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Relationship between alternating sums of powers of integers and sums of powers of integers

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Abstract: In this note, we consider the alternating sums of powers of integers. We write alternating sum of powers of integers as the linear combination of sums of powers of integers. As the coefficients, the special value of the Euler polynomial appears.
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1 Introduction

For a non-negative integer $m \in \mathbb{Z}$ and a positive integer $n \in \mathbb{Z}$, we define

$$S_m(n) = \sum_{i=1}^{n} i^m, \quad \Omega_m(n) = \sum_{i=0}^{n-1} (-1)^i (n-i)^m$$

It is well known that $S_m(n)$ can be written as the polynomial of the variable n as follows (see [2], for example):

$$S_m(n) = \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} B_j n^{m+1-j}. \tag{1}$$

Here, B_j is the j-th Bernoulli number defined by $\displaystyle\sum_{j=0}^{l}\binom{l+1}{j}B_j = l+1$, inductively.

According to Kim [5, Theorem 1] (see also Cereceda [3, Section 4]), $\Omega_m(n)$ can be written as a polynomial of the variable n as follows:

$$\Omega_m(n) = \begin{cases} \dfrac{1}{2}\left(n^m - \displaystyle\sum_{j=1}^{m-1}\binom{m}{j}E_j(0)n^{m-j} + \left((-1)^n - 1\right)E_m(0)\right), & \text{if } m \geq 1, \\ \dfrac{\left(1+(-1)^{n-1}\right)}{2}, & \text{if } m = 0. \end{cases} \tag{2}$$

Here, $E_j(x)$ is the j-th Euler polynomial defined by $\dfrac{2e^{xt}}{e^t+1} = \displaystyle\sum_{j=0}^{\infty}E_j(x)\frac{t^j}{j!}$.

In this note, we investigate the relationship between $\Omega_m(n)$ and $S_m(n)$ and have the following theorem.

Theorem 1.1. *For positive integers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we have the following identity*:

$$\Omega_m(n) = -\sum_{j=0}^{m-1}\binom{m}{j}E_{m-j}(0)S_j(n) + \frac{(-1)^n - 1}{2}E_m(0).$$

In [6], we wrote $S_m(n)$ as the linear combination of $\Omega_j(n)$ (see also Theorem 3.1). Therefore, we can see that Theorem 1.1 is the converse of Theorem 3.1.

Remark 1.1. *In Theorem 2.3 of* [1]*, Antonippillai proved the following identity*:

$$(-1)^{n+1}\Omega_m(n) = \begin{cases} \dfrac{n+1}{2} + \displaystyle\sum_{j=1}^{m-1}\binom{m}{j}2^jS_j\left(\frac{n-1}{2}\right), & \text{if } n \geq 1 \text{ is odd,} \\ (-1)^m\dfrac{n}{2} + \displaystyle\sum_{j=1}^{m-1}(-1)^{m-j}\binom{m}{j}2^jS_j\left(\frac{n}{2}\right), & \text{if } n \geq 0 \text{ is even.} \end{cases} \tag{3}$$

As Theorem 1.1, the identity (3) *represents $\Omega_m(n)$ as the linear combination of $S_j(n)$.*

This note is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, by combining Theorems 1.1 and 3.1, we state the relationships between B_k and $E_j(0)$.

2 Proof of Theorem 1.1

Both $\Omega_m(n)$ and $S_{m-1}(n)$ are degree m polynomials. On the other hand, $\Omega_m(n)$ has the constant term $\dfrac{(-1)^n - 1}{2}E_m(0)$ and $S_{m-1}(n)$ does not have the constant term. Therefore, we set

$$\Omega_m(n) = \sum_{j=0}^{m-1}a_jS_j(n) + \frac{(-1)^n - 1}{2}E_m(0), \tag{4}$$

where $a_j \in \mathbb{Q}$. From (4) and the definition of $S_j(n)$, we have

$$\begin{aligned}\Omega_m(n+1) - \Omega_m(n) &= a_0 + \sum_{j=1}^{m-1} a_j (n+1)^j + (-1)^{n+1} E_m(0) \\ &= \sum_{j=1}^{m-1} a_j (n+1)^j + \left(a_0 + (-1)^{n+1} E_m(0)\right).\end{aligned} \tag{5}$$

Also, from the definition of $\Omega_m(n)$,

$$\Omega_m(n+1) + \Omega_m(n) = (n+1)^m \tag{6}$$

holds if $m \geq 1$. By summing (5) and (6), we have

$$\Omega_m(n+1) = \frac{1}{2}(n+1)^m + \frac{1}{2}\sum_{j=1}^{m-1} a_j (n+1)^j + \frac{a_0 + (-1)^{n+1} E_m(0)}{2} \tag{7}$$

if $m \geq 1$. On the other hand, from (2),

$$\Omega_m(n+1) = \frac{1}{2}(n+1)^m - \frac{1}{2}\sum_{j=1}^{m-1} \binom{m}{j} E_j(0)(n+1)^{m-j} + \frac{\left(-1 + (-1)^{n+1}\right) E_m(0)}{2} \tag{8}$$

holds if $m \geq 1$. By comparing Equations (7) and (8), we obtain

$$a_j = -\binom{m}{j} E_{m-j}(0)$$

for $j = 0, \ldots, m-1$. This completes the proof of Theorem 1.1. $\square$

Corollary 2.1. *For a non-negative integer $m \in \mathbb{Z}$ and a positive integer $n \in \mathbb{Z}$, the following identity holds*:

$$S_m(n) = \frac{E_{m+1}(n+1) + E_{m+1}(0)}{m+1} + \frac{2}{m+1}\sum_{j=0}^{m-1} \binom{m+1}{j} E_{m+1-j}(0) S_j(n).$$

Proof. If $m \geq 0$ and $n \geq 1$, from the identities (2) and Theorem 1.1, we have

$$\begin{aligned}-\Omega_{m+1}(n) + \frac{(-1)^n - 1}{2} E_{m+1}(0) &= -\frac{1}{2} n^{m+1} + \frac{1}{2}\sum_{j=1}^{m} \binom{m+1}{j} E_j(0) n^{m+1-j}, \\ -\Omega_{m+1}(n) + \frac{(-1)^n - 1}{2} E_{m+1}(0) &= \sum_{j=0}^{m} \binom{m+1}{j} E_{m+1-j}(0) S_j(n).\end{aligned} \tag{9}$$

From (9), we have

$$S_m(n) = \frac{1}{m+1}\left(n^{m+1} - \sum_{j=1}^{m} \binom{m+1}{j} E_j(0) n^{m+1-j}\right) + \frac{2}{m+1}\sum_{j=0}^{m-1} \binom{m+1}{j} E_{m+1-j}(0) S_j(n). \tag{10}$$

The contents of the parentheses in the first term on the right-hand side of (10) can be transformed as follows:

$$
\begin{aligned}
n^{m+1} - \sum_{j=1}^{m} \binom{m+1}{j} E_j(0) n^{m+1-j} &= 2n^{m+1} - \sum_{j=0}^{m} \binom{m+1}{j} E_j(0) n^{m+1-j} \\
&= 2n^{m+1} + E_{m+1}(0) - \sum_{j=0}^{m+1} \binom{m+1}{j} E_j(0) n^{m+1-j} \qquad (11) \\
&= 2n^{m+1} + E_{m+1}(0) - E_{m+1}(n) \\
&= E_{m+1}(n+1) + E_{m+1}(0).
\end{aligned}
$$

Here, the third equation of (11) uses the relationship

$$E_m(x) = \sum_{j=0}^{m} \binom{m}{j} E_j(0) x^{m-j}, \quad \text{if } m \geq 0.$$

It is obtained from the definition of the Euler polynomial $\dfrac{2e^{xt}}{e^t+1} = \displaystyle\sum_{j=0}^{\infty} E_j(x) \frac{t^j}{j!}$. The fourth equation comes from the identity

$$E_{m+1}(x+1) + E_{m+1}(x) = 2x^{m+1}, \quad \text{if } m \geq 0$$

(see [4], for example). This completes the proof of Corollary 2.1. $\square$

Remark 2.1. *By using* (1) *and Theorem 3.1, we have the following identity, which is similar to Corollary 2.1*:

$$\Omega_{m+1}(n) = \frac{B_{m+1}(n+1) - B_{m+1}}{2} - \sum_{j=1}^{m-1} \binom{m+1}{j} B_{m+1-j} \Omega_j(n).$$

Here, m and n are positive integers and $B_j(x)$ is the j-th Bernoulli polynomial defined by $\dfrac{te^{xt}}{e^t-1} = \displaystyle\sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!}$.

3 Relationships between B_k and $E_j(0)$

Relating to Theorem 1.1, the following theorem is known.

Theorem 3.1 ([6]). *For a non-negative integer $m \in \mathbb{Z}$ and a positive integer $n \in \mathbb{Z}$, we have the following identity*:

$$S_m(n) = \begin{cases} \dfrac{2}{m+1} \displaystyle\sum_{\substack{0 \leq j \leq m \\ j \neq 1}} \binom{m+1}{j} B_j \Omega_{m+1-j}(n), & \text{if } m \geq 1, \\ 2B_0\Omega_1(n) - \Omega_0(n), & \text{if } m = 0. \end{cases}$$

From Theorem 1.1, if we define an $m \times m$ matrix A by

$$A = -\begin{pmatrix} \binom{m}{1}E_1(0) & 0 & \cdots & 0 & 0 & 0 \\ \binom{m}{2}E_2(0) & \binom{m-1}{1}E_1(0) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \binom{m}{m-2}E_{m-2}(0) & \binom{m-1}{m-3}E_{m-3}(0) & \cdots & \binom{3}{1}E_1(0) & 0 & 0 \\ \binom{m}{m-1}E_{m-1}(0) & \binom{m-1}{m-2}E_{m-2}(0) & \cdots & \binom{3}{2}E_2(0) & \binom{2}{1}E_1(0) & 0 \\ E_m(0) & E_{m-1}(0) & \cdots & E_3(0) & E_2(0) & E_1(0) \end{pmatrix}$$

and a $1 \times m$ matrix b by

$$b = \frac{(-1)^n - 1}{2}\begin{pmatrix} E_m(0) & E_{m-1}(0) & \cdots & E_3(0) & E_2(0) & E_1(0) \end{pmatrix},$$

we have

$$\begin{pmatrix} \Omega_m(n) & \cdots & \Omega_1(n) \end{pmatrix} = \begin{pmatrix} S_{m-1}(n) & \cdots & S_0(n) \end{pmatrix} A + b.$$

From Theorem 3.1, if we define an $m \times m$ matrix C by

$$C = \begin{pmatrix} \frac{2}{m}B_0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2}{m-1}B_0 & \cdots & 0 & 0 & 0 \\ \frac{2}{m}\binom{m}{2}B_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{m}\binom{m}{m-3}B_{m-3} & \frac{2}{m-1}\binom{m-1}{m-4}B_{m-4} & \cdots & \frac{2}{3}B_0 & 0 & 0 \\ \frac{2}{m}\binom{m}{m-2}B_{m-2} & \frac{2}{m-1}\binom{m-1}{m-3}B_{m-3} & \cdots & 0 & B_0 & 0 \\ \frac{2}{m}\binom{m}{m-1}B_{m-1} & \frac{2}{m-1}\binom{m-1}{m-2}B_{m-2} & \cdots & \frac{2}{3}\binom{3}{2}B_2 & 0 & 2B_0 \end{pmatrix}$$

and a $1 \times m$ matrix d by

$$d = \frac{(-1)^n - 1}{2}\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

we have

$$\begin{pmatrix} S_{m-1}(n) & \cdots & S_0(n) \end{pmatrix} = \begin{pmatrix} \Omega_m(n) & \cdots & \Omega_1(n) \end{pmatrix} C + d.$$

From the equation $AC = CA = I_m$, we have the following corollary. Here, I_m is the identity matrix.

Corollary 3.1. *For a positive integer $m \in \mathbb{Z}$, we take positive integers $i, j \in \mathbb{Z}$ such that $1 \le j \le i \le m$. Then, the following relations hold:*

$$\sum_{k=j+2}^{i} \binom{m+1-j}{k-j}\binom{m+1-k}{i+1-k} B_{k-j} E_{i+1-k}(0) + \binom{m+1-j}{i+1-j} E_{i+1-j}(0) = -\frac{m+1-j}{2}\delta_{i,j}$$

and

$$\sum_{k=j}^{i-2} \frac{1}{m+1-k}\binom{m+1-j}{k+1-j}\binom{m+1-k}{i-k} B_{i-k} E_{k-j+1}(0) + \frac{1}{m+1-i}\binom{m+1-j}{i+1-j} E_{i+1-j}(0) = -\frac{1}{2}\delta_{i,j}.$$

Here, $\delta_{i,j}$ is the Kronecker delta.

Remark 3.1. *When $j = 1$ and $i = m$, both identities in Corollary 3.1 reduce to the equation*:

$$\sum_{k=1}^{m-2} \binom{m}{k} B_{m-k} E_k(0) = -E_m(0)$$

when $m \ge 2$. This equation corresponds to the case $x = 0$ of the formula (13) in [4].

4 Conclusion

In this note, we investigated the relationship between alternating sums of powers of integers and sums of powers of integers. By combining this note with previous researchs such as [1] and [4], it is expected that research on alternating sums of powers of integers and the relationship between the Bernoulli numbers and the Euler polynomials will progress further.

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