Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 4, 728–735
DOI: 10.7546/nntdm.2025.31.4.728-735

On some classes of binary matrices

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Received: 21 July 2025 **Revised:** 21 September 2025
Accepted: 10 October 2025 **Online First:** 27 October 2025

Abstract: The work considers the set $\mathcal{L}_n^k$ of all $n \times n$ binary matrices having the same number of k units in each row and each column. The article specifically focuses on the matrices whose rows and columns are sorted lexicographically. We examine some particular cases and special properties of this matrices. Finally, we demonstrate the relationship between the Fibonacci numbers and the cardinality of two classes of $\mathcal{L}_n^k$-matrices with lexicographically sorted rows and columns.
Keywords: Binary matrix, Sorted lexicographically, Decreasing order, Increasing order, Fibonacci numbers.
2020 Mathematics Subject Classification: 15B34, 11B39.

1 Preliminaries and notations

A *binary* (or *Boolean*, or (0,1)-*matrix*) is a matrix whose all elements belong to the set $\mathbb{B} = \{0, 1\}$. Let m and n be positive integers. With $\mathbb{B}_{n\times m}$ we will denote the set of all $n \times m$ binary matrices and with $\mathbb{B}_n$ we will denote the set of all binary n-vectors.

If n and k are integers such that $n \geq 2$, $0 \leq k \leq n$, then we will call $\mathcal{L}_n^k$-*matrices* all $n \times n$ binary matrices in each row and each column of which there are exactly k unity elements. The set $\mathcal{L}_n^k$ is well known as the set of adjacency matrices of bipartite k-regular graphs of order n.

The set

$$\mathcal{L}_n^k \subset \mathbb{B}_{n\times n}$$

is the set of all $\mathcal{L}_n^k$-matrices.

Let $A = (a_{ij}) \in \mathbb{B}_{n\times m}$. With $r(A)$ we will denote the ordered n-tuple

$$r(A) = \langle x_1, x_2, \ldots, x_n \rangle,$$

where $0 \leq x_i \leq 2^m - 1$, $i = 1, 2, , \ldots n$ and x_i are nonnegative integers written in binary notation with the help of the i-th row of A, i.e.,

$$x_i = \sum_{j=1}^{m} a_{ij} 2^{m-j}, \quad i = 1, 2, , \ldots n.$$

Similarly, with $c(A)$ we will denote the ordered m-tuple

$$c(A) = \langle y_1, y_2, \ldots, y_m \rangle,$$

where $0 \leq y_j \leq 2^n - 1$, $j = 1, 2, \ldots m$ and y_j are nonnegative integers written in binary notation with the help of the j-th column of A, i.e.,

$$y_j = \sum_{i=1}^{n} a_{ij} 2^{n-i}, \quad j = 1, 2, , \ldots m.$$

Let $A \in \mathbb{B}_{n\times m}$, $r(A) = \langle x_1, x_2, \ldots, x_n \rangle$ and $c(A) = \langle y_1, y_2, \ldots, y_m \rangle$. Then by $\mathfrak{C}_{n\times m}$ and with $\mathfrak{D}_{n\times m}$ we will denote the sets:

$$\mathfrak{C}_{n\times m} = \{A \in \mathbb{B}_{n\times m} \mid x_1 \leq x_2 \leq \cdots \leq x_n \quad \text{and} \quad y_1 \leq y_2 \leq \cdots \leq y_m\} \subset \mathbb{B}_{n\times m},$$
$$\mathfrak{D}_{n\times m} = \{A \in \mathbb{B}_{n\times m} \mid x_1 \geq x_2 \geq \cdots \geq x_n \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_m\} \subset \mathbb{B}_{n\times m}.$$

In other words, $A \in \mathfrak{C}_{n\times m}$ if and only if the rows and columns of A are sorted in lexicographical nondecreasing order and $A \in \mathfrak{D}_{n\times m}$ if and only if the rows and columns of A are sorted in lexicographical nonincreasing order.

Example 1.1.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in \mathfrak{C}_{3\times 4}, \textit{ because } r(A) = \langle 7, 11, 12 \rangle \textit{ and } c(A) = \langle 3, 5, 6, 6 \rangle.$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathfrak{D}_{3\times 4}, \textit{ because } r(B) = \langle 8, 6, 1 \rangle \textit{ and } c(A) = \langle 4, 2, 2, 1 \rangle.$$

We define the sets

$$\Gamma_n^k = \mathfrak{C}_{n\times n} \cap \mathcal{L}_n^k,$$
$$\Delta_n^k = \mathfrak{D}_{n\times n} \cap \mathcal{L}_n^k$$

and the functions

$$\gamma(n,k) = \left|\Gamma_n^k\right|,$$
$$\delta(n,k) = \left|\Delta_n^k\right|.$$

In this paper we will demonstrate the relationship between the functions γ, δ and the Fibonacci numbers.

As is well known (see for example [1] or [3]), the sequence $\{f_n\}_{n=0}^{\infty}$ of *Fibonacci numbers* is defined by the recurrence relation

$$f_0 = f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n = 2, 3, \dots$$

2 Some properties of the sets Γ_n^k and Δ_n^k

In general,

$$\gamma(n,k) \neq \delta(n,k)$$

Indeed, according to [7, Sequence A229162] and [5], some values of the integer sequence, obtained using a computer program are

$$\{\gamma(n,k)\}_{n=1}^{10} = A229162 = \{0,0,1,1,3,25,272,4070,79221,1906501\}.$$

It is not difficult to see that all Γ_5^3-matrices are as follows:

$$\begin{pmatrix} 0&0&1&1&1\\ 0&0&1&1&1\\ 1&1&0&0&1\\ 1&1&0&1&0\\ 1&1&1&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&1&1\\ 0&1&0&1&1\\ 1&0&0&1&1\\ 1&1&1&0&0\\ 1&1&1&0&0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0&0&1&1&1\\ 0&1&0&1&1\\ 1&0&1&0&1\\ 1&1&0&1&0\\ 1&1&1&0&0 \end{pmatrix}$$

On the other hand, according to [7, Sequence A181344] all Δ_5^3-matrices are as follows:

$$\begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&1&0&0&1\\ 0&0&1&1&1\\ 0&0&1&1&1 \end{pmatrix}, \quad \begin{pmatrix} 1&1&1&0&0\\ 1&1&1&0&0\\ 1&0&0&1&1\\ 0&1&0&1&1\\ 0&0&1&1&1 \end{pmatrix}, \quad \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&0&1\\ 0&1&0&1&1\\ 0&0&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&0&1&1\\ 0&1&1&0&1\\ 0&0&1&1&1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1&1&1&0&0\\ 1&0&0&1&1\\ 1&0&0&1&1\\ 0&1&1&1&0\\ 0&1&1&0&1 \end{pmatrix}$$

So $\gamma(5,3) = 3$ and $\delta(5,3) = 5$, which proves that in general $\gamma(n,k) \neq \delta(n,k)$. .

Proposition 2.1. *Let* $A = (a_{ij}) \in \mathfrak{C}_{n\times m}$*. Then there exist integers* s, t*, such that* $1 \le s \le m$*,* $1 \le t \le n$ *and*

$$a_{11} = a_{12} = \cdots = a_{1s} = 0, \quad a_{1,s+1} = a_{1,s+2} = \cdots = a_{1m} = 1, \tag{1}$$

$$a_{11} = a_{21} = \cdots = a_{t1} = 0, \quad a_{t+1,1} = a_{t+2,1} = \cdots = a_{n1} = 1. \tag{2}$$

Proof. Let $r(A) = \langle x_1, x_2, \dots x_n\rangle$ and $c(A) = \langle y_1, y_2, \dots y_m\rangle$. We assume that there exist integers p and q, such that $1 \le p < q \le m$, $a_{1p} = 1$ and $a_{1q} = 0$. In this case, $y_p > y_q$, which contradicts the condition that columns of A are sorted in lexicographical non decreasing order. We have proven (1). Similarly, we prove (2) as well. □

Corollary 2.1. *Let* $A = (a_{ij}) \in \mathfrak{C}_{n\times m}$*. Then there exist integers* s, t*,* $0 \le s \le m$*,* $0 \le t \le n$*, such that*

$$x_1 = 2^s - 1$$

and

$$y_1 = 2^t - 1,$$

where s *equals the number of units in the first row and* t *equals the number of units in the first column of* A*.*

Numbers of the form $M_n = 2^n - 1$, for a positive integer n, are generally known as *Mersenne numbers*. [2]

Proposition 2.2 (Dual of Proposition 2.1)**.** *Let* $A = (a_{ij}) \in \mathfrak{D}_{n\times m}$*. Then there exist integers* s, t*, such that* $1 \le s \le m$*,* $1 \le t \le n$ *and*

$$a_{11} = a_{12} = \cdots = a_{1s} = 1, \quad a_{1,s+1} = a_{1,s+2} = \cdots = a_{1m} = 0, \tag{3}$$

$$a_{11} = a_{21} = \cdots = a_{t1} = 1, \quad a_{t+1,1} = a_{t+2,1} = \cdots = a_{n1} = 0. \tag{4}$$

Corollary 2.2 (Dual of Corollary 2.1)**.** *Let* $A = (a_{ij}) \in \mathfrak{D}_{n\times m}$*. Then there exist integers* s, t*,* $0 \le s \le m$*,* $0 \le t \le n$*, such that*

$$x_1 = (2^s - 1)2^{m-s} = 2^m - 2^{m-s}$$

and

$$y_1 = (2^t - 1)2^{n-t} = 2^n - 2^{n-t},$$

where s *equals the number of units in the first row and* t *equals the number of units in the first column of* A*.*

Theorem 2.1. *Let* n *and* k*, be integers, such that* $n \ge 1$*,* $0 \le k \le n$*. Then*

$$\gamma(n, n-k) = \delta(n, k).$$

Proof. Let $a \in \mathbb{B} = \{0, 1\}$. Then with $\overline{a}$ we will denote

$$\overline{a} = \begin{cases} 1, & \text{if} \quad a = 0; \\ 0, & \text{if} \quad a = 1. \end{cases}$$

Obviously $\overline{\overline{a}} = a$.

If $u = \langle u_1, u_2, \ldots, u_n \rangle \in \mathbb{B}_n$, then with $\overline{u}$ we will denote $\overline{u} = \langle \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n \rangle$. If $A = (a_{ij}) \in \mathbb{B}_{n\times m}$, then with $\overline{A}$ we will denote $\overline{A} = (\overline{a}_{ij})$.

Let $u = \langle u_1, u_2, \ldots, u_n \rangle$, $v = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{B}_n$. Then it is easy to see that $u < v$ if and only if $\overline{u} > \overline{v}$. Therefore, a matrix $A = (a_{ij}) \in \mathfrak{C}_{n\times m}$ if and only if the matrix $\overline{A} = (\overline{a}_{ij}) \in \mathfrak{D}_{n\times m}$.

Finally, we take into account the fact that the matrix $A = (a_{ij}) \in \mathcal{L}_n^{n-k}$ if and only if the matrix $\overline{A} = (\overline{a}_{ij}) \in \mathcal{L}_n^k$. $\square$

Theorem 2.2. *Let n be an integer, $n \geq 2$ and let $A = (a_{ij}) \in \Delta_n^2 \subset \mathcal{L}_n^2$. Then A has the form:*

$$A = \begin{pmatrix} B & O' \\ O'' & C \end{pmatrix}, \tag{5}$$

where B and C are square binary matrices,

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{6}$$

or B has the form:[*]

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7}$$

$C \in \Delta_s^2 \subset \mathcal{L}_s^2$ for some s such that $2 \leq s \leq n-2$, or C does not exist. All the elements of matrices O' and O'', which are outside of the submatrices B and C, are equal to 0.

Proof. From $A \in \mathcal{L}_n^2$ and from Proposition 2.2 it follows that $a_{11} = a_{12} = a_{21} = 1$, $a_{i1} = 0$ and $a_{1j} = 0$ for $3 \leq i, j \leq n$.

i) If $a_{22} = 1$, then B has the form (6). If the matrix C exists, then it is easy to see that $C \in \Delta_{n-2}^2 \subset \mathcal{L}_{n-2}^2$ matrix.

ii) Let $a_{22} = 0$, i.e., A is of the form

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & a_{23} & & \\ 0 & a_{32} & & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}.$$

Let $c(A) = \langle y_1, y_2, \ldots, y_m \rangle$. We suppose that $a_{23} = 0$. Since $A \in \mathcal{L}_n^2$, there exists an integer t such that $3 < t \leq n$ and $a_{2t} = 1$. In this case, it is easy to see that $y_3 < y_t$, which is impossible

[*]In other words, B is a tridiagonal matrix.

because $A \in \mathfrak{D}_{n\times n}$. Therefore, $a_{23} = 1$. Similarly, $a_{32} = 1$. Therefore, when $a_{22} = 0$, A is represented as

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \\ 0 & 1 & a_{33} & \\ \vdots & & & \end{pmatrix}.$$

We consider again the two possible cases for a_{33}: $a_{33} = 1$ or $a_{33} = 0$. When $a_{33} = 1$, the statement is proved. When $a_{33} = 0$, we do the same reasoning as above. This process cannot continue indefinitely, since n is a finite integer. Therefore, there exists an integer t, $2 \le t \le n$ such that $a_{t\,t} = a_{t-1\,t} = a_{t\,t-1} = 1$, i.e., in the upper left corner of A there is a submatrix of the form (7). And in this case, it is easy to see that if the matrix C exists, then $C \in \Delta_s^2 \subset \mathcal{L}_s^2$ for some s such that $2 \le s \le n-2$. $\square$

Corollary 2.3. *Let n be an integer, $n \ge 2$. Then*

$$\delta(n,2) = \gamma(n, n-2) = \text{number of all ordered } s\text{−tuples of integers}$$

$$\langle p_1, p_2, \ldots, p_s\rangle, \quad 1 \le s \le \left[\frac{n}{2}\right],$$

such that $2 \le p_i \le n$, $i = 1, 2, \ldots s$ and

$$p_1 + p_2 + \cdots + p_s = n.$$

A similar theorem can be formulated and proved for the set $\Gamma_n^{n-2} \subset \mathcal{L}_n^{n-2}$, $n \ge 2$.

Example 2.1. *The following matrices will play a crucial role in the inductive proof of Theorem 3.1, which we will present in Section 3.*

i) There is only one Δ_2^2 matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

ii) There is only one Δ_3^2 matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

iii) There are two Δ_4^2 matrices:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \textit{ and } \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

iv) There are three Δ_5^2 matrices:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

3 Δ_n^k and Γ_n^k matrices in relation to the Fibonacci numbers

Theorem 3.1. *Let n be a nonnegative integer. Then*

$$f_n = \delta(n+2, 2), \tag{8}$$

where f_n is the n-th element of the Fibonacci sequence.

Proof. For $n = 0,\ 1,\ 2$ and 3, see Example 2.1.

Let n be an integer, $n \geq 2$ and let $A = (a_{ij}) \in \Delta^2_{n+2}$. From Theorem 2.2 it follows that A is presented in the form (5) and the set Δ^2_{n+2} is a partition into two disjoint subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ such that the set $\mathcal{M}_1$ consists of matrices A whose upper left corner is a submatrix B of the type (6) and the set $\mathcal{M}_2$ consists of matrices A whose upper left corner is a submatrix B of the type (7).

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \varnothing, \qquad \mathcal{M}_1 \cup \mathcal{M}_2 = \Delta^2_{n+2}.$$

Therefore,

$$\left|\Delta^2_{n+2}\right| = |\mathcal{M}_1| + |\mathcal{M}_2|\,. \tag{9}$$

i) Let $A \in \mathcal{M}_1$. In A, we remove the first and second rows and the first and second columns. We obtain the matrix $C \in \mathcal{L}^2_n$. From Theorem 2.2 it follows that $C \in \Delta^2_n$.

Conversely, let $C = (c_{i\,j}) \in \Delta^2_n$, $n \geq 2$. From C we obtain the matrix $A = (a_{ij}) \in \mathcal{L}^2_{n+2}$ as follows: $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $a_{1j} = a_{2j} = 0$ for $3 \leq j \leq n+2$ and $a_{i1} = a_{i2} = 0$ for $3 \leq i \leq n+2$. For each $i, j \in \{3, 4, \ldots, n+2\}$ we assume $a_{ij} = c_{i-2\,j-2}$. It is easy to see that the so obtained matrix $A \in \Delta^2_{n+2}$.

Therefore,

$$|\mathcal{M}_1| = \delta(n, 2) \tag{10}$$

for any integer $n \geq 2$.

ii) Let $A \in \mathcal{M}_2$, i.e., $A \in \Delta^2_{n+2}$ is of the form

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{pmatrix}.$$

We change a_{22} from 0 to 1 and remove the first row and the first column of A. In this way we obtain a matrix, which can be easily seen to belong to the set Δ^2_{n+1}.

Conversely, let $D = (d_{i\,j}) \in \Delta^2_{n+1}$. According to Proposition 2.1, $d_{11} = d_{12} = d_{21} = 1$. We change d_{11} from 1 to 0. In D, we add a first row and a first column and get the matrix $A = (a_{ij}) \in \mathcal{L}^2_{n+2}$, such that $a_{11} = a_{12} = a_{21} = 1$, $a_{1j} = 0$ for $j = 3, 4, \ldots, n+2$, $\alpha_{i1} = 0$ for $i = 3, 4, \ldots, n+2$ and $a_{s+1\,t+1} = d_{st}$ for $s, t \in \{1, 2, \ldots, n+1\}$. It is easy to see that the resulting matrix $A \in \mathcal{M}_2 \subset \Delta^2_{n+2}$.

Therefore,

$$|\mathcal{M}_2| = \delta(n+1,2) \tag{11}$$

for every integer $n \geq 2$.

From (9), (10) and (11) it follows that when $n \geq 2$

$$\delta(n+2,2) = \left|\Delta^2_{n+2}\right| = |\mathcal{M}_1| + |\mathcal{M}_2| = \delta(n,2) + \delta(n+1,2).$$

This completes the proof. □

Corollary 3.1. [6] *From Theorem 3.1 and Theorem 2.1 it follows:*

$$f_n = \gamma(n+2,n), \tag{12}$$

where f_n is the n-th element of the Fibonacci sequence.

Equations (8) and (12) obviously are different. Thus, the result obtained in this paper differs from the result defined and proven in [6] concerning similar problem.

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