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# On $\psi$-amicable numbers and their generalizations

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**Abstract:** In this article, we study the properties of $\psi$-amicable numbers. We prove that their asymptotic density relative to the positive integers is zero. We also propose generalizations of $\psi$-amicable numbers.
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## 1 Notations, definitions and formulas

The letter $p$, with or without a subscript, will always denote prime number. Let $n > 1$ be positive integer with prime factorization

$$n = p_1^{a_1} \cdots p_r^{a_r} .$$

We define the Dedekind function $\psi(n)$ by the formula

$$\psi(n) = n \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right) \quad \text{and} \quad \psi(1) = 1\,. \tag{1}$$

Recall that

$$\psi(n) = \sum_{d\,|\,n} \frac{n\mu^2(d)}{d}\,, \tag{2}$$

where $\mu(n)$ is the Möbius function. The proof of (2) can be found in [5]. We shall use the convention that a congruence, $m \equiv n \pmod{d}$ will be written as $m \equiv n\,(d)$. A positive integer $n$ is said to be $\psi$-abundant if $\psi(n) > 2n$. A primitive $\psi$-abundant number is defined as an $\psi$-abundant number none of whose proper divisors is $\psi$-abundant. Thus every $\psi$-abundant number is a multiple of a primitive $\psi$-abundant numbers. Throughout this paper we denote $\nu = \log\log n$ and $s_\psi(n) = \psi(n) - n$.

# 2 Introduction and statement of the results

Two different natural numbers $a$ and $b$ are said to be $\psi$-amicable if

$$\psi(a) = \psi(b) = a + b\,. \tag{3}$$

In 2019, Amiram Eldar contributed sequences A323329 and A323330 to the OEIS [6], listing the smaller and larger members, respectively, of the $\psi$-amicable pairs. The smallest $\psi$-amicable pair is (1330, 1550). Apparently, this definition is analogous to the classical definition of amicable pairs, which uses the sum-of-divisors function $\sigma$. In Section 5, we introduce the notion of $\psi$-amicable $k$-tuples. In Section 6, we provide another definition of the same concept. Our main result concerns the density of $\psi$-amicable pairs. We prove that their asymptotic density is zero.

**Theorem 2.1.** *Let $M(n)$ denote the number of $\psi$-amicable pairs $(a, b)$ with $a < b$ and $a \le n$. Then $M(n) = o(n)$ as $n \to \infty$.*

Our approach is based on the method of Erdős' [3]. We essentially reproduce his argument, adapting it to Dedekind's $\psi$-function, with only minor technical modifications.

# 3 Lemmas

**Lemma 3.1.** *Let $q_i$ be a sequence of prime numbers satisfying*

$$\sum_{i=1}^{\infty} \frac{1}{q_i} = \infty\,.$$

*Denote by $v_q(n)$ the number of $q_i$ dividing $n$. Then the density of integers $n$ with $v_q(n) < A$ is $0$ for every $A$.*

*Proof.* See ( [3], Lemma 1). □

**Lemma 3.2.** *The number of integers $m \le n$ which do not satisfy all of the following three conditions:*

*(1) If $p^a \mid m$ and $a > 1$, then $p^a < (\log n)^{10}$ ;*

*(2) The number of distinct prime factors of $m$ is less than $10\nu$ ;*

*(3) The greatest prime factor of $m$ is greater than $n^{1/(20\nu)}$ ;*

*is $o\Big(\frac{n}{\log^2 n}\Big)$.*

*Proof.* See [2, Lemma 1]. □

**Lemma 3.3.** *Let $A$ be any constant. Then the density of integers $n$ for which*

$$\psi(n) \not\equiv 0 \left( \Big( \prod_{p \le A} p \Big)^A \right)$$

*is $0$ for every $A$.*

*Proof.* It suffices to show that the density of integers $n$ for which there exists a prime $p \le A$ such that $\psi(n) \not\equiv 0 \left(p^A\right)$ is $0$. Let $q_1, q_2, \dots$ be primes satisfying $q_i \equiv -1 \,(p)$. It is well known that

$$\sum_{i=1}^{\infty} \frac{1}{q_i} = \infty.$$

Hence, by Lemma 3.1, the density of integers divisible by fewer than $A$ of the $q_i$ is $0$. If $n$ is divisible by at least $A$ of the $q_i$, then (1) gives us $\psi(n) \equiv 0 \left(p^A\right)$. Therefore the density of the integers with $\psi(n) \not\equiv 0 \left(p^A\right)$ is $0$. □

**Lemma 3.4.** *Denote*

$$\psi_A(n) = \sum_{\substack{d \mid n \\ d \le A}} \frac{n\mu^2(d)}{d} \,. \tag{4}$$

*Then for every $\varepsilon > 0$ and $\eta > 0$, there exists $A_0$ such that for $A > A_0$, the number of integers $n < x$ for which $\psi(n) - \psi_A(n) > \eta n$ is less than $\varepsilon x$.*

*Proof.* Using (2) and (4), we have

$$\sum_{n=1}^{x} \big(\psi(n) - \psi_A(n)\big) = \sum_{n=1}^{x} \sum_{\substack{d \mid n \\ d > A}} \frac{n\mu^2(d)}{d} = \sum_{d_1 > A} \mu^2(d_1) \sum_{d_2 \le x/d_1} d_2 < \sum_{d > A} \frac{x^2}{d^2} < \frac{x^2}{A} \,. \tag{5}$$

If Lemma 3.4 were not true, we would have $\psi(n) - \psi_A(n) > \eta n$ for at least $\varepsilon x$ integers $d \le x$. Thus

$$\sum_{n=1}^{x} \big(\psi(n) - \psi_A(n)\big) > \eta \sum_{d \le \varepsilon x} d > \frac{\eta \varepsilon^2 x^2}{4} \,. \tag{6}$$

For $A > \frac{4}{\eta\varepsilon^2}$ (6) contradicts (5), which proves Lemma 3.4. □

**Lemma 3.5.** *A primitive $\psi$-abundant number not exceeding $n$, which satisfies the three conditions of Lemma 3.2, necessarily has a prime divisor between $(\log n)^{10}$ and $n^{1/(40\nu)}$, provided $n$ is sufficiently large.*

*Proof.* Assume that $m = ab$ is such a primitive $\psi$-abundant number, where all prime factors of $a$ are less than $(\log n)^{10}$ and all prime factors of $b$ are greater than $n^{1/(40\nu)}$. We have

$$\frac{\psi(m)}{m} \geq 2 \tag{7}$$

and

$$\frac{\psi(a)}{a} < 2\,. \tag{8}$$

Now (8) and Lemma 3.2 imply

$$\frac{\psi(a)}{a} \leq 2 - \frac{1}{a} < 2 - \frac{1}{(\log n)^{100\nu}} \tag{9}$$

On the other hand, by (2) and Lemma 3.2, we obtain

$$\frac{\psi(b)}{b} = \sum_{d\,|\,b} \frac{\mu^2(d)}{d} = \prod_{p\,|\,b} \left(1 + \frac{1}{p}\right) < \left(1 + \frac{1}{n^{1/(40\nu)}}\right)^{10\nu} < 1 + \frac{20\nu}{n^{1/(40\nu)}} \tag{10}$$

if $n$ is sufficiently large. Now (9) and (10) yield

$$\frac{\psi(m)}{m} = \frac{\psi(a)}{a}\frac{\psi(b)}{b} < 2$$

for sufficiently large $n$, which contradicts (7). □

# 4 Proof of Theorem 2.1

Denote by $(a_i, b_i)$, $a_i < b_i$, $i = 1, 2, \ldots$ the sequence of pairs of $\psi$-amicable numbers. It is sufficient to prove that the sequence $a_i$, $i = 1, 2, \ldots$ has density 0.

We split the sequence $a_i$ into two classes. Let $A = A(\varepsilon)$ be sufficiently large. In the first class are the $a_i$ for which there exists a $p \leq A$ with $\psi(a_i) \not\equiv 0 \left(p^A\right)$. It follows from Lemma 3.3 that the density of the $a_i$ of the first class is 0. For the $a_i$ of the second class $\psi(a_i) \equiv 0 \left(p^A\right)$ for every $p \leq A$. It is easy to see that if $d \leq A$ and $d \,|\, a_i$ then $\psi(a_i) - a_i \equiv 0\,(d)$. Therefore $\psi(a_i) - a_i = b_i \equiv 0\,(d)$. From Lemma 3.4 it follows that except for at most $\varepsilon n$ of the $a_i$ not exceeding $n$ we have

$$\frac{\psi_A(a_i)}{a_i} \geq \frac{\psi(a_i)}{a_i} - \eta\,. \tag{11}$$

By (2), (4), (11) and the fact that every divisor $d \leq A$ of $a_i$ also divides $b_i$, we get

$$\frac{\psi(b_i)}{b_i} = \sum_{d\,|\,b_i} \frac{\mu^2(d)}{d} \geq \sum_{\substack{d\,|\,a_i \\ d \leq A}} \frac{\mu^2(d)}{d} = \frac{\psi_A(a_i)}{a_i} \geq \frac{\psi(a_i)}{a_i} - \eta\,. \tag{12}$$

Now (3) and (12) lead to

$$\eta \geq \frac{\psi(a_i)}{a_i} - \frac{\psi(b_i)}{b_i} = \frac{b_i}{a_i} - \frac{a_i}{b_i}\,.$$

Hence

$$1 < \frac{b_i}{a_i} < 1 + \eta\,.$$

The last inequality and (3) give us

$$2 < \frac{\psi(a_i)}{a_i} < 2 + \eta\,. \tag{13}$$

Bearing in mind Lemma 3.2, we may assume that each $a_i$ from (13) has a primitive $\psi$-abundant divisor satisfying all of the three conditions of Lemma 3.2. Let $a_1, a_2, \ldots, a_k$ denote all distinct numbers from (13) such that $a_i \leq n$. According to Lemma 3.5, each $a_i$ has a prime factor $p_i$ between $(\log n)^{10}$ and $n^{1/(40\nu)}$. Thus $a_i = p_i c_i$, where $c_i < n/(\log n)^{10}$. Suppose that $c_i = c_j$ for some $i \neq j$. Then $p_i \neq p_j$. We have

$$\frac{\psi(a_i)}{a_i} = \frac{\psi(p_i)\psi(c_i)}{a_i} = \frac{\psi(c_i)}{c_i}\frac{p_i+1}{p_i}$$

and

$$\frac{\psi(a_j)}{a_j} = \frac{\psi(p_j)\psi(c_j)}{a_j} = \frac{\psi(c_j)}{c_j}\frac{p_j+1}{p_j}\,,$$

which together imply

$$\frac{\psi(a_i)}{a_i}\frac{a_j}{\psi(a_j)} = \frac{p_j(p_i+1)}{p_i(p_j+1)}\,. \tag{14}$$

Without loss of generality, assume that $p_j > p_i$. Now (14) yields

$$\frac{\psi(a_i)}{a_i}\frac{a_j}{\psi(a_j)} > 1 + \frac{1}{p_i(p_j+1)} > \frac{1}{n^{1/(20\nu)}} \tag{15}$$

On the other hand, from (13) it follows that

$$\frac{\psi(a_i)}{a_i}\frac{a_j}{\psi(a_j)} < 1 + \frac{\eta}{2}\,,$$

which contradicts (15) for $\eta$ sufficiently small. Consequently, $c_i \neq c_j$ for $i \neq j$, which means that the number of $a_i \leq n$ from (13) is equal to the number of $c_i$, which is $o(n)$. This completes the proof of Theorem 2.1.

# 5 $\psi$-amicable $k$-tuples

Dickson [1] and Mason [4] introduced a definition of amicable $k$-tuples using the sum-of-divisors function $\sigma$. We now provide an analogous definition based on the function $\psi$. We say that the natural numbers $n_1, \ldots, n_k$ form an $\psi$-amicable $k$-tuple if

$$\psi(n_1) = \psi(n_2) = \cdots = \psi(n_k) = n_1 + n_2 + \cdots + n_k\,.$$

When $n_1 < n_2 < \cdots < n_k$, we have that

$$kn_1 < \psi(n_j) < kn_k$$

for each $j \in [1,k]$, which means that $n_1$ is $k$-$\psi$-abundant. The next theorem will help us search for $\psi$-amicable $k$-tuples.

**Theorem 5.1.** *Suppose the natural numbers $N_1, \ldots, N_k$ and $a$ satisfy $(a, N_1) = \cdots = (a, N_k) = 1$ and*

$$\frac{\psi(a)}{a} = \frac{N_1 + \cdots + N_k}{\psi(N_1)} = \cdots = \frac{N_1 + \cdots + N_k}{\psi(N_k)}\,.$$

*Then $aN_1, \ldots, aN_k$ are an $\psi$-amicable $k$-tuple.*

*Proof.* This follows directly from the multiplicativity of $\psi$. □

Several $\psi$-amicable triples are listed below.

List 1. $\psi$-amicable triples

| $\psi$-amicable triples |
|---|
| $(79170, 80850, 81900)$, $(150150, 158340, 175350)$, $(158340, 161700, 163800)$, $(237510, 242550, 245700)$, $(300300, 316680, 350700)$, $(316680, 323400, 327600)$, $(395850, 404250, 409500)$, $(450450, 474810, 526260)$, $(450450, 475020, 526050)$, $(468930, 483210, 499380)$, $(474810, 485940, 490770)$, $(475020, 485100, 491400)$, $(554190, 565950, 573300)$, $(570570, 662340, 702450)$, $(600600, 633360, 701400)$, $(622440, 641550, 671370)$, $(633360, 646800, 655200)$, $(641550, 646800, 647010)$, $(644280, 644280, 646800)$, $(696150, 696150, 784980)$, $(712530, 727650, 737100)$ |

The OEIS [6] sequences A385852, A386901 and A386933 consist of the first, second and third components of $\psi$-amicable triples, respectively.

# 6 Another definition of $\psi$-amicable $k$-tuples

The following definition is analogous to that given by Yanney [7], formulated for $\sigma$-function. We say that the natural numbers $n_1, \ldots, n_k$ form a $\psi$-amicable $k$-tuple if

$$\psi(n_1) = \psi(n_2) = \cdots = \psi(n_k) = \frac{1}{k-1}(n_1 + n_2 + \cdots + n_k)\,.$$

When $k = 3$, we have

$$\left|\begin{array}{l} n_1 = s_\psi(n_2) + s_\psi(n_3) \\ n_2 = s_\psi(n_1) + s_\psi(n_3) \\ n_3 = s_\psi(n_1) + s_\psi(n_2) \end{array}\right. .$$

Several $\psi$-amicable triples are listed below.

List 2. $\psi$-amicable triples

| $\psi$-amicable triples |
|---|
| $(2, 2, 2)$, $(4, 4, 4)$, $(6, 9, 9)$, $(8, 8, 8)$, $(16, 16, 16)$, $(18, 27, 27)$, $(28, 33, 35)$, $(32, 32, 32)$, $(64, 64, 64)$, $(54, 81, 81)$, $(70, 99, 119)$, $(105, 124, 155)$, $(128, 128, 128)$, $(110, 135, 187)$, $(165, 176, 235)$, $(150, 275, 295)$, $(200, 225, 295)$, $(182, 245, 245)$, $(162, 243, 243)$, $(256, 256, 256)$, $(238, 255, 371)$, $(240, 385, 527)$, $(280, 345, 527)$, $(310, 315, 527)$, $(310, 345, 497)$, $(315, 320, 517)$, $(315, 320, 517)$, $(382, 385, 385)$, $(364, 441, 539)$, $(512, 512, 512)$, $(512, 512, 512)$, $(468, 715, 833)$, $(520, 663, 833)$, $(585, 598, 833)$, $(644, 705, 955)$, $(590, 675, 895)$, $(486, 729, 729)$, $(795, 862, 935)$, (800, 885, 1195) |

The OEIS [6] sequence A387291 consists of the first elements of $\psi$-amicable triples. When $k = 4$, we have

$$\left|\begin{array}{l} n_1 = s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_4) \\ n_2 = s_\psi(n_1) + s_\psi(n_3) + s_\psi(n_4) \\ n_3 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_4) \\ n_4 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_3) \end{array}\right. .$$

Several $\psi$-amicable quadruples are listed below.

List 3. $\psi$-amicable quadruples

| $\psi$-amicable quadruples |
|---|
| $(3,3,3,3)$, $(4,4,5,5)$, $(6,8,11,11)$, $(8,8,9,11)$, $(9,9,9,9)$, $(12,14,23,23)$, |
| $(30,44,71,71)$, $(44,46,55,71)$, $(45,45,55,71)$, $(51,55,55,55)$, $(68,68,81,107)$, |
| $(81,81,81,81)$, $(99,99,115,119)$, $(75,95,95,95)$, $(96,128,161,191)$, $(105,155,155,161)$, |
| $(112,112,161,191)$, $(100,116,145,179)$, $(114,158,209,239)$, $(152,152,177,239)$, |
| $(152,158,171,239)$, $(171,171,175,203)$, $(188,188,235,253)$, $(164,166,205,221)$, |
| $(190,236,295,359)$, $(225,261,275,319)$, $(243,243,243,243)$, $(186,254,329,383)$, |
| $(204,230,431,431)$, $(230,284,391,391)$, $(238,272,355,431)$, $(255,255,355,431)$ |

The OEIS [6] sequence A387292 lists the first components of $\psi$-amicable quadruples. When $k = 5$, we have

$$\left|\begin{array}{l} n_1 = s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_4) + s_\psi(n_5) \\ n_2 = s_\psi(n_1) + s_\psi(n_3) + s_\psi(n_4) + s_\psi(n_5) \\ n_3 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_4) + s_\psi(n_5) \\ n_4 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_5) \\ n_5 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_4) \end{array}\right. .$$

Several $\psi$-amicable quintuples are listed below.

List 4. $\psi$-amicable quintuples

| $\psi$-amicable quintuples |
|---|
| $(4,5,5,5,5)$, $(6,9,11,11,11)$, $(8,9,9,11,11)$, $(12,15,23,23,23)$, |
| $(28,35,35,47,47)$, $(32,33,33,47,47)$, $(30,45,71,71,71)$, $(36,55,55,71,71)$, |
| $(40,51,55,71,71)$, $(44,51,51,71,71)$, $(45,46,55,71,71)$, $(78,117,143,167,167)$, |
| $(98,117,123,167,167)$, $(104,117,117,167,167)$, $(84,141,161,191,191)$, |
| $(112,155,155,155,191)$, $(124,161,161,161,161)$, $(158,175,209,209,209)$, |
| $(158,177,177,209,239)$, $(140,253,253,253,253)$, $(176,235,235,253,253)$, |
| $(174,225,323,359,359)$, $(174,261,323,323,359)$, $(200,261,261,359,359)$ |

The OEIS [6] sequence A387486 consists of the first elements of $\psi$-amicable quintuples.

When $k = 6$, we have

$$\left|\begin{array}{l} n_1 = s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_4) + s_\psi(n_5) + s_\psi(n_6) \\ n_2 = s_\psi(n_1) + s_\psi(n_3) + s_\psi(n_4) + s_\psi(n_5) + s_\psi(n_6) \\ n_3 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_4) + s_\psi(n_5) + s_\psi(n_6) \\ n_4 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_5) + s_\psi(n_6) \\ n_5 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_4) + s_\psi(n_6) \\ n_6 = s_\psi(n_1) + s_\psi(n_2) + s_\psi(n_3) + s_\psi(n_4) + s_\psi(n_5) \end{array}\right| .$$

Several $\psi$-amicable sextuples are listed below.

List 5. $\psi$-amicable sextuples

| $\psi$-amicable sextuples |
|---|
| $(5,5,5,5,5,5)$, $(8,8,11,11,11,11)$, $(9,9,9,11,11,11)$, $(12,16,23,23,23,23)$ |
| $(14,14,23,23,23,23)$, $(24,28,47,47,47,47)$, $(25,25,25,25,25,25)$, $(32,32,35,47,47,47)$, |
| $(30,46,71,71,71,71)$, $(33,33,33,47,47,47)$, $(36,40,71,71,71,71)$, $(45,51,51,71,71,71)$, |
| $(46,46,55,71,71,71)$, $(98,98,143,167,167,167)$, $(117,123,123,143,167,167)$, |
| $(84,112,191,191,191,191)$, $(105,141,141,191,191,191)$, $(128,128,161,161,191,191)$, |
| $(141,141,141,155,191,191)$, $(155,161,161,161,161,161)$, $(152,152,209,209,239,239)$, |
| $(152,158,203,209,239,239)$, $(158,158,203,203,239,239)$, $(171,171,171,209,239,239)$, |

The OEIS [6] sequence A387636 lists the first components of $\psi$-amicable sextuples.

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