

On the equation $F(n^k - 1) = D$

I. Kátai¹ , B. M. M. Khanh²  and B. M. Phong³ 

¹ Department of Computer Algebra, University of Eötvös Loránd
1117 Budapest, Hungary
e-mail: katai@inf.elte.hu

² Department of Computer Algebra, University of Eötvös Loránd
1117 Budapest, Hungary
e-mail: mbuiminh@yahoo.com

³ Department of Computer Algebra, University of Eötvös Loránd
1117 Budapest, Hungary
e-mail: bui@inf.elte.hu

Received: 12 June 2025

Accepted: 1 October 2025

Revised: 28 September 2025

Online First: 12 October 2025

Abstract: We prove that if F is a completely multiplicative function and $k \in \{2, 3\}$ such that the equation $F(n^k - 1) = 1$ holds for every $n \in \mathbb{N}$, $n > 1$, then F is the identity function. A similar result is proved for the equation $F(n^4 - 1) = 1$ assuming a suitable conjecture concerning prime numbers. The equation $F(n^3 + 1) = 1$ is also studied.

Keywords: Completely multiplicative function, Identity function, Functional equation, Dirichlet character.

2020 Mathematics Subject Classification: 11A07, 11A25, 11N25, 11N64.

1 Introduction

Let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the set of primes, positive integers, integers and complex numbers, respectively. Let $\mathbb{N}_1 := \mathbb{N} \setminus \{1\}$. We denote by $\mathcal{M}$ and $\mathcal{M}^*$ the set of all complex-valued multiplicative and completely multiplicative functions, respectively. For each $a \in \mathbb{Z}$ and $p \in \mathcal{P}$



let $\left(\frac{a}{p}\right)$ be the Legendre symbol. We denote by $P(m)$ the largest prime divisor of $m \in \mathbb{N}$ and by p_k the k -th prime number. For fixed integers $a_1, a_2, \dots, a_k \in \mathbb{Z} \setminus \{0\}$ let

$$l(a_1, a_2, \dots, a_k) := \left\{ p \in \mathcal{P} \mid \left(\frac{a_1}{p}\right) = \left(\frac{a_2}{p}\right) = \dots = \left(\frac{a_k}{p}\right) = -1 \right\}$$

and

$$\mathcal{H}(a_1, a_2, \dots, a_k) := \mathcal{P} \setminus l(a_1, a_2, \dots, a_k).$$

Let $\mathbb{E}(n) = 1$ for every $n \in \mathbb{N}$.

For $\varkappa(x) \in \mathbb{Z}[x]$ such that $\varkappa(n) \in \mathbb{N}$ for all sufficiently large $n \in \mathbb{N}$, we study the equation:

$$F(\varkappa(n)) = D \text{ for every } n \in \mathbb{N}, \text{ where } D \in \mathbb{C} \setminus \{0\}, F \in \mathcal{M}^*. \quad (1)$$

In particular, we examine the cases where:

$$\varkappa(n) \in \{n^2 - 1, n^2 + 1, n^3 - 1, n^4 - 1, n^3 + 1\}.$$

In this paper, we present four main theorems addressing these equations. Under specific conjectures regarding prime numbers, we derive further results characterizing the identity function among completely multiplicative functions.

For the case $\varkappa(n) = n^2 + 1$, it was proved in Theorem 1 (a) [2] that if $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy

$$F(n^2 + 1) = D \text{ for every } n \in \mathbb{N},$$

then $D = 1$, $F(2) = 1$ and $F(p) = 1$ for every $p \in \mathcal{P}, p \equiv 1 \pmod{4}$.

In [1], for $\varkappa(n) = n^3 + 1$ and $\varkappa(n) = n^4 - 1$, the following conjectures were proposed:

Conjecture 1. [1, Conjecture 1.3.] *If the function $F \in \mathcal{M}^*$ satisfies*

$$F(n^3 + 1) = 1 \text{ for every } n \in \mathbb{N},$$

then $F = \mathbb{E}$.

Conjecture 2. [1, Conjecture 1.2.] *If the function $F \in \mathcal{M}^*$ satisfies*

$$F(n^4 - 1) = 1 \text{ for every } n \in \mathbb{N}_1,$$

then $F = \mathbb{E}$.

In this note, we prove the following results.

Theorem 1. *Assume that the function $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy*

$$F(n^2 - 1) = D \text{ for every } n \in \mathbb{N}_1,$$

then $D = 1$ and $F = \mathbb{E}$.

Theorem 2. *Assume that the function $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy*

$$F(n^3 - 1) = D \text{ for every } n \in \mathbb{N}_1,$$

then $D = 1$ and $F = \mathbb{E}$.

For $\varkappa(n) = n^3 + 1$ and $\varkappa(n) = n^4 - 1$, we prove Conjecture 1 and Conjecture 2 under a suitable assumption concerning the sets $l(-2, -3)$ and $l(-1, -3, -7)$ of prime numbers.

In [2], we also proposed:

Conjecture 3. *Let $p \in \mathcal{P}$ and $p > p_{93} = 487$. Then there exists such an $m \in \mathbb{N}$ for which $p \mid m^2 - 1$ and*

$$P\left(\frac{m^4 - 1}{p}\right) < p.$$

For the proof of our result concerning $\varkappa(n) = n^4 - 1$, we require the following weaker conjecture:

Conjecture 4. *Let $p \in \mathcal{P}$, $p > p_{39} = 167$ and $p \in l(-1, -3, -7)$. Then there exists such a $t \in \mathbb{N}$ for which $p \mid t^2 - 1$ and*

$$P\left(\frac{t^4 - 1}{p}\right) < p.$$

Using *Maple*, we have verified that Conjecture 3 holds for every p_k , $93 < k \leq 10^6$ and in the interval of integers $k \in \{1, \dots, 93\}$, we cannot check that Conjecture 3 holds for $p = p_k$, where

$$k \in \mathcal{F} := \{1, 2, 3, 4, 5, 6, 11, 13, 14, 15, 16, 18, 24, 25, 28, 29, 30, 33, 39, 56, 67, 74, 93\}.$$

Let

$$\overline{\mathcal{F}} := \mathcal{F} \cap l(-1, -3, -7) = \{p_{15} = 47, p_{39} = 167\}.$$

Then Conjecture 4 were verified for all $p = p_k$, $39 < k \leq 10^6$ and in the interval of integers $k \in \{1, \dots, 39\}$, Conjecture 4 holds for every k except for $p = p_k \in \overline{\mathcal{F}}$.

Theorem 3. *Assume that the function $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy*

$$F(n^4 - 1) = D \text{ for every } n \in \mathbb{N}_1,$$

then $D = 1$. Moreover, if Conjecture 4 holds, then $F = \mathbb{E}$.

For the proof of our result concerning $\varkappa(n) = n^3 + 1$, we require the following conjecture:

Conjecture 5. *For every prime $p > p_{15} = 47$ and $p \in l(-2, -3)$, there exists a positive integer l such that $p \mid l + 1$ and*

$$P\left(\frac{l^3 + 1}{p}\right) < p.$$

It is clear to show that

$$l(-2, -3) = \{p \in \mathcal{P} \mid p \equiv 23 \pmod{24}\}.$$

Using *Maple*, we verified that Conjecture 5 holds for every $p_k \in l(-2, -3)$, where $15 < k \leq 10^6$ and in the range $k \in \{1, \dots, 15\}$, we cannot check that Conjecture 5 holds for

$$p = p_k \in \{p_9 = 23, p_{15} = 47\}.$$

Theorem 4. *Assume that the function $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy*

$$F(n^3 + 1) = D \text{ for every } n \in \mathbb{N},$$

then $D = 1$. Moreover, if Conjecture 5 holds, then $F = \mathbb{E}$.

2 Proof of Theorem 1

First we note that if $\varkappa(n) = n^2 + An + B$ ($A, B \in \mathbb{Z}$), then

$$\varkappa(n)\varkappa(n+1) = \varkappa(n + \varkappa(n)). \quad (2)$$

Assume that $F \in \mathcal{M}^*$, $D \in \mathbb{C} \setminus \{0\}$ and $\varkappa(n) = n^2 - 1$ satisfy

$$F(n^2 - 1) = F(\varkappa(n)) = D \text{ for every } n \in \mathbb{N}_1.$$

Together with (2), this implies that

$$D^2 = F(\varkappa(n))F(\varkappa(n+1)) = F(\varkappa(n + \varkappa(n))) = D,$$

therefore $D = 1$ and

$$F(n^2 - 1) = F(\varkappa(n)) = 1 \text{ for every } n \in \mathbb{N}_1.$$

For each square-free Q , it is well-known that the Pell equation $x^2 - Qy^2 = 1$ has infinitely many solutions. Let us fix one such solution and write

$$u^2 - Qv^2 = 1.$$

Then it follows that

$$(u + \sqrt{Q}v)(1 + \sqrt{Q}) = (u + Qv) + \sqrt{Q}(u + v)$$

and

$$(u + Qv)^2 - Q(u + v)^2 = 1 - Q.$$

Hence,

$$(u + Qv)^2 - 1 = Q((u + v)^2 - 1),$$

which shows that the equation

$$k^2 - 1 = Q(\ell^2 - 1)$$

has infinitely many solutions in $k, \ell \in \mathbb{N}$. Then

$$F(Q) = F(Q)F(\ell^2 - 1) = F(k^2 - 1) = 1.$$

Since Q was arbitrary square-free and $F \in \mathcal{M}^*$, it follows that $F(n) = 1$ for all $n \in \mathbb{N}$.

This completes the proof of Theorem 1. □

3 Proof of Theorem 2

Assume that $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy

$$F(n^3 - 1) = D \text{ for every } n \in \mathbb{N}_1. \quad (3)$$

Substitute $n = m + 1$ into Equation (3). Then

$$D = F((m + 1)^3 - 1) = F(m)F((m + 1)^2 + (m + 1) + 1),$$

which implies $F(m) \neq 0$ for all $m \in \mathbb{N}$, since $D \neq 0$.

Let $\varkappa(n) = n^2 + n + 1$. Using the identity

$$F(n^3 - 1) = F(n - 1)F(\varkappa(n)) = D,$$

we solve for $F(\varkappa(n))$:

$$F(\varkappa(n)) = \frac{D}{F(n - 1)} \quad \text{for all } n \in \mathbb{N}_1. \quad (4)$$

Using (2) and (4), we compute that

$$\frac{D}{F(n - 1)} \frac{D}{F(n)} = \frac{D}{F(n^2 + 2n)} = \frac{D}{F(n)F(n + 2)},$$

which leads to

$$F(n + 2) = CF(n - 1) \quad \text{for all } n \in \mathbb{N}_1, \quad (5)$$

where $C = \frac{1}{D}$.

Substitute $n = 3k + 1$ into (5):

$$F(3k + 3) = CF(3k), \quad \text{so } F(k + 1) = CF(k).$$

This recurrence yields:

$$F(2) = C, \quad F(4) = CF(3) = C^2F(2) = C^3.$$

Therefore,

$$C^2 = F(2)^2 = F(4) = C^3,$$

which implies $C = 1$ and $D = 1$.

Finally, from the recurrence $F(k + 1) = CF(k) = F(k)$, we obtain that $F(k) = 1$ for all $k \in \mathbb{N}$. Thus we proved $F = \mathbb{E}$ and the proof of Theorem 2 is finished. $\square$

4 Proof of Theorem 3

Assume that $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy

$$F(n^4 - 1) = D \quad \text{for all } n \in \mathbb{N}_1.$$

According to Theorem 1.2(a) in [1], this implies that $D = 1$, so we have

$$F(n^4 - 1) = 1 \quad \text{for all } n \in \mathbb{N}_1. \quad (6)$$

Substituting $n = m + 1$ into (6), we have

$$1 = F((m + 1)^4 - 1) = F(m)F(m + 2)F((m + 1)^2 + 1),$$

which implies

$$F(m) \neq 0 \quad \text{for all } m \in \mathbb{N}. \quad (7)$$

Let $\varkappa(n) = n^2 + 1$. From Equation (6), we also obtain:

$$F(\varkappa(n)) = \frac{1}{F(n - 1)F(n + 1)} \quad \text{for all } n \in \mathbb{N}_1. \quad (8)$$

Using this and Identity (2), we compute:

$$\frac{1}{F(n - 1)F(n + 1)} \frac{1}{F(n)F(n + 2)} = \frac{1}{F(n)F(n + 1)F(n^2 + n + 2)},$$

which simplifies to

$$F(n^2 + n + 2) = F(n - 1)F(n + 2) \quad \text{for all } n \in \mathbb{N}_1. \quad (9)$$

Now let $\varkappa(n) = n^2 + n + 2$. Using Identity (2) and Equation (9), we deduce:

$$F(n - 1)F(n + 2)F(n)F(n + 3) = F(n + 1)^2 F(n^2 + 2n + 4).$$

Thus,

$$F(n^2 + 2n + 4) = \frac{F(n - 1)F(n)F(n + 2)F(n + 3)}{F(n + 1)^2}.$$

Replacing n with $n - 1$ yields:

$$F(n^2 + 3) = \frac{F(n - 2)F(n - 1)F(n + 1)F(n + 2)}{F(n)^2} \quad \text{for all } n > 2. \quad (10)$$

Let $T_n = F(n^4 - 1) - 1$. Then (6) shows that $T_n = 0$ for every $n \in \mathbb{N}$. Consider the following system of equations

$$\begin{cases} T(2) &= F(3)F(5) - 1 = 0 \\ T(3) &= F(2)^4 F(5) - 1 = 0 \\ T(4) &= F(3)F(5)F(17) - 1 = 0 \\ T(5) &= F(2)^4 F(3)F(13) - 1 = 0 \\ T(7) &= F(2)^5 F(3)F(5)^2 - 1 = 0 \\ T(8) &= F(3)^2 F(5)F(7)F(13) - 1 = 0 \\ T(21) &= F(2)^4 F(5)F(11)F(13)F(17) - 1 = 0. \end{cases}$$

Solving this system, we obtain

$$F(2) = F(3) = F(5) = F(7) = F(11) = F(13) = F(17) = 1.$$

Proceeding by induction, assume that $F(n) = 1$ for all $n < P$, and let us prove $F(P) = 1$.

If $P \notin \mathcal{P}$, then trivially $F(P) = 1$.

Now consider $P \in \mathcal{P}$. We distinguish two cases: $P \in \mathcal{H}(-1, -3, -7)$ or $P \in l(-1, -3, -7)$.

Case I. $P \in \mathcal{H}(-1, -3, -7)$. Then at least one of the following holds:

$$(Ia) \left(\frac{-1}{P} \right) = 1,$$

$$(Ib) \left(\frac{-3}{P} \right) = 1,$$

$$(Ic) \left(\frac{-7}{P} \right) = 1.$$

(Ia) First consider the case $\left(\frac{-1}{P} \right) = 1$. Then there is a positive integer $n < P$ such that $n^2 + 1 \equiv 0 \pmod{P}$. Then $n^2 + 1 = PQ$ with $Q \in \mathbb{N}, Q < P$. Since $P > 17$ and $n^2 + 1 \equiv 0 \pmod{P}$, we have $n \leq P - 2$. Consequently $F(Q) = 1, F(n) = 1$ and $F(n + 1) = 1$. Thus we infer from (8) that

$$F(P) = F(P)F(Q) = F(PQ) = F(n^2 + 1) = \frac{1}{F(n - 1)F(n + 1)} = 1.$$

Hence, the case (Ia) is proved.

(Ib) Now we consider the case $\left(\frac{-3}{P}\right) = 1$. Thus there is a positive integer $n < P$ such that $n^2 + 3 \equiv 0 \pmod{P}$. Then $n^2 + 3 = PQ$ with $Q \in \mathbb{N}, Q < P$. Since $P > 17$ and $n^2 + 3 \equiv 0 \pmod{P}$, we have $n < P - 2$, consequently

$$F(Q) = 1, F(n-2) = 1, F(n-1) = 1, F(n) = 1, F(n+1) = 1, F(n+2) = 1.$$

Thus we infer from (10) that

$$F(P) = F(P)F(Q) = F(PQ) = F(n^2+3) = \frac{F(n-2)F(n-1)F(n+1)F(n+2)}{F(n)^2} = 1.$$

Thus, the proof of the case (Ib) is complete.

(Ic) Finally we consider the case $\left(\frac{-7}{P}\right) = 1$. Since

$$n^2 + n + 2 \equiv 0 \pmod{P} \iff (2n+1)^2 \equiv -7 \pmod{P},$$

there is a positive integer $n < P$ such that $n^2 + n + 2 \equiv 0 \pmod{P}$. Then $n^2 + n + 2 = PQ$ with $Q \in \mathbb{N}, Q < P$. Since $P > 17$ and $n^2 + n + 2 \equiv 0 \pmod{P}$, we have $n < P - 2$, consequently

$$F(Q) = 1, F(n-1) = 1, F(n+2) = 1.$$

Thus we infer from (9) that

$$F(P) = F(P)F(Q) = F(PQ) = F(n^2 + n + 2) = F(n-1)F(n+2) = 1.$$

This completes the proof of the case (Ic).

Therefore the case $P \in \mathcal{H}$ is proved.

Case II. $P \in l(-1, -3, -7)$. Assume that $P \in l(-1, -3, -7)$ and $F(n) = 1$ for every $n \in \mathbb{N}, n < P$. We can apply Conjecture 5 provided that we verify $F(P) = 1$ for $P \in \{p_{15} = 47, p_{39} = 167\}$.

If $P = 47$, then we infer from

$$F(27) = F(2)^4 F(5)F(7)F(13)F(73) - 1 = 0$$

and

$$T(46) = F(3)^2 F(5)F(29)F(47)F(73) - 1 = 0.$$

These imply that $F(73) = 1$ and $F(47) = 1$.

If $P = 167$, then from our assumption, the following relations

$$T(81) = F(2)^6 F(5)F(17)F(41)F(193) - 1 = 0,$$

$$T(192) = F(5)F(73)F(101)F(191)F(193) - 1 = 0$$

and the relation

$$T(2673) = F(2)^6 F(5)F(7)F(13)F(17)F(53)F(61)F(167)F(191)) - 1 = 0$$

imply that

$$F(193) = 1, F(191) = 1 \text{ and } F(167) = 1.$$

Thus we proved that $P > p_{39} = 167$.

Now we deduce from Conjecture 5 that there is $t \in \mathbb{N}$ such that

$$P \mid t^2 - 1 \text{ and } P\left(\frac{t^4 - 1}{P}\right) < P.$$

Then

$$F\left(\frac{t^4 - 1}{P}\right) = 1$$

and

$$F(P) = F(P)F\left(\frac{t^4 - 1}{P}\right) = F(t^4 - 1) = 1$$

The last relation proves Theorem 3. □

5 Proof of Theorem 4

Assume that $F \in \mathcal{M}^*$ and $D \in \mathbb{C} \setminus \{0\}$ satisfy

$$F(n^3 + 1) = D \text{ for every } n \in \mathbb{N}.$$

According to Theorem 1.2(b) in [1], we have $D = 1$. Hence

$$F(n^3 + 1) = 1 \text{ for every } n \in \mathbb{N}. \quad (11)$$

Similarly to earlier arguments, we conclude $F(n) \neq 0$ for every $n \in \mathbb{N}$. Define $\varkappa(n) = n^2 - n + 1$. Then from Identity (11) we obtain

$$F(\varkappa(n)) = \frac{1}{F(n+1)} \text{ for every } n \in \mathbb{N}. \quad (12)$$

Using Identity (2), we derive from (12) that

$$\frac{1}{F(n+1)} \frac{1}{F(n+2)} = \frac{1}{F(n^2 + 2)},$$

which implies

$$F(n^2 + 2) = F(n+1)F(n+2) \text{ for every } n \in \mathbb{N}. \quad (13)$$

We now aim to prove that $F(n) = 1$ for every $n \leq 13$. From (11), define

$$W(n) = F(n+1)F(n^2 - n + 1) - 1 = 0.$$

We can compute $W(n)$ for $n \in \{1, 2, 3, 4, 8, 10, 12, 31\}$:

$$\left\{ \begin{array}{lcl} W(1) & = & F(2) - 1 = 0 \\ W(2) & = & F(3)^2 - 1 = 0 \\ W(3) & = & F(2)^2 F(7) - 1 = 0 \\ W(4) & = & F(5) F(13) - 1 = 0 \\ W(8) & = & F(3)^3 F(19) - 1 = 0 \\ W(10) & = & F(7) F(11) F(13) - 1 = 0 \\ W(12) & = & F(7) F(13) F(19) - 1 = 0 \\ W(31) & = & F(2)^5 F(7)^2 F(19) - 1 = 0. \end{array} \right.$$

From these, it follows that $F(2) = 1, F(7) = 1, F(19) = 1, F(13) = 1, F(5) = 1, F(3) = 1, F(11) = 1$ follow from $W(1), W(3), W(31), W(12), W(4), W(2)$ and $W(8), W(10)$.

Therefore, we have proved that $F(n) = 1$ for every $n \leq 13$.

Now we proceed by induction and assume $F(n) = 1$ for all $n < P$ with $P > 13$. We prove $F(P) = 1$.

If $P \notin \mathcal{P}$, then $F(P) = 1$ trivially. Assume $P \in \mathcal{P}$.

(I) First we consider the case $P \notin l(-2, -3)$. Then $\left(\frac{-2}{P}\right) = 1$ or $\left(\frac{-3}{P}\right) = 1$.

If $\left(\frac{-2}{P}\right) = 1$, then there exist $n, Q \in \mathbb{N}, n < P$ such that

$$n^2 + 2 = PQ.$$

Since $n < P$ and $n^2 + 2 = PQ$, we have $Q < P, n < P - 2$, and so $F(Q) = 1$ and $F(n + 1) = 1$. We infer from (13) that

$$F(P) = F(P)F(Q) = F(n^2 + 2) = F(n + 1)F(n + 2) = 1,$$

which proves $F(P) = 1$ for the case $\left(\frac{-2}{P}\right) = 1$.

If $\left(\frac{-3}{P}\right) = 1$, then there exist $n, Q \in \mathbb{N}, n < P$ such that

$$n^2 - n + 1 = PQ,$$

because

$$n^2 - n + 1 \equiv 0 \pmod{P} \Leftrightarrow (2n - 1)^2 \equiv -3 \pmod{P}.$$

Since $n < P$ and $n^2 - n + 1 = PQ$, we have $Q < P, n < P - 2$, and so $F(Q) = 1$, $F(n + 1) = 1$ and $F(n + 2) = 1$. We infer from (12) that

$$F(P) = F(P)F(Q) = F(n^2 - n + 1) = \frac{1}{F(n + 1)} = 1,$$

which proves $F(P) = 1$ for the case $\left(\frac{-3}{P}\right) = 1$.

(II) Now we consider the case $P \in l(-2, -3)$.

Since $P \in \mathcal{P}$ and $P > 13$, we infer from Conjecture 5 that there exists a positive integer l such that $p \mid l + 1$ and $P\left(\frac{l^3 + 1}{p}\right) < p$.

To apply Conjecture 5, we first verify the special cases:

$$F(P) = 1 \text{ if } P \in \{p_9 = 23, p_{15} = 47\}.$$

(a1) If $P = p_9 = 23$, then from $W(6)$ and $W(68)$, we compute:

$$F(6) = F(7)F(31) - 1 = 0 \text{ and } W(68) = F(23)F(3)^2F(7)^2F(31) - 1 = 0.$$

Since $F(n) = 1$ for all $n < P = 23$, the last relations imply that $F(31) = 1$ and $F(23) = 1$.

(a2) If $P = p_{15} = 47$, then $F(n) = 1$ for every $n < 47$. Thus we infer from E_{27} and E_{46} that

$$W(14) = F(5)F(3)^2F(61) - 1 = F(61) - 1 = 0$$

and

$$W(563) = F(2)^2F(3)^2F(7)F(13)F(19)F(47)F(61) - 1 = F(47)F(61) - 0 = 0.$$

Consequently

$$F(61) = 1 \text{ and } F(P) = F(47) = 1.$$

From now on, assume that $P > 47$. Then by Conjecture 5 there is l such that $P \mid l + 1$ and $P\left(\frac{l^3+1}{P}\right) < P$. Thus, our assumptions imply that

$$F\left(\frac{l^3+1}{P}\right) = 1,$$

consequently

$$F(P) = F(P)F\left(\frac{l^3+1}{P}\right) = F(l^3+1) = 1.$$

This completes the proof of Theorem 4. □

References

- [1] Kátai, I., Khanh, B. M. M., & Phong, B. M. (2023). On the equation $F(n^3) = F(n^3 - 1) + D$ and some conjectures. *Publicationes Mathematicae Debrecen*, 103(1–2), 257–267.
- [2] Kátai, I., Khanh, B. M. M., & Phong, B. M. (2025). On the equation $F(n^2+1) = bF(n^2-1) + c$. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Computatorica*, 58, 177–189.