Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 4, 736–746
DOI: 10.7546/nntdm.2025.31.4.736-746

Three Diophantine equations concerning the polygonal numbers

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Received: 24 April 2025 **Revised:** 6 October 2025
Accepted: 26 October 2025 **Online First:** 28 October 2025

Abstract: Many authors investigated the problem about the linear combination of two polygonal numbers being a perfect square, i.e., the Diophantine equation

$$mP_k(x) + nP_k(y) = z^2,$$

where $P_k(x)$ denotes the x-th k-polygonal number and m, n are positive integers. In this note, we continue the study of this problem in another direction and consider three Diophantine equations

$$mP_k(x) - 1 = z^2, \quad mP_k(x) - nP_k(y) = z^2, \quad mP_k(x) - nP_k(y) = 1.$$

By the theory of Pell equations and congruences, we obtain some conditions such that the above three Diophantine equations have infinitely many positive integer solutions.

Keywords: Polygonal number, Diophantine equation, Pell equation, Positive integer solution.
2020 Mathematics Subject Classification: 11D09, 11D72.

1 Introduction

The x-th k-polygonal number [3, p. 5] is given by the formula

$$P_k(x) = \frac{x\left((k-2)(x-1)+2\right)}{2},$$

where $x \geq 1, k \geq 3$. In the first chapter of [4] and D3 of [7], the authors collected many results and properties about the polygonal numbers.

In 2005, Bencze [1] asked a question: find all positive integers n such that $1 + \frac{9}{2}n(n+1) = 1 + 9P_3(n)$ is a perfect square. In 2007, Le [11] answered Bencze's problem and obtained all such n which are

$$\frac{1}{2}\left(\frac{1}{6}\left(a^{2i+1} + b^{2i+1}\right) - 1\right),$$

where $a = 3 + \sqrt{8}, b = 3 - \sqrt{8}$, and $i \in \mathbb{Z}^+$. In 2013, Hu [8] studied the Diophantine equation

$$1 + nP_3(y-1) = z^2,$$

where

$$n = \begin{cases} \dfrac{t^2 \pm 1}{2}, & t \equiv 1 \pmod 2, t \geq 3, \\ \dfrac{t^2 \pm 2}{2}, & t \equiv 0 \pmod 2, t \geq 2, \\ \dfrac{t(t-1)}{2}, & t \geq 2. \end{cases}$$

There are more research in this direction, we can refer to [2, 6, 13–15].

In 2019, Peng [12] showed that if $2n$ is not a perfect square, then $1 + nP_3(y-1) = z^2$ has infinitely many positive integer solutions. Meanwhile, she studied the problem about the linear combination of two triangular numbers is a perfect square, i.e., the Diophantine equation

$$mP_3(x-1) + nP_3(y-1) = z^2,$$

where $m, n \in \mathbb{Z}^+$. Under some conditions, she obtained an infinity of positive integer solutions of the above Diophantine equation.

In 2020, Jiang and Li [9] considered the linear combination of two polygonal numbers is a perfect square and gave a positive answer to Question 4.1 of Peng [12]. They showed that if $k \geq 5$, $2(k-2)n$ is not a perfect square, and there is a positive integer solution (Y', Z') of $Y^2 - 2(k-2)nZ^2 = (k-4)^2n^2 - 8(k-2)n$ satisfying

$$Y' + (k-4)n \equiv 0 \pmod{2(k-2)n}, \quad Z' \equiv 0 \pmod 2,$$

then the Diophantine equation $1 + nP_k(y) = z^2$ has infinitely many positive integer solutions (y, z). Moreover, they studied the Diophantine equation

$$mP_k(x) + nP_k(y) = z^2,$$

where $m, n \in \mathbb{Z}^+$, and proved that when $\frac{ntr}{2}$ is a perfect square, where $t = r(k-2) - 1$, there exist infinitely many pairs (a, b) of positive integers such that $mP_k(x) + nP_k(y) = z^2$ has integer parametric solutions $(x, ax+b, u(cx+d))$, where $k \geq 5$. Further, they obtained two general results:

1) If $k \geq 5$, $2(k-2)(m+n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pellian equation
$$X^2 - 2(k-2)(m+n)Z^2 = (k-4)^2(m+n)^2 - 4(k-2)^2mnr^2$$
has a positive integer solution (X_0, Z_0) satisfying
$$X_0 - 2(k-2)nr + (k-4)(m+n) \equiv 0 \pmod{2(k-2)(m+n)}, \quad Z_0 \equiv 0 \pmod{2},$$
then $mP_k(x) + nP_k(y) = z^2$ has infinitely many positive integer solutions.

2) Let $k \geq 5$, $m = 2(u^2-4u-4)^2$, $n = 2(u^2+4u-4)^2$. If $2(k-2)$ is not a perfect square, and the Pell equation $X^2 - 8(k-2)(u^2+4)^2Z^2 = 1$ has a positive integer solution (U_0, V_0) satisfying $U_0 - 1 \equiv 0 \pmod{2(k-2)}$, then $mP_k(x) + nP_k(y) = z^2$ has infinitely many positive integer solutions.

2 Main results

We consider the positive integer solutions of the following Diophantine equations
$$mP_k(x) - 1 = z^2, \tag{1}$$
and
$$mP_k(x) - nP_k(y) = z^2, \tag{2}$$
where $k \geq 3$ and $k, m, n \in \mathbb{Z}^+$.

By the theory of Pell equation, we have the following theorems.

Theorem 2.1. *If $k \geq 3$, $2(k-2)m$ is not a perfect square, and there is a positive integer solution (X', Z') of the Pell equation $X^2 - 2(k-2)mZ^2 = (k-4)^2m^2 + 8(k-2)m$ satisfying the congruence condition*
$$X' \equiv (k-4)m \pmod{2(k-2)m}, \quad Z' \equiv 0 \pmod{2},$$
then Equation (1) has infinitely many positive integer solutions (x, z). In particular, if $k \geq 3$, $m = r^2+1$, and $2(r^2+1)(k-2)$ is not a perfect square, then Equation (1) has infinitely many positive integer solutions (x, z).

Theorem 2.2. *When $k \geq 3$ and $n = (r(k-2)+1)m$, if $\frac{(r(k-2)+1)mr}{2}$ is a perfect square, denoted by u^2, then there exist infinitely many pairs (a, b) such that Equation (2) has integer parametric solutions $(ay+b, y, u(cy+d))$, where y and a are positive integers, b is an integer.*

Moreover, we obtain

Theorem 2.3. *If $k \geq 3$, $2(k-2)(a^2m-n) > 0$ is not a perfect square, $a, b \in \mathbb{Z}^+$, and the Pell equation*
$$Y^2 - 2(k-2)(a^2m-n)Z^2 = (k-4)^2(am-n)^2 + 4bmn(k-2)(ak+bk-4a-2b-k+4)$$
has a positive integer solution (Y', Z') satisfying the congruence condition
$$Y' \equiv 2abm(k-2) - (k-4)(am-n) \pmod{2(k-2)(a^2m-n)}, \quad Z' \equiv 0 \pmod{2},$$
then Equation (2) has infinitely many positive integer solutions $(ay+b, y, z)$.

If we fix $z=1$ in Equation (2), we get another interesting Diophantine equation

$$mP_k(x) - nP_k(y) = 1. \tag{3}$$

By the same method, we have

Theorem 2.4. *If $k \geq 3$, mn is not a perfect square, and there is a positive integer solution (X', Y') of the Pell equation $X^2 - mnY^2 = (k-4)^2m^2 - ((k-4)^2n - 8k + 16)m$ satisfying the congruence condition*

$$X' \equiv -(k-4)m \pmod{2(k-2)m}, \quad Y' \equiv -(k-4) \,(\mathrm{mod}\{2(k-2)),$$

then Equation (3) has infinitely many positive integer solutions (x, y). In particular, if $k \geq 3$, $n = m-1$, then Equation (3) has infinitely many positive integer solutions (x, y).

3 Preliminaries

To prove these theorems, it needs some results about the solutions of Pell equation.

Lemma 3.1. [10] *Let D be a positive integer which is not a perfect square, then the Pell equation $x^2 - Dy^2 = 1$ has infinitely many positive integer solutions. If (U, V) is the least positive integer solution of the Pell equation $x^2 - Dy^2 = 1$, then all positive integer solutions are given by*

$$x_s + y_s\sqrt{D} = \left(U + V\sqrt{D}\right)^s,$$

where s is an arbitrary integer.

Lemma 3.2. [10] *Let D be a positive integer which is not a perfect square, N be a nonzero integer, and (U, V) is the least positive integer solution of $x^2 - Dy^2 = 1$. If (x_0, y_0) is a positive integer solution of $x^2 - Dy^2 = N$, then an infinity of positive integer solutions are given by*

$$x_s + y_s\sqrt{D} = \left(x_0 + y_0\sqrt{D}\right)\left(U + V\sqrt{D}\right)^s,$$

where s is an arbitrary integer.

Lemma 3.3. [5] *Let D be a positive integer which is not a perfect square, m_1, m_2 are positive integers, and N be a nonzero integer. If the Pell equation $x^2 - Dy^2 = N$ has a positive integer solution satisfying*

$$u_0 \equiv a \pmod{m_1}, \quad v_0 \equiv b \pmod{m_2},$$

then it has infinitely many positive integer solutions satisfying

$$u \equiv a \pmod{m_1}, \quad v \equiv b \pmod{m_2}.$$

4 Proofs of the Theorems

Proof of Theorem 2.1. 1) Multiplying Equation (1) by $8(k-2)m$, we have

$$(m(2(k-2)x - (k-4)))^2 - 2(k-2)m(2z)^2 = (k-4)^2m^2 + 8(k-2)m.$$

Setting $X = m(2(k-2)x - (k-4)),\ Z = 2z$, we get the Pell equation

$$X^2 - 2(k-2)mZ^2 = (k-4)^2m^2 + 8(k-2)m. \tag{4}$$

By Lemma 3.1, if $k \geq 3$ and $2(k-2)m$ is not a perfect square, the Pell equation $X^2-2(k-2)mZ^2=1$ always has an infinite number of positive integer solutions. By Lemma 3.2, if Equation (4) has a positive integer solution, it has infinitely many positive integer solutions. Assume that Equation (4) has a positive integer solution (X', Z') satisfying the congruence condition

$$X' \equiv (k-4)m \pmod{2m(k-2)}, \quad Z' \equiv 0 \pmod 2.$$

By Lemma 3.3, Equation (4) has infinitely many positive integer solutions (X, Z) satisfying the above condition, which leads to infinitely many $x, z \in \mathbb{Z}^+$. Hence, under the above conditions Equation (1) has infinitely many positive integer solutions (x, z).

2) When $m = r^2+1$, Equation (4) becomes

$$X^2 - 2(k-2)(r^2+1)Z^2 = k^2(r^2+1)^2 - 2(k-2)(r^2+1)(2r)^2. \tag{5}$$

It is easy to see that $(X', Z') = (k(r^2+1), 2r)$ is an integer solution of Equation (5). And suppose (u, v) is the least positive integer solution of $X^2 - 2(k-2)(r^2+1)Z^2 = 1$. By Lemma 3.2, an infinity of positive integer solutions of Equation (5) are given by

$$X_s + Z_s\sqrt{2(k-2)(r^2+1)} = \left(k(r^2+1) + 2r\sqrt{2(k-2)(r^2+1)}\right) \times \left(u + v\sqrt{2(k-2)(r^2+1)}\right)^s, \quad s \geq 0.$$

Then

$$\begin{cases} X_s = 2uX_{s-1} - X_{s-2}, & X_0 = k(r^2+1),\ \ X_1 = (4krv + ku - 8rv)(r^2+1), \\ Z_s = 2uZ_{s-1} - Z_{s-2}, & Z_0 = 2r,\ \ Z_1 = kr^2v + kv + 2ru. \end{cases}$$

From $X = (r^2+1)(2(k-2)x - (k-4)),\ Z = 2z$, we have

$$\begin{cases} x_s = 2ux_{s-1} - x_{s-2} - \dfrac{(k-4)(u-1)}{k-2}, & x_0 = 1,\ \ x_1 = \dfrac{4krv + ku - 8rv + k - 4}{2(k-2)}, \\ z_s = 2uz_{s-1} - z_{s-2}, & z_0 = r,\ \ z_1 = \dfrac{kr^2v + kv + 2ru}{2}. \end{cases}$$

Using the recurrence relations of x_s and z_s twice, we get

$$\begin{cases} x_{s+2} = 2(2u^2-1)x_s - x_{s-2} - \dfrac{2(u^2-1)(k-4)}{k-2}, \\ z_{s+2} = 2(2u^2-1)z_s - z_{s-2}. \end{cases}$$

By $u^2 - 2(k-2)(r^2+1)v^2 = 1$, we get

$$\begin{cases} x_{s+2} = 2(2u^2-1)x_s - x_{s-2} - 4(k-4)(r^2+1)v^2, \\ z_{s+2} = 2(2u^2-1)z_s - z_{s-2}. \end{cases}$$

Replacing s by $2s$, we have

$$\begin{cases} x_{2s+2} = 2(2u^2-1)x_{2s} - x_{2s-2} - 4(k-4)(r^2+1)v^2, & x_0 = 1, \\ & x_2 = 2(r^2+1)kv^2 + 4ruv + 1, \\ z_{2s+2} = 2(2u^2-1)z_{2s} - z_{2s-2}, & z_0 = r, \\ & z_2 = 2ru^2 + kv(r^2+1)u - r. \end{cases}$$

Thus, if $k \geq 3$, $m = r^2+1$, and $2(k-2)(r^2+1)$ is not a perfect square, Equation (1) has infinitely many positive integer solutions $(x_{2s}, z_{2s}),\ s \geq 0$. □

Example 4.1. When $k = 5,\ r = 2,\ m = 2^2+1 = 5$, then $2(k-2)m = 30$ is not a perfect square. $(u, v) = (11, 2)$ is the least positive integer solution of $X^2 - 30Z^2 = 1$. $(X', Z') = (25, 4)$ is the least positive integer solution of $X^2 - 30Z^2 = 145$, then

$$\begin{cases} X_s = 22X_{s-1} - X_{s-2}, & X_0 = 25,\ \ X_1 = 515, \\ Z_s = 22Z_{s-1} - Z_{s-2}, & Z_0 = 4,\ \ Z_1 = 94. \end{cases}$$

So

$$\begin{cases} x_{2s+2} = 482x_{2s} - x_{2s-2} - 80, & x_0 = 1,\ \ x_2 = 377, \\ z_{2s+2} = 482z_{2s} - z_{2s-2}, & z_0 = 2,\ \ z_2 = 1032. \end{cases}$$

Therefore, Equation (1) has infinitely many positive integer solutions $(x_{2s}, z_{2s}),\ s \geq 0$.

Proof of Theorem 2.2. Taking $n = tm$ and $x = ay + b$, then Equation (2) reduces to

$$\begin{aligned} &\frac{m(k-2)(a^2-t)}{2}y^2 + \frac{m\left(2(k-2)ab - (k-4)(a-t)\right)}{2}y \\ &\qquad + \frac{mb((k-2)b-(k-4))}{2} = z^2. \end{aligned} \tag{6}$$

Consider

$$g(y) = \frac{m(k-2)(a^2-t)}{2}y^2 + \frac{m(2(k-2)ab-(k-4)(a-t))}{2}y + \frac{mb((k-2)b-(k-4))}{2}$$

as a quadratic polynomial of y, if $g(y) = 0$ has a root with multiplicity 2, the discriminant of $g(y)$ is zero, i.e.,

$$\frac{m^2}{4}\left(4t(k-2)^2b^2 + 4t(k-2)(k-4)(a-1)b + (k-4)^2(a-t)^2\right) = 0.$$

Solving it for b, we get

$$b = \frac{\left(-at + t + \sqrt{t(t-1)a^2 - t^2(t-1)}\right)(k-4)}{2(k-2)t}.$$

To find $b \in \mathbb{Z}$, we take $t(t-1)a^2 - t^2(t-1) = v^2$, then

$$v^2 - t(t-1)a^2 = -t^2(t-1). \tag{7}$$

It is easy to see that the pair $(v', a') = (t(t-1), t)$ is a solution of Equation (7), and the pair $(v, a) = (2t-1, 2)$ solves the Pell equation $v^2 - t(t-1)a^2 = 1$. So an infinity of positive integer solutions of Equation (7) are given by

$$v_s + a_s\sqrt{t(t-1)} = \left(t(t-1) + t\sqrt{t(t-1)}\right)\left(2t - 1 + 2\sqrt{t(t-1)}\right)^s,\ s \geq 0.$$

Thus,

$$\begin{cases} v_s = 2(2t-1)v_{s-1} - v_{s-2}, & v_0 = t(t-1),\ \ v_1 = t(4t-1)(t-1), \\ a_s = 2(2t-1)a_{s-1} - a_{s-2}, & a_0 = t,\ \ a_1 = t(4t-3). \end{cases}$$

According to the recurrence relation of a_s, we have

$$a_s \in \mathbb{Z}^+.$$

Then

$$b_s = \frac{(-t(a_s-1)+v_s)(k-4)}{2(k-2)t}.$$

Further, we get

$$b_s = 2(2t-1)b_{s-1} - b_{s-2} - \frac{2(t-1)(k-4)}{k-2}, \quad b_0 = 0,\ b_1 = -\frac{(k-4)(t-1)}{k-2}.$$

According to the recurrence relation of b_s, we have

$$b_s \equiv 0 \pmod{t-1}.$$

In order for b_s to be an integer, we need $t \equiv 1 \pmod{k-2}$. Taking $t = r(k-2)+1$, we have

$$b_0 = 0,\ b_1 = -r(k-4),$$

so b_s is an integer for each $s \geq 0$.

Equation (6) now becomes

$$\frac{(r(k-2)+1)mr}{2}(cy+d)^2 = z^2,$$

where c, d are determined by $g(y)$.

If $k \geq 3$ and $\frac{(r(k-2)+1)mr}{2}$ is a perfect square, denoted by u^2, then there exist infinitely many pairs (a, b) such that Equation (2) has integer parametric solutions $(ay+b, y, u(cy+d))$, where y and a are positive integers, b is an integer. □

Example 4.2. When $k = 5,\ r = 1,\ m = 2,\ n = 8,\ \frac{(r(k-2)+1)mr}{2} = 4$ is a perfect square. Taking $a_0 = 4,\ b_0 = -1$, Equation (2) has integer parametric solutions $(4y-1, y, 3y-1)$, where y is a positive integer.

Proof of Theorem 2.3. Letting $x = ay+b,\ a, b \in \mathbb{Z}^+$, Equation (2) equals

$$\begin{aligned}(2(k-2)(a^2m-n)y+&2abm(k-2)-(k-4)(am-n))^2 - 2(k-2)(a^2m-n)(2z)^2 \\ &= (k-4)^2(am-n)^2 + 4bmn(k-2)(ak+bk-4a-2b-k+4).\end{aligned}$$

Taking $Y = 2(k-2)(a^2m-n)y + 2abm(k-2) - (k-4)(am-n),\ Z = 2z$, we get

$$Y^2 - 2(k-2)(a^2m-n)Z^2 = (k-4)^2(am-n)^2 + 4bmn(k-2)(ak+bk-4a-2b-k+4). \quad (8)$$

By Lemma 3.1, if $2(k-2)(a^2m-n)$ is not a perfect square, the Pell equation

$$Y^2 - 2(k-2)(a^2m-n)Z^2 = 1$$

has infinitely many positive integer solutions. By Lemma 3.2, if Equation (8) has a positive integer solution, it has an infinite number of positive integer solutions. Assume that Equation (8) has a positive integer solution (Y', Z') satisfying the congruence condition

$$Y' \equiv 2abm(k-2) - (k-4)(am-n) \pmod{2(k-2)(a^2m-n)}, \quad Z' \equiv 0 \pmod{2}.$$

By Lemma 3.3, Equation (8) has infinitely many positive integer solutions (Y, Z) satisfying the above condition, which leads to infinitely many $y, z \in \mathbb{Z}^+$. Then there exist infinitely many $x = ay + b \in \mathbb{Z}^+$. Hence, under the above conditions Equation (2) has an infinite number of positive integer solutions $(ay + b, y, z)$. $\square$

Example 4.3. 1) When $k = 5,\ a = 1,\ b = 2,\ m = 5,\ n = 2$, Equation (8) becomes

$$Y^2 - 18Z^2 = 1449. \tag{9}$$

It has a positive integer solution $(Y', Z') = (93, 20)$ satisfying

$$Y' \equiv 57 \pmod{18}, \quad Z' \equiv 0 \pmod{2}.$$

Note that $(u, v) = (17, 4)$ is the least positive integer solution of $Y^2 - 18Z^2 = 1$. By Lemma 3.3, Equation (9) has infinitely many positive integer solutions (Y, Z) satisfying the above condition, which leads to infinitely many $y, z \in \mathbb{Z}^+$. Then there are infinitely many $x = y + 2 \in \mathbb{Z}^+$. Hence, Equation (2) has an infinite number of positive integer solutions $(y + 2, y, z)$.

2) When $k = 5,\ a = 2,\ b = 1,\ m = 4,\ n = 3$, Equation (8) becomes

$$Y^2 - 78Z^2 = 601. \tag{10}$$

It has a positive integer solution $(Y', Z') = (43099, 4880)$ satisfying

$$Y' \equiv 43 \pmod{78}, \quad Z' \equiv 0 \pmod{2}.$$

Note that $(u, v) = (53, 6)$ is the least positive integer solution of $Y^2 - 78Z^2 = 1$. By Lemma 3.3, Equation (10) has infinitely many positive integer solutions (Y, Z) satisfying the above condition, which leads to infinitely many $y, z \in \mathbb{Z}^+$. Then there are infinitely many $x = 2y + 1 \in \mathbb{Z}^+$. Hence, Equation (2) has an infinite number of positive integer solutions $(2y + 1, y, z)$.

Proof of Theorem 2.4. 1) Multiplying Equation (3) by $8(k - 2)m$, we have

$$(2m(k-2)x - m(k-4))^2 - mn(2(k-2)y - (k-4))^2 = (k-4)^2m^2 - ((k-4)^2n - 8k + 16)m.$$

Setting $X = 2m(k - 2)x - m(k - 4),\ Y = 2(k - 2)y - (k - 4)$, we get the Pell equation

$$X^2 - mnY^2 = (k - 4)^2m^2 - ((k - 4)^2n - 8k + 16)m. \tag{11}$$

By Lemma 3.1, if mn is not a perfect square, the Pell equation $X^2 - mnY^2 = 1$ always has an infinite number of positive integer solutions. By Lemma 3.2, if Equation (11) has a positive integer solution, it has infinitely many positive integer solutions. Assume that Equation (11) has a positive integer solution (X', Y') satisfying the congruence condition

$$X' \equiv -(k - 4)m \pmod{2(k - 2)m}, \quad Y' \equiv -(k - 4) \pmod{2(k - 2)}.$$

By Lemma 3.3, Equation (11) has infinitely many positive integer solutions (X, Y) satisfying the above condition, which leads to infinitely many $x, y \in \mathbb{Z}^+$. Hence, under the above conditions Equation (3) has infinitely many positive integer solutions (x, y).

2) When $n = m - 1$, Equation (11) reduces to

$$X^2 - m(m-1)Y^2 = k^2 m. \tag{12}$$

It is easy to see that $(X, Y) = (2m - 1, 2)$ is the least positive integer solution of $X^2 - m(m - 1)Y^2 = 1$. Note that $(X', Y') = (km, k)$ is a positive integer solution of Equation (12) and satisfies the congruence condition

$$X' \equiv -(k-4)m \pmod{2(k-2)m}, \quad Y' \equiv -(k-4) \pmod{2(k-2)}.$$

By Lemma 3.3, Equation (12) has infinitely many positive integer solutions (X, Y) satisfying the above condition, which leads to infinitely many $x, y \in \mathbb{Z}^+$. Hence, when $k \geq 3, n = m - 1$, Equation (3) has infinitely many positive integer solutions (x, y). $\square$

Example 4.4. 1) When $k = 3,\ n = m - 1$, Equation (12) becomes

$$X^2 - m(m-1)Y^2 = 9m. \tag{13}$$

By Lemma 3.2, an infinity of positive integer solutions of Equation (13) are given by

$$X_s + Y_s\sqrt{m(m-1)} = \left(3m + 3\sqrt{m(m-1)}\right)\left(2m - 1 + 2\sqrt{m(m-1)}\right)^s,\ s \geq 0.$$

Then

$$\begin{cases} X_s = 2(2m-1)X_{s-1} - X_{s-2}, & X_0 = 3m,\ X_1 = 3m(4m-3), \\ Y_s = 2(2m-1)Y_{s-1} - Y_{s-2}, & Y_0 = 3,\ Y_1 = 12m - 3. \end{cases}$$

From $X = 2mx + m,\ Y = 2y + 1$, we have

$$\begin{cases} x_s = 2(2m-1)x_{s-1} - x_{s-2} + 2m - 2, & x_0 = 1,\ x_1 = 6m - 5, \\ y_s = 2(2m-1)y_{s-1} - y_{s-2} + 2m - 2, & y_0 = 1,\ y_1 = 6m - 2. \end{cases}$$

Thus, when $k = 3,\ n = m - 1$, Equation (3) has infinitely many positive integer solutions $(x_s, y_s),\ s \geq 0$.
When $k = 4, 6$, we can give the same results by the recurrence relation.

2) When $k = 5, n = m - 1$, Equation (12) reduces to

$$X^2 - m(m-1)Y^2 = 25m. \tag{14}$$

By Lemma 3.2, an infinity of positive integer solutions of Equation (14) are given by

$$X_s + Y_s\sqrt{m(m-1)} = \left(5m + 5\sqrt{m(m-1)}\right)\left(2m - 1 + 2\sqrt{m(m-1)}\right)^s,\ s \geq 0.$$

Then

$$\begin{cases} X_s = 2(2m-1)X_{s-1} - X_{s-2}, & X_0 = 5m,\ X_1 = 5m(4m-3), \\ Y_s = 2(2m-1)Y_{s-1} - Y_{s-2}, & Y_0 = 5,\ Y_1 = 20m - 5. \end{cases}$$

From $X = 6mx - m,\ Y = 6y - 1$, we have

$$\begin{cases} x_s = 2(2m-1)x_{s-1} - x_{s-2} - \dfrac{2m-2}{3}, & x_0 = 1,\ x_1 = \dfrac{10m-7}{3}, \\ y_s = 2(2m-1)y_{s-1} - y_{s-2} - \dfrac{2m-2}{3}, & y_0 = 1,\ y_1 = \dfrac{10m-2}{3}. \end{cases}$$

Although by Lemma 3.3, Equation (14) has infinitely many positive integer solutions (X', Y') satisfying the congruence condition

$$X' \equiv -m \pmod{6m}, \quad Y' \equiv -1 \pmod{6}.$$

However, for general positive integer $m \geq 2$, we cannot obtain infinitely many positive integer solutions by the similar way in 1).

When $m = 3, n = 2$, we have the recurrence relations

$$\begin{cases} x_s = 10x_{s-1} - x_{s-2} - \dfrac{4}{3}, & x_0 = 1, \ \ x_1 = \dfrac{23}{3}, \\ y_s = 10y_{s-1} - y_{s-2} - \dfrac{4}{3}, & y_0 = 1, \ \ y_1 = \dfrac{28}{3}. \end{cases}$$

By the property of congruence, it is easy to check that

$$x_{6i-1}, x_{6i} \in \mathbb{Z}^+, \ y_{2i} \in \mathbb{Z}^+.$$

So for $k = 5, \ m = 3, \ n = 2$, Equation (3) has an infinity of positive integer solutions (x_{6i}, y_{6i}), $i \geq 1$.

5 Conclusion

We continue the study about the linear combination of two polygonal numbers is a perfect square [9, 12]. By the theory of Pell equation and congruence, we show that the following three Diophantine equations

$$mP_k(x) - 1 = z^2, \quad mP_k(x) - nP_k(y) = z^2, \quad mP_k(x) - nP_k(y) = 1$$

have infinitely many positive integer solutions under some conditions. This note extends the ideas of the existing research and enriches the results of polygonal numbers and polynomial Diophantine equations.

Acknowledgements

This research was supported by the National Natural Science Foundation of China (Grant No. 11501052), the Natural Science Foundation of Hunan Province (Project No. 2022JJ40464), and Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science and Technology).

References

[1] Bencze, M. (2005). Proposed problem 7508. *Octogon Mathematical Magazine*, 13(1B), 678.

[2] Chen, J. P. (2012). The squares with the form $p^2 + \frac{4n(n+1)s^2k^2}{s^2-1}$. *Natural Science Journal of China West Normal University*, 33(2), 196–198, 217.

[3] Deza, E., & Deza, M. M. (2012). *Figurate Numbers*. Word Scientific, Singapore.

[4] Dickson, L. E. (2005). *History of the Theory of Numbers. Vol. II: Diophantine Analysis*. Dover Publications, New York.

[5] Eggan, L. C., Eggan, P. C., & Selfridge, J. L. (1982). Polygonal products of polygonal numbers and the Pell equation. *The Fibonacci Quarterly*, 20(1), 24–28.

[6] Guan, X. G. (2011). The squares with the form $1 + \frac{4n(n+1)s^2}{s^2-1}$. *Natural Science Journal of Ningxia Teachers University*, 32(3), 97–107.

[7] Guy, R. K. (2004). *Unsolved Problems in Number Theory*. Springer-Verlag, New York.

[8] Hu, M. J. (2013). The positive integer solutions of the Diophantine equation $1 + n\binom{x}{2} = y^2$. *Journal of Zhejiang International Studies University*, 4, 70–76.

[9] Jiang, M., & Li, Y. C. (2020). The linear combination of two polygonal numbers is a perfect square. *Notes on Number Theory and Discrete Mathematics*, 26(2), 105–115.

[10] Ke, S., & Sun, Q. (2011). *About Indeterminate Equation*. Harbin Institute of Technology Press, Harbin.

[11] Le, M. H. (2007). The squares with the form $1 + 9n(n+1)/2$. *Natural Science Journal of Hainan University*, 25(1), 13–14.

[12] Peng, J. Y. (2019). The linear combination of two triangular numbers is a perfect square. *Notes on Number Theory and Discrete Mathematics*, 25(3), 1–12.

[13] Ran, Y. X. (2008). The square number with the form $\frac{1+25n(n+1)}{2}$. *Journal of Tianshui Normal University*, 28(5), 18–19.

[14] Sun, Z. H. (2009). On the number of representations of n by $ax(x-1)/2 + by(y-1)/2$. *Journal of Number Theory*, 129(5), 971–989.

[15] Wu, H. M. (2011). The square numbers with the form $1 + p^2\frac{x(x+1)}{2}$. *Journal of Zhanjiang Normal College*, 32(3), 20–22.