangle,\end{aligned} \tag{22}$$

and that (see (1))

$$\begin{aligned}(a^\dagger a)_{n,\lambda}|m\rangle &= (a^\dagger a)(a^\dagger a-\lambda)\cdots(a^\dagger a-(n-1)\lambda)|m\rangle \\ &= m(m-\lambda)\cdots(m-(n-1)\lambda)|m\rangle \\ &= (m)_{n,\lambda}|m\rangle.\end{aligned} \tag{23}$$

By (21) and (22), we get

$$(a^\dagger)^n a^n|m\rangle = (a^\dagger a)_n|m\rangle, \quad (n \geq 0). \tag{24}$$

From (3) and (24), we have (see (5))

$$\begin{aligned}(a^\dagger)^n a^n = (a^\dagger a)_n &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)(a^\dagger a)_{l,\lambda} \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_\lambda (-1)^{n-l}(a^\dagger a)_{l,\lambda}, \quad (n \geq 0),\end{aligned}$$

which is the inverse of the expansion in (20).

We now show another method of obtaining normal ordering of degenerate integral powers of the number operator $a^\dagger a$. By (8) and (21), we get

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{l=0}^{k}\begin{Bmatrix} k \\ l \end{Bmatrix}_\lambda (a^\dagger)^l a^l|m\rangle = \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\frac{t^k}{k!}\begin{Bmatrix} k \\ l \end{Bmatrix}_\lambda (a^\dagger)^l a^l|m\rangle \\ &= \sum_{l=0}^{\infty}\frac{1}{l!}(e_\lambda(t)-1)^l(a^\dagger)^l a^l|m\rangle = \sum_{l=0}^{\infty}\frac{(m)_l}{l!}(e_\lambda(t)-1)^l|m\rangle \\ &= \sum_{l=0}^{m}\binom{m}{l}(e_\lambda(t)-1)^l|m\rangle = e_\lambda^m(t)|m\rangle.\end{aligned} \tag{25}$$

On the other hand, by (2) and (23), we get

$$\begin{aligned}e_\lambda^{a^\dagger a}(t)|m\rangle &= \sum_{k=0}^{\infty}\frac{t^k}{k!}(a^\dagger a)_{k,\lambda}|m\rangle = \sum_{k=0}^{\infty}\frac{t^k}{k!}(m)_{k,\lambda}|m\rangle \\ &= e_\lambda^m(t)|m\rangle.\end{aligned} \tag{26}$$

From (25) and (26), we note that

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\begin{Bmatrix} k \\ l \end{Bmatrix}_\lambda (a^\dagger)^l a^l\frac{t^k}{k!}|m\rangle &= e_\lambda^{a^\dagger a}(t)|m\rangle \\ &= \sum_{k=0}^{\infty}(a^\dagger a)_{k,\lambda}\frac{t^k}{k!}|m\rangle.\end{aligned} \tag{27}$$

By comparing the coefficients on both sides of (27), we again get the expansion in (20):

$$(a^\dagger a)_{k,\lambda} = \sum_{l=0}^{k}\begin{Bmatrix} k \\ l \end{Bmatrix}_\lambda (a^\dagger)^l a^l, \quad (k \geq 0). \tag{28}$$

The normal ordering of degenerate integral powers of number operator is particularly useful in coherent state representation. First, we recall that the coherent states satisfy (see [3])

$$a|\alpha\rangle = \alpha|\alpha\rangle,\ \langle\alpha|a^\dagger = \langle\alpha|\bar{\alpha},\ \langle\alpha|\alpha\rangle = 1, \quad (\alpha\in\mathbb{C}). \tag{29}$$

Second, discarding the phase factors, the coherent expansions are given by:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \langle\alpha| = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\bar{\alpha}^n}{\sqrt{n!}}|n\rangle. \tag{30}$$

Third, the eigenstates $|n\rangle$, for $n\in\mathbb{N}$, satisfy

$$a^\dagger a|n\rangle = n|n\rangle, \quad \langle m|n\rangle = \delta_{m,n}, \tag{31}$$

where $\delta_{m,n}$ is the Kronecker's symbol. We now compute the diagonal coherent state elements of the operator $(a^\dagger a)_{k,\lambda}$ in two different ways. On the one hand, using (28) and (29), we have

$$\begin{aligned}
\langle\alpha|(a^\dagger a)_{k,\lambda}|\alpha\rangle &= \sum_{l=0}^{k}\begin{Bmatrix}k\\l\end{Bmatrix}_\lambda \langle\alpha|(a^\dagger)^l a^l|\alpha\rangle \\
&= \sum_{l=0}^{k}\begin{Bmatrix}k\\l\end{Bmatrix}_\lambda |\alpha|^{2l}\langle\alpha|\alpha\rangle \\
&= \sum_{l=0}^{k}\begin{Bmatrix}k\\l\end{Bmatrix}_\lambda |\alpha|^{2l}, \quad (k\geq 0).
\end{aligned} \tag{32}$$

On the other hand, by the coherent expansion in (30) and (31), we get

$$\begin{aligned}
\langle\alpha|(a^\dagger a)_{k,\lambda}|\alpha\rangle &= e^{-|\alpha|^2/2}e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{\bar{\alpha}^m}{\sqrt{m!}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle m|(a^\dagger a)_{k,\lambda}|n\rangle \\
&= e^{-|\alpha|^2}\sum_{m=0}^{\infty}\frac{\bar{\alpha}^m}{\sqrt{m!}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}(n)_{k,\lambda}\langle m|n\rangle \\
&= e^{-|\alpha|^2}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{m!}(m)_{k,\lambda}.
\end{aligned} \tag{33}$$

Thus, from (32) and (33), we get

$$\sum_{l=0}^{k}\begin{Bmatrix}k\\l\end{Bmatrix}_\lambda |\alpha|^{2l} = \frac{1}{e^{|\alpha|^2}}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{m!}(m)_{k,\lambda}. \tag{34}$$

In particular, letting $\alpha = 1$ in (34) gives the following Dobinski-like formula:

$$\phi_{k,\lambda} = \frac{1}{e}\sum_{m=0}^{\infty}\frac{(m)_{k,\lambda}}{m!}, \quad (k\geq 0),$$

where $\phi_{k,\lambda} = \sum_{l=0}^{k}\begin{Bmatrix}k\\l\end{Bmatrix}_\lambda$ is the k-th degenerate Bell number.

3 Conclusion

This investigation successfully derived the normal ordering expansion for degenerate integral powers of the number operator, $(a^\dagger a)_{n,\lambda}$, providing a clear algebraic representation in terms of powers of the normal ordered operator $(a^\dagger)^l a^l$. The consistency of the results was established by employing two independent methods: one relying on a recurrence relation for the expansion coefficients and the other utilizing the action of the exponential operator $e_\lambda^m(t)$ on number states. Crucially, we also determined the inverse expansion, which is essential for transforming between ordered forms in quantum field theory and quantum optics. The most significant finding, however, comes from analyzing the diagonal coherent state elements. By computing $\langle \alpha | (a^\dagger a)_{k,\lambda} | \alpha \rangle$ in two distinct ways, we established a non-trivial combinatorial identity. When this identity is evaluated at $\alpha = 1$, it directly yields a Dobinski-like formula for the degenerate Bell numbers. This result connects the abstract algebraic properties of degenerate number operator powers to the field of combinatorics, offering a new perspective on these important number sequences and validating the utility of operator ordering techniques in establishing novel combinatorial identities.

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