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(G, F)-points on $\mathbb{Q}$-algebraic varieties

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Abstract: Let $\mathbb{Q}$ be the field of rational numbers, and let C be an algebraically closed field containing $\mathbb{Q}$. Let $G \in \mathbb{Q}[x, y, z]$ be a polynomial, then the zero set of G is $Z(G) = \{P \in C^n \mid G(P) = 0\}$. A set $V \subset C^n$ is called a $\mathbb{Q}$-algebraic variety if $V = Z(G)$ for some polynomial G in $\mathbb{Q}[x, y, z]$. The set $V(G) = \{P \in \mathbb{Q}^3 \mid G(P) = 0\}$ is called the set of $\mathbb{Q}$-rational points of V. Let

$$\begin{aligned} F: \quad & \mathbb{Q}^3 \to \mathbb{Q}^3, \\ & (x, y, z) \mapsto (f(x), f(y), f(z)) \end{aligned}$$

be a vector function, where $f \in \mathbb{Q}[x]$. It is easy to show that the function obtained by the composition of G and F, denoted as $G \circ F$, is still in $\mathbb{Q}[x, y, z]$. Moreover, let $V(G \circ F)$ be the set of $\mathbb{Q}$-rational points of the $\mathbb{Q}$-algebraic variety corresponding to $G \circ F$, i.e., $V(G \circ F) = \{P \in \mathbb{Q}^3 \mid G \circ F(P) = 0\}$. A rational point P is called a (G, F)-point on $V(G)$ if P belongs to the intersection of $V(G)$ and $V(G \circ F)$, that is $P \in V(G) \cap V(G \circ F)$. Denote $\langle G, F\rangle$ as the set consisting of all (G, F)-points on $V(G)$. Obviously, $\langle G, F\rangle$ is the set of $\mathbb{Q}$-rational points of a $\mathbb{Q}$-algebraic variety, that is, $\langle G, F\rangle = \{P \in \mathbb{Q}^3 \mid G(P) = 0 \text{ and } G \circ F(P) = 0\}$. In this paper, we consider the algebraic variety $\langle G, F\rangle$ for some specific functions G and F. For these specific functions G and F, we prove that $\langle G, F\rangle$ will be isomorphic to a certain elliptic curve. We also analyze some properties of these elliptic curves.

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1 Introduction

Diophantine geometry aims to study the rational points on algebraic varieties. However, it is not an easy task to determine the rational points on an algebraic variety. In other words, it is not easy to find the rational solutions of certain Diophantine equations. The famous Mordell Conjecture states that there are only finitely many rational points on a curve with genus greater than 1. In 1983, the Mordell Conjecture was finally proven by Faltings. Curves with genus equal to 0 can be parameterized. Therefore, we are concerned with curves with genus equal to 1. The representative of curves with genus equal to 1 is the elliptic curve. However, the structure of rational points on elliptic curves is still not fully understood.

Several authors (see [1–3, 6–11, 13]) investigated the rational solutions of the Diophantine equation

$$f(x)f(y) = f(z)^2,$$

where $f \in \mathbb{Q}[x]$. In this studies, the transformation

$$x = T, \;\; y = u^2T, \;\; z = uT$$

was frequently used. It means that they actually gave the rational solutions of the Diophantine system

$$\begin{cases} f(x)f(y) = f(z)^2, \\ xy = z^2. \end{cases} \tag{1}$$

Naturally, we can consider the rational solutions of the general Diophantine system

$$\begin{cases} G(f(x), f(y), f(z)) = 0, \\ G(x, y, z) = 0, \end{cases} \tag{2}$$

where $G \in \mathbb{Q}[x, y, z]$ and $f \in \mathbb{Q}[x]$. Let us redescribe the Diophantine system (2). Let $G \in \mathbb{Q}[x, y, z]$ be a polynomial, and let $V(G)$ be the set of $\mathbb{Q}$-rational points of the $\mathbb{Q}$-algebraic variety corresponding to G, i.e., $V(G) = \{P \in \mathbb{Q}^3 \mid G(P) = 0\}$. Let

$$\begin{aligned} F: \quad & \mathbb{Q}^3 \to \mathbb{Q}^3, \\ & (x, y, z) \mapsto (f(x), f(y), f(z)) \end{aligned}$$

be a vector function, where $f \in \mathbb{Q}[x]$. It is easy to show that the function obtained by the composition of G and F, denoted as $G \circ F$, is still in $\mathbb{Q}[x, y, z]$.

Moreover, let $V(G \circ F)$ be the set of $\mathbb{Q}$-rational points of the $\mathbb{Q}$-algebraic variety corresponding to $G \circ F$, i.e., $V(G \circ F) = \{P \in \mathbb{Q}^3 \mid G \circ F(P) = 0\}$. A rational point P is called a (G, F)-point on $V(G)$ if P belongs to the intersection of $V(G)$ and $V(G \circ F)$, that is $P \in V(G) \cap V(G \circ F)$. Denote $\langle G, F\rangle$ as the set consisting of all (G, F)-points on $V(G)$. Obviously, $\langle G, F\rangle$ is the set of $\mathbb{Q}$-rational points of a $\mathbb{Q}$-algebraic variety, that is, $\langle G, F\rangle = \{P \in \mathbb{Q}^3 \mid G(P) = 0 \text{ and } G \circ F(P) = 0\}$.

When $G(x,y,z) = xy - z^2$, $\langle G, F\rangle$ represents all the rational solutions of the Diophantine system (1). From $G(x,y,z) = xy - z^2 = 0$, we obtain $z = \sqrt{xy}$, which means that z is the geometric mean of x and y.

For some specific functions G and F, the set $V(G \circ F)$ may be an empty set or may have only trivial solutions. When $G = x + y - z$ and $f(x) = x^n$, by Fermat's Last Theorem, $V(G \circ F)$ has only trivial solutions. When $G = xy - z^2$ and $f(x) = x^k - 1$, Bennett [1] proved that $V(G \circ F)$ has only trivial solutions. When $G = x^2 + y^2 - z^2$ and $f(x) = x(x+1)/2$, Sierpiński [4] introduced (given by Zarankiewicz) a nontrivial positive integer solution $(x,y,z) = (132, 143, 164)$ for $V(G \circ F)$.

In 2019, Zhang and Chen [12] studied the rational solutions of Diophantine equation of harmonic mean

$$f(z) = \frac{2}{\frac{1}{f(x)} + \frac{1}{f(y)}}.$$

In this paper, we consider the set $\langle G, F\rangle$ for some specific functions G and F. For the function G, we consider that $G = xy - z^2$, or $G = (x+y)z - 2xy$. For $G = (x+y)z - 2xy = 0$, it means that z is the harmonic mean of x and y. For the vector function F, we consider the components f of F to be $f = ax^2 + bx + c$, $f = ax + b + cx^{-1}$, $f = x(ax^2 + bx + c)$, respectively. For these specific functions G and F, we prove that $\langle G, F\rangle$ will be isomorphic to a certain elliptic curve. We also analyze some properties of these elliptic curves.

2 The main results

By the theory of elliptic curves, we prove

Theorem 2.1. *Let $G = xy - z^2$ and $f(x) = ax^2 + bx + c \in \mathbb{Q}[x]$ with $abc \neq 0$. When $4ac - b^2 \neq 0$, $\langle G, F\rangle$ is birationally equivalent to the elliptic curve*

$$\mathcal{E}_1 : Y^2 = X^3 + 27a^2b^2c^2(3ac - b^2)X + 27a^3b^4c^3(9ac - 2b^2).$$

When $4ac - b^2 = 0$, $\langle G, F\rangle$ is a curve with genus 0 and its parameterization is given by

$$y = -\frac{bct^2}{(2ct + b)^2}, \quad z = -\frac{(bt + 4ct + 2b)ct}{(2ct + b)(bt + 2ct + b)},$$

where t is a rational number.

Proof. When $G = xy - z^2$ and $f(x) = ax^2 + bx + c$, the Diophantine system (2) is equivalent to

$$(y - z)^2 \left(abyz^2 + acy^2 + 2acyz + acz^2 + bcy\right) = 0. \tag{3}$$

Since $y = z$ is trivial, we only need to consider the rational points on the curve

$$\mathcal{C}_1 : abyz^2 + acy^2 + 2acyz + acz^2 + bcy = 0.$$

By the map φ_1:

$$\begin{aligned} X &= \frac{3abc(3acy+6acz-b^2y+2bc)}{by+c}, \\ Y &= \frac{27a^2bc^2(abcy^2+3abcyz-b^3yz-ac^2y-ac^2z+b^2cy-b^2cz-bc^2)}{(by+c)^2}, \end{aligned} \tag{4}$$

with the inverse map φ_1^{-1}:

$$\begin{aligned} y &= -\frac{c(108a^3b^2c^3-18a^2b^4c^2+9Xa^2c^2-3Xab^2c+6Yac+X^2)}{b(X-9a^2c^2+3ab^2c)^2}, \\ z &= -\frac{9a^2b^2c^2+3Xac+Y}{3ba(X-9a^2c^2+3ab^2c)}, \end{aligned} \tag{5}$$

we can transform $\mathcal{C}_1$ into the elliptic curve

$$\mathcal{E}_1 : \ Y^2 = X^3 + 27a^2b^2c^2(3ac-b^2)X + 27a^3b^4c^3(9ac-2b^2).$$

The discriminant of $\mathcal{E}_1$ is

$$\Delta_1(E) = -531441a^8b^6c^8(4ac-b^2).$$

Hence, if $4ac-b^2 \neq 0$, then $\Delta_1(E) \neq 0$, so $\mathcal{E}_1$ is non-singular. Therefore, (4) and (5) give a bijection between $\langle G, F\rangle$ and the elliptic curve $\mathcal{E}_1$, so $\langle G, F\rangle$ is birationally equivalent to the elliptic curve $\mathcal{E}_1$.

When $4ac-b^2=0$, the Diophantine system (2) is equivalent to

$$b^2yz^2 + bcy^2 + 2bcyz + bcz^2 + 4c^2y = 0. \tag{6}$$

The curve given by (6) is a curve of genus 0, and its parameterization is given by

$$y = -\frac{bct^2}{(2ct+b)^2}, \quad z = -\frac{(bt+4ct+2b)ct}{(2ct+b)(bt+2ct+b)},$$

where t is a rational number. This completes the proof. $\square$

Remark 2.1. *Since $xy=z^2$, when $f = x^k(ax^2+bx+c), k \in \mathbb{Z}$, we can obtain exactly the same result as that of Theorem 1.1. Particularly, when $k=1$ or $k=-1$, we obtain $f = x(ax^2+bx+c)$ and $f = ax+b+cx^{-1}$, respectively.*

Theorem 2.2. *When $b^2 \neq kac$ $(k=-1,1,3)$, the elliptic curve $\mathcal{E}_1$ has a positive rank. When $b^2 = kac$ $(k=-1,1,3)$, the rank of the elliptic curve $\mathcal{E}_1$ is 0.*

Proof. It is easy to check that the elliptic curve $\mathcal{E}_1$ contains two rational points

$$P_0 = (-3ab^2c, 0), \quad P_1 = (6ab^2c, 27a^2b^2c^2).$$

By the group law, we have

$$P_2 = [2]P_1 = \left(\frac{3(3ac-b^2)(ac-3b^2)}{4},\ -\frac{27(ac-b^2)(a^2c^2+4ab^2c-b^4)}{8}\right),$$

$$P_3 = [3]P_1 = \left(\frac{6ab^2c(13a^4c^4+24a^3b^2c^3-22a^2b^4c^2+b^8)}{(a^2c^2-6ab^2c+b^4)^2},\right.$$
$$\left. -\frac{27a^2b^2c^2(3ac-b^2)(ac+b^2)(a^4c^4+24a^3b^2c^3-22a^2b^4c^2+16ab^6c-3b^8)}{(a^2c^2-6ab^2c+b^4)^3}\right)$$

and

$$\begin{aligned} P_4 &= [4]P_1 \\ &= \left(\frac{3(3ac-b^2)X_4}{16(ac-b^2)^2(a^2c^2+4ab^2c-b^4)^2}, \frac{27(a^4c^4-20a^3b^2c^3+6a^2b^4c^2-4ab^6c+b^8)Y_4}{64(ac-b^2)^3(a^2c^2+4ab^2c-b^4)^3}\right),\end{aligned}$$

where

$$\begin{aligned} X_4 &= a^7c^7-45a^6b^2c^6+365a^5b^4c^5-121a^4b^6c^4 \\ &\quad +307a^3b^8c^3-151a^2b^{10}c^2+31ab^{12}c-3b^{14}, \\ Y_4 &= a^8c^8+80a^7b^2c^7-180a^6b^4c^6+656a^5b^6c^5 \\ &\quad -282a^4b^8c^4-80a^3b^{10}c^3+76a^2b^{12}c^2-16ab^{14}c+b^{16}. \end{aligned}$$

Let the line go through the points P_0 and P_1, intersecting $\mathcal{E}_1$ at P_5, then

$$P_5 = -(P_0+P_1) = \left(3ac(3ac-b^2), 27a^3c^3\right).$$

Similarly,

$$P_6 = -(P_0+P_2) = \left(\frac{3ab^2c(11a^2c^2+2ab^2c-b^4)}{(ac-b^2)^2}, -\frac{54(a^2c^2+4ab^2c-b^4)a^3b^2c^3}{(ac-b^2)^3}\right),$$

$$P_7 = -(P_0+P_3) = \left(\frac{3ac(3a^5c^5-45a^4b^2c^4+102a^3b^4c^3-34a^2b^6c^2+7ab^8c-b^{10})}{(ac+b^2)^2(3ac-b^2)^2},\right.$$
$$\left. -\frac{27c^3a^3(a^4c^4+24a^3b^2c^3-22a^2b^4c^2+16ab^6c-3b^8)(a^2c^2-6ab^2c+b^4)}{(3ac-b^2)^3(ac+b^2)^3}\right),$$

and

$$P_8 = -(P_0+P_4) = \left(\frac{3ab^2cX_8}{(a^4c^4-20a^3b^2c^3+6a^2b^4c^2-4ab^6c+b^8)^2},\right.$$
$$\left.\frac{108c^3b^2a^3(ac-b^2)(a^2c^2+4ab^2c-b^4)Y_8}{(a^4c^4-20a^3b^2c^3+6a^2b^4c^2-4ab^6c+b^8)^3}\right),$$

where

$$\begin{aligned} X_8 &= 47a^8c^8+328a^7b^2c^7-460a^6b^4c^6-1096a^5b^6c^5 \\ &\quad +1290a^4b^8c^4-392a^3b^{10}c^3+20a^2b^{12}c^2+8ab^{14}c-b^{16}, \\ Y_8 &= a^8c^8+80a^7b^2c^7-180a^6b^4c^6+656a^5b^6c^5 \\ &\quad -282a^4b^8c^4-80a^3b^{10}c^3+76a^2b^{12}c^2-16ab^{14}c+b^{16}. \end{aligned}$$

Let $X(P)$ denote the X-coordinate of the point P. When $a \neq 0$, we give the conditions such that $X(P_i) = X(P_j),\ 0 \leq i < j \leq 8$ in the following Table 1.

Table 1. Conditions such that $X(P_i) = X(P_j), 0 \leq i < j \leq 8$.

X-coordinate	$k\ (b^2 = kab)$
$X(P_0) = X(P_i),\ i = 1, ..., 8$	$-1,\ 1,\ 3$
$X(P_1) = X(P_i),\ i = 2, ..., 8$	$-1,\ 1,\ 3$
$X(P_2) = X(P_i),\ i = 3, ..., 8$	$-1,\ 1,\ 3$
$X(P_3) = X(P_i),\ i = 4, ..., 8$	1
$X(P_4) = X(P_i),\ i = 5, ..., 8$	$-1,\ 3$
$X(P_5) = X(P_i),\ i = 6, ..., 8$	1
$X(P_6) = X(P_i),\ i = 7, 8$	$-1,\ 3$
$X(P_7) = X(P_8),\ i = 8$	none

Table 1 shows that when $b^2 \neq kac\ (k = -1, 1, 3)$, the points P_0 and $\pm P_i,\ i = 1, ..., 8$, are different. By Mazur's theorem (see p. 58 of [5]) about the quantity of rational points and the rank of elliptic curve: If an elliptic curve E defined over $\mathbb{Q}$ has more than 16 different rational points, then it has infinitely many rational points and its rank has at least one. Therefore, $\mathcal{E}_1$ has a positive rank, and thus there are infinitely many rational points on $\mathcal{E}_1$.

When $b^2 = kac\ (k = -1, 1, 3)$, we have $a = \frac{b^2}{ck}$, and then

$$\mathcal{E}_1' :\ Y^2 = \ X^3 - \frac{27(k-3)b^8}{k^3}X - \frac{27(2k-9)b^{12}}{k^4}.$$

By the transformation

$$U = \frac{Y}{b^6}, \quad V = \frac{X}{b^4}, \tag{7}$$

we get

$$\mathcal{E}_{(k)} :\ U^2 = V^3 - \frac{27(k-3)}{k^3}V - \frac{27(2k-9)}{k^4}.$$

1) When $k = -1$, we get

$$\mathcal{E}_{(-1)} :\ U^2 = V^3 - 108V + 297.$$

Using the package of *Magma*, the rank of $\mathcal{E}_{(-1)}$ is 0, and the only rational points on $\mathcal{E}_{(-1)}$ are

$$(3, 0),\ (-6, \pm 27)\ (12, \pm 27).$$

2) When $k = 1$, we get

$$\mathcal{E}_{(1)} :\ U^2 = V^3 + 54V + 189.$$

The rank of $\mathcal{E}_{(1)}$ is 0, and the only rational points on $\mathcal{E}_{(1)}$ are

$$(-3, 0),\ (6, \pm 27).$$

3) When $k=3$, we get

$$\mathcal{E}_{(3)}:\ U^2=V^3+1.$$

The rank of $\mathcal{E}_{(3)}$ is 0, and the only rational points on $\mathcal{E}_{(3)}$ are

$$(-1,0),\ (0,\pm1),\ (2,\pm3).$$

Therefore, when $b^2\neq kac$ $(k=-1,1,3)$, the elliptic curve $\mathcal{E}_1$ has a positive rank. When $b^2=kac$ $(k=-1,1,3)$, the rank of the elliptic curve $\mathcal{E}_1$ is 0. $\square$

Theorem 2.3. *Let $G=(x+y)z-2xy$ and $f(x)=ax^2+bx+c\in\mathbb{Q}[x]$ with $abc\neq0$. When $2ac-b^2\neq0$ and $4ac-b^2\neq0$, $\langle G,F\rangle$ is birationally equivalent to the elliptic curve*

$$\mathcal{E}_2: Y^2=X^3-3a^4c^4X+a^4c^4(2a^2c^2-4ab^2c+b^4).$$

When $2ac-b^2=0$, $\langle G,F\rangle$ is the union of two curves, which are

$$z=-\frac{2c}{b},\ \ z=\frac{2cy(by+2c)}{b^2y^2+2c^2}.$$

When $4ac-b^2=0$, $\langle G,F\rangle$ is a curve with genus 0 and its parameterization is given by

$$\begin{aligned}y&=\frac{24ct(2bt-12ct-c)}{68b^2t^2+48bct^2-144c^2t^2+4bct-24c^2t-c^2},\\ z&=-\frac{48ct(8bt-12ct-c)}{100b^2t^2-192bct^2+144c^2t^2-16bct+24c^2t+c^2},\end{aligned}$$

where t is a rational number.

Proof. When $G=(x+y)z-2xy$ and $f(x)=ax^2+bx+c$, the Diophantine system (2) is equivalent to

$$(y-z)^2\left(a^2y^2z^2-2acy^2-2acyz+acz^2-2bcy+bcz\right)=0. \tag{8}$$

Since $y=z$ is trivial, we only need to consider the rational points on the curve

$$\mathcal{C}_2: a^2y^2z^2-2acy^2-2acyz+acz^2-2bcy+bcz=0.$$

By the map φ_2:

$$\begin{aligned}X&=\frac{c(a^2by^2z+a^2cy^2+abcz+b^2c)}{y^2},\\ Y&=\frac{bc^2(a^3y^3z+a^2by^3+a^2by^2z+2a^2cy^2+a^2cyz+abcy+abcz+b^2c)}{y^3},\end{aligned} \tag{9}$$

with the inverse map φ_2^{-1}:

$$y=\frac{bc(-a^3c^3+a^2b^2c^2+Xac+Y)}{a^4c^4-2a^3b^2c^3-2Xa^2c^2+X^2},\ \ z=\frac{2bc(-2a^3c^3-a^2b^2c^2+2Xac+Y)}{a^4c^4+4a^3b^2c^3-2Xa^2c^2+X^2}, \tag{10}$$

we can transform $\mathcal{C}_2$ into the elliptic curve

$$\mathcal{E}_2:\ Y^2=X^3-3a^4c^4X+a^4c^4(2a^2c^2-4ab^2c+b^4).$$

The discriminant of $\mathcal{E}_2$ is

$$\Delta_2(E) = 27a^8c^8b^2(4ac - b^2)(2ac - b^2)^2.$$

Hence, if $2ac - b^2 \neq 0$ and $4ac - b^2 \neq 0$, then $\Delta_2(E) \neq 0$, so $\mathcal{E}_2$ is non-singular. Therefore, (9) and (10) give a bijection between $\langle G, F \rangle$ and the elliptic curve $\mathcal{E}_2$, so $\langle G, F \rangle$ is birationally equivalent to the elliptic curve $\mathcal{E}_2$.

When $2ac - b^2 = 0$, the Diophantine system (2) is equivalent to

$$(bz + 2c)(b^2y^2z - 2bcy^2 - 4c^2y + 2c^2z) = 0,$$

which leads to

$$z = -\frac{2c}{b}, \quad z = \frac{2cy(by + 2c)}{b^2y^2 + 2c^2}.$$

When $4ac - b^2 = 0$, the Diophantine system (2) is equivalent to

$$b^3y^2z^2 - 8bc^2y^2 - 8bc^2yz + 4bc^2z^2 - 32c^3y + 16c^3z = 0. \tag{11}$$

The curve given by (11) is a curve of genus 0, and its parameterization is given by

$$\begin{aligned} y &= \frac{24ct(2bt - 12ct - c)}{68b^2t^2 + 48bct^2 - 144c^2t^2 + 4bct - 24c^2t - c^2}, \\ z &= -\frac{48ct(8bt - 12ct - c)}{100b^2t^2 - 192bct^2 + 144c^2t^2 - 16bct + 24c^2t + c^2}, \end{aligned}$$

where t is a rational number. This completes the proof. □

Theorem 2.4. *The elliptic curve $\mathcal{E}_2$ has a positive rank.*

Proof. It is easy to check that the elliptic curve $\mathcal{E}_2$ contains the following rational point

$$P = \left(-a^2c^2, -c^2(2ac - b^2)a^2\right).$$

By the group law, we get the following eight points

$$[2]P, \quad [3]P, \quad [4]P, \quad [5]P, \quad [6]P, \quad [7]P, \quad [8]P, \quad [9]P.$$

The points $[2]P$ and $[3]P$ are as follows

$$\begin{aligned} [2]P &= \left(2a^2c^2, a^2c^2(2ac - b^2)\right), \\ [3]P &= \left(\frac{7}{9}a^2c^2 - \frac{16}{9}ab^2c + \frac{4}{9}b^4, -\frac{(2ac - b^2)(5a^2c^2 - 32ab^2c + 8b^4)}{27}\right). \end{aligned}$$

We omit the expressions for the other six points because they will not be used directly. It is easy to verify that when $2ac - b^2 \neq 0$, the points P and $\pm[i]P,\ i = 2, ..., 9$, are different. By Mazur's theorem (see p. 58 of [5]) about the quantity of rational points and the rank of elliptic curve: If an elliptic curve E defined over $\mathbb{Q}$ has more than 16 different rational points, then it has infinitely many rational points and its rank has at least one. Therefore, $\mathcal{E}_2$ has a positive rank, and thus there are infinitely many rational points on $\mathcal{E}_2$. □

Theorem 2.5. *Let* $G = (x + y)z - 2xy$ *and* $f(x) = ax + b + cx^{-1} \in \mathbb{Q}[x, x^{-1}]$ *with* $abc \neq 0$. *When* $4ac - b^2 \neq 0$, $\langle G, F\rangle$ *is birationally equivalent to the elliptic curve*

$$\mathcal{E}_3 : Y^2 = X^3 - 3a^2c^6X - a^2c^8(2ac - b^2).$$

When $4ac - b^2 = 0$, $\langle G, F\rangle$ *is a curve with genus* 0 *and its parameterization is given by*

$$y = \frac{8ct(9bt + 6ct + b)}{(17bt + 6ct + b)(5bt + 6ct + b)}, \ \ z = -\frac{(17bt + 6ct + b)(9bt + 6ct + b)c}{16b^3t^2},$$

where t *is a rational number.*

Proof. When $G = (x + y)z - 2xy$ and $f(x) = ax + b + cx^{-1}$, the Diophantine system (2) is equivalent to

$$(y - z)^2 \left(aby^2z^2 + 3acy^2z - 2c^2y + c^2z\right) = 0. \tag{12}$$

Since $y = z$ is trivial, we only need to consider the rational points on the curve

$$\mathcal{C}_3 : aby^2z^2 + 3acy^2z - 2c^2y + c^2z = 0.$$

By the map φ_3:

$$X = \frac{c^2(aby^2z + 2acy^2 + c^2)}{y^2}, \quad Y = \frac{c^4(aby^3 + aby^2z + 3acy^2 + c^2)}{y^3}, \tag{13}$$

with the inverse map φ_3^{-1}:

$$y = \frac{c^2(Y + abc^4)}{(X + ac^3)(X - 2ac^3)}, \quad z = \frac{2c^2(Y - abc^4)}{(X + ac^3)^2}, \tag{14}$$

we can transform $\mathcal{C}_3$ into the elliptic curve

$$\mathcal{E}_3 : \ Y^2 = X^3 - 3a^2c^6X - a^2c^8(2ac - b^2).$$

The discriminant of $\mathcal{E}_3$ is

$$\Delta_3(E) = 27a^4c^{16}b^2(4ac - b^2).$$

Hence, if $4ac - b^2 \neq 0$, then $\Delta_3(E) \neq 0$, so $\mathcal{E}_3$ is non-singular. Therefore, (13) and (14) give a bijection between $\langle G, F\rangle$ and the elliptic curve $\mathcal{E}_3$, so $\langle G, F\rangle$ is birationally equivalent to the elliptic curve $\mathcal{E}_3$.

When $4ac - b^2 = 0$, the Diophantine system (2) is equivalent to

$$b^3y^2z^2 + 3b^2cy^2z - 8c^3y + 4c^3z = 0. \tag{15}$$

The curve given by (15) is a curve of genus 0, and its parameterization is given by

$$y = \frac{8ct(9bt + 6ct + b)}{(17bt + 6ct + b)(5bt + 6ct + b)}, \ \ z = -\frac{(17bt + 6ct + b)(9bt + 6ct + b)c}{16b^3t^2},$$

where t is a rational number. This completes the proof. □

Theorem 2.6. *When $b^2 \neq kac$ $(k = \frac{27}{8}, \frac{27}{4})$, the elliptic curve $\mathcal{E}_3$ has a positive rank. When $b^2 = kac$ $(k = \frac{27}{8}, \frac{27}{4})$, the rank of the elliptic curve $\mathcal{E}_3$ is 0.*

Proof. It is easy to check that the elliptic curve $\mathcal{E}_3$ contains the following rational point

$$P = (2ac^3, -abc^4).$$

By the group law, we get the following eight points

$$[2]P, \quad [3]P, \quad [4]P, \quad [5]P, \quad [6]P, \quad [7]P, \quad [8]P, \quad [9]P.$$

The point P_2 is as follows

$$[2]P = \left(\frac{ac^3(81ac - 16b^2)}{4b^2}, \frac{c^4a(729a^2c^2 - 216ab^2c + 8b^4)}{8b^3} \right).$$

We omit the expressions for the other seven points because they will not be used directly. It is easy to verify that when $b^2 \neq kac$ $(k = \frac{27}{8}, \frac{27}{4})$, the points P and $\pm[i]P,\ i = 2, ..., 9$, are different. By Mazur's theorem (see p. 58 of [5]) about the quantity of rational points and the rank of elliptic curve: If an elliptic curve E defined over $\mathbb{Q}$ has more than 16 different rational points, then it has infinitely many rational points and its rank has at least one. Therefore, $\mathcal{E}_3$ has a positive rank, and thus there are infinitely many rational points on $\mathcal{E}_3$.

When $b^2 = kac$ $(k = \frac{27}{8}, \frac{27}{4})$, we have $a = \frac{b^2}{ck}$, and then

$$\mathcal{E}_3' : \ Y^2 = \ X^3 - \frac{3b^4c^4}{k^2}X + \frac{b^6c^6(k-2)}{k^3}.$$

By the transformation

$$U = \frac{Y}{b^3c^3}, \quad V = \frac{X}{b^2c^2}, \tag{16}$$

we get

$$\mathcal{E}_{(k)} : \ U^2 = V^3 - \frac{3}{k^2}V + \frac{k-2}{k^3}.$$

1) When $k = \frac{27}{8}$, we get

$$\mathcal{E}_{(\frac{27}{8})} : \ U^2 = V^3 - \frac{64}{243}V + \frac{704}{19683}.$$

Using the package of *Magma*, the rank of $\mathcal{E}_{(\frac{27}{8})}$ is 0.

2) When $k = \frac{27}{4}$, we get

$$\mathcal{E}_{(\frac{27}{4})} : \ U^2 = V^3 - \frac{16}{243}V + \frac{304}{19683}.$$

The rank of $\mathcal{E}_{(\frac{27}{4})}$ is 0.

Therefore, when $b^2 \neq kac$ $(k = \frac{27}{8}, \frac{27}{4})$, the elliptic curve $\mathcal{E}_3$ has a positive rank. When $b^2 = kac$ $(k = \frac{27}{8}, \frac{27}{4})$, the rank of the elliptic curve $\mathcal{E}_3$ is 0. $\square$

Theorem 2.7. *Let $G = (x+y)z - 2xy$ and $f(x) = x(ax^2+bx+c) \in \mathbb{Q}[x]$ with $abc \neq 0$. When $ac-b^2 \neq 0, 3ac-b^2 \neq 0$ and $4ac-b^2 \neq 0$, $\langle G, F\rangle$ is birationally equivalent to the elliptic curve*

$$\mathcal{E}_4 : Y^2 = X^3 - 3a^6c^2X - a^6(2ac-b^2)(a^2c^2-4ab^2c+b^4).$$

When $ac-b^2 = 0$, $\langle G, F\rangle$ is a curve with genus 0 and its parameterization is given by

$$y = -\frac{c(2t^2+6t+3)}{3(t+1)b}, \quad z = -\frac{c(2t^2+6t+3)}{3b(t+1)^2},$$

where t is a rational number. When $3ac-b^2 = 0$, $\langle G, F\rangle$ is the union of two curves, which are

$$z = -\frac{2c}{b}, \quad z = \frac{y(by+2c)}{c}.$$

When $4ac-b^2 = 0$, $\langle G, F\rangle$ is a curve with genus 0 and its parameterization is given by

$$y = -\frac{2ct(bt+6ct+3b)}{3(bt+2ct+b)(2ct+b)}, \quad z = -\frac{4(2bt+6ct+3b)ct}{3(bt+2ct+b)^2},$$

where t is a rational number.

Proof. When $G = (x+y)z - 2xy$ and $f(x) = x(ax^2+bx+c)$, the Diophantine system (2) is equivalent to

$$(y-z)^2\left(3a^2y^2z + 2aby^2 + 2abyz - abz^2 - 2acy + acz + 2b^2y - b^2z\right) = 0. \tag{17}$$

Since $y = z$ is trivial, we only need to consider the rational points on the curve

$$\mathcal{C}_4 : 3a^2y^2z + 2aby^2 + 2abyz - abz^2 - 2acy + acz + 2b^2y - b^2z = 0.$$

By the map φ_4:

$$\begin{aligned} X &= \frac{2a^3cy^2 - a^2b^2y^2 - a^2bcz + ab^3z + a^2c^2 - 2ab^2c + b^4}{y^2}, \\ Y &= \frac{(ac-b^2)Y_1}{y^3}, \end{aligned} \tag{18}$$

where

$$Y_1 = 3a^3cy^2 - a^2b^2y^2 + a^2b^2yz - a^2bcy - a^2bcz + ab^3y + ab^3z + a^2c^2 - 2ab^2c + b^4,$$

with the inverse map φ_4^{-1}:

$$\begin{aligned} y &= -\frac{(ac-b^2)(-2a^4bc + a^3b^3 + Xab - Y)}{(a^3c - 2a^2b^2 + X)(-2a^3c + a^2b^2 + X)}, \\ z &= -\frac{2(ac-b^2)(-a^4bc - a^3b^3 + 2Xab - Y)}{(a^3c - 2a^2b^2 + X)^2}, \end{aligned} \tag{19}$$

we can transform $\mathcal{C}_4$ into the elliptic curve

$$\mathcal{E}_4 : \; Y^2 = X^3 - 3a^6c^2X - a^6(2ac-b^2)(a^2c^2-4ab^2c+b^4).$$

The discriminant of $\mathcal{E}_4$ is

$$\Delta_4(E) = 27a^{12}b^2(4ac-b^2)(3ac-b^2)^2(ac-b^2)^2.$$

Hence, if $ac-b^2\neq 0, 3ac-b^2\neq 0$ and $4ac-b^2\neq 0$, then $\Delta_4(E)\neq 0$, so $\mathcal{E}_4$ is non-singular. Therefore, (18) and (19) give a bijection between $\langle G,F\rangle$ and the elliptic curve $\mathcal{E}_4$, so $\langle G,F\rangle$ is birationally equivalent to the elliptic curve $\mathcal{E}_4$.

When $ac-b^2=0$, the Diophantine system (2) is equivalent to

$$3by^2z+2cy^2+2cyz-cz^2=0. \tag{20}$$

The curve given by (20) is a curve of genus 0, and its parameterization is given by

$$y=-\frac{c(2t^2+6t+3)}{3(t+1)b},\quad z=-\frac{c(2t^2+6t+3)}{3b(t+1)^2},$$

where t is a rational number. When $3ac-b^2=0$, the Diophantine system (2) is equivalent to

$$(bz+2c)(by^2+2cy-cz)=0,$$

which leads to

$$z=-\frac{2c}{b},\quad z=\frac{y(by+2c)}{c}.$$

When $4ac-b^2=0$, the Diophantine system (2) is equivalent to

$$3b^2y^2z+8bcy^2+8bcyz-4bcz^2+24c^2y-12c^2z=0. \tag{21}$$

The curve given by (21) is a curve of genus 0, and its parameterization is given by

$$y=-\frac{2ct(bt+6ct+3b)}{3(bt+2ct+b)(2ct+b)},\quad z=-\frac{4(2bt+6ct+3b)ct}{3(bt+2ct+b)^2},$$

where t is a rational number. This completes the proof. $\square$

Theorem 2.8. *When $b^2\neq\frac{3}{2}ac$, the elliptic curve $\mathcal{E}_4$ has a positive rank. When $b^2=\frac{3}{2}ac$, the rank of the elliptic curve $\mathcal{E}_4$ is 0.*

Proof. It is easy to check that the elliptic curve $\mathcal{E}_4$ contains two rational points

$$P_0=(2a^3c-a^2b^2,0),\quad P_1=(-a^3c,a^3b(3ac-b^2)).$$

By the group law, we get the following seven points

$$P_2=[2]P_1,\quad P_3=[3]P_1,\quad P_4=[4]P_1,$$
$$P_5=P_1+P_0,\quad P_6=P_2+P_0,\quad P_7=P_3+P_0,\quad P_8=P_4+P_0.$$

The point P_2 is as follows

$$P_2=(2a^3c,-3cba^4+a^3b^3).$$

We omit the expressions for the other six points because they will not be used directly. It is easy to verify that when $b^2\neq\frac{3}{2}ac$, the points P and $\pm[i]P,\ i=2,...,9$, are different. By Mazur's theorem (see p. 58 of [5]) about the quantity of rational points and the rank of elliptic curve: If an elliptic curve E defined over $\mathbb{Q}$ has more than 16 different rational points, then it has infinitely many rational points and its rank has at least one. Therefore, $\mathcal{E}_4$ has a positive rank, and thus there are infinitely many rational points on $\mathcal{E}_4$.

When $b^2 = \frac{3}{2}ac$, we have $a = \frac{2b^2}{3c}$, and then

$$\mathcal{E}_4' : \ Y^2 = \ X^3 - \frac{64b^{12}}{243c^4}X + \frac{704b^{18}}{19683c^6}.$$

By the transformation

$$U = \frac{c^3Y}{b^9}, \quad V = \frac{c^2X}{b^6}, \tag{22}$$

we get

$$\mathcal{E}_4'' : \ U^2 = V^3 - \frac{64}{243}V + \frac{704}{19683}.$$

Using the package of *Magma*, the rank of $\mathcal{E}_4''$ is 0. $\square$

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