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On split Narayana and Narayana–Lucas hybrid quaternions

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Abstract: In this paper, we introduce the novel concepts of split Narayana quaternions and split Narayana–Lucas quaternions within the innovative framework of hybrid numbers. We explore their deep connections with Narayana and Narayana–Lucas quaternions, uncovering new perspectives in this mathematical domain. Furthermore, we establish several fundamental properties, including recurrence relations, Binet formulas, generating functions, exponential generating functions, and other significant identities associated with these newly defined quaternions. Finally, to better illustrate these theoretical findings, we also provide a numerical simulation of split Narayana quaternions and split Narayana–Lucas hybrid quaternions.
Keywords: Narayana quaternions, Narayana–Lucas, Hybrid number, Split quaternions.
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1 Introduction

Number theory is one of the most interesting and prominent areas of research in the field of mathematics. One can observe numerous applications of this fascinating area of research across various disciplines such as cryptography, computer science, quantum physics, and many others [16]. In recent years, many researchers and scientists have shown their keen interest in the study of number sequences [21, 23]. Number theory and quaternions share an interesting connection through the exploration of certain number-theoretic properties within the context of an algebraic framework. Specifically, number theory provides a foundation for understanding the arithmetic properties of quaternions formed due to the sequences. Hamilton [9] was the first to introduce the concept of quaternions as:

$$\begin{aligned} H = \{p + iq + jr + ks \,|p, q, r, s \in \mathbb{R}, i^2 = j^2 = k^2 = -1, \\ ij = -ji = k, jk = -kj = i, ki = -ik = j\}. \end{aligned}$$

where i, j, k are the quaternionic units and are non-commutative under the multiplication rule. In applied mathematics, quaternions are particularly useful in the fields of computer science, physics, differential geometry, quantum physics, engineering, algebra, and the computation of rotating motions in three dimensions. Numerous research works have explored the connections between the algebra of sequences and quaternions or octonions, and for more details, one can refer to [2, 3, 6–12, 15, 18–20, 26, 27].

Quaternions and split quaternions are both extensions of the concept of complex number sequences, but they differ in their algebraic properties and structure. The concept of a split quaternion or co-quaternion is based upon the four-dimensional non-commutative associative algebra, which was first put forth by James Cockle [4]. Split quaternions contain nilpotent elements, non-trivial idempotent, and zero divisors, unlike the Hamilton quaternions algebra.

The split quaternions $q^{'}$ may be represented as:

$$\begin{aligned} q^{'} = \{p + iq + jr + ks \,|p, q, r, s \in \mathbb{R}, i^2 = -1, j^2 = k^2 = 1, \\ ij = -ji = k, jk = -kj = -i, ki = -ik = j\}. \end{aligned}$$

where i, j, k are the split quaternionic units and are non-commutative under multiplication rule. In this context, the vector and scalar components of split quaternions are denoted by $\vec{V_{q^{'}}} = iq + jr + ks$ and $S_{q^{'}} = p$, respectively. As a result, $q^{'} = S_{q^{'}} + \vec{V_{q^{'}}}$ can be used to represent a split quaternion. Define $q^{'}_n = p_n + iq_n + jr_n + ks_n, (n = 0, 1)$.

Therefore the addition, subtraction and Multiplication of split quaternions are, respectively, defined as:

$$\begin{aligned} q^{'}_0 \mp q^{'}_1 &= (p_0 + iq_0 + jr_0 + ks_0) \mp (p_1 + iq_1 + jr_1 + ks_1) \\ &= (p_0 \mp p_1) + i(q_0 \mp q_1) + j(r_0 \mp r_1) + k(s_0 \mp s_1) \end{aligned}$$

$$\begin{aligned} q^{'}_0 . q^{'}_1 &= (p_0 + iq_0 + jr_0 + ks_0).(p_1 + iq_1 + jr_1 + ks_1) \\ &= (p_0p_1 - q_0q_1 + r_0r_1 + s_0s_1) + i(p_0q_1 + q_0p_1 - r_0s_1 + s_0r_1) \\ &\quad + j(p_0r_1 - q_0s_1 + r_0p_1 + s_0q_1) + k(p_0s_1 + q_0r_1 - r_0q_1 + s_0p_1) \\ &= S_{q^{'}_0}S_{q^{'}_1} + g(\vec{V_{q^{'}_0}}, \vec{V_{q^{'}_1}}) + S_{q^{'}_0}\vec{V_{q^{'}_1}} + S_{q^{'}_1}\vec{V_{q^{'}_0}} + \vec{V_{q^{'}_0}} \wedge \vec{V_{q^{'}_1}} \end{aligned}$$

where $g(\vec{V}_{q_0'}, \vec{V}_{q_1'}) = -q_0q_1 + r_0r_1 + s_0s_1$ and

$$\vec{V}_{q_0'} \wedge \vec{V}_{q_1'} = \begin{vmatrix} -i & j & k \\ q_0 & r_0 & s_0 \\ q_1 & r_1 & s_1 \end{vmatrix}$$

A split quaternions conjugate $\overline{q'}$ is given by

$$\overline{q'} = p - iq - jr - ks.$$

The formula for a norm of split quaternions $\mathcal{N}_{q'}$ is defined as:

$$\mathcal{N}_{q'} = q' \cdot \overline{q'} = p^2 + q^2 - r^2 - s^2.$$

A particular class of quaternionic numbers connected to the Narayana sequence and Narayana–Lucas sequences are referred to as split Narayana and Narayana–Lucas quaternions, respectively. These quaternions are distinguished by having split components, which means that one or more of the quaternion components have fictional values. They are useful in quaternion algebra and have uses in many branches of mathematics and physics. Tokeşer *et al.* [28] introduced the concept of split Pell and Pell–Lucas quaternions along with several identities related to these quaternions. Additionally, a number of researchers have conducted investigations on other split quaternion sequence types; these might be consulted [1, 5, 22].

Recently, Özdemir [17] introduced the concept of hybrid numbers, which are a combination of real, complex, dual and hyperbolic numbers. The set of hybrid numbers H is defined as:

$$H = \{z = a + b\iota + c\epsilon + dh; a, b, c, d \in \mathbb{R}\} \tag{1}$$

where ι, ϵ, h are operators such that $\iota^2 = -1, \epsilon^2 = 0, h^2 = 1, \iota h = -h\iota = \epsilon + \iota$. The conjugate of hybrid numbers z is defined as: $\overline{z} = \overline{a + b\iota + c\epsilon + dh} = a - b\iota - c\epsilon - dh$. The character of hybrid number z is defined as the real number $C(z) = z\overline{z} = \overline{z}z = a^2 + b^2 - 2bc - d^2$ and the norm of hybrid number z is defined as $\sqrt{|C(z)|}$ and denoted by $\|z\|$ [17]. In the development of science and technology, integer sequences have played a significant role. Consequently, it is extensively used, particularly in mathematics and various other fields of science. Many areas of mathematics can benefit from the use of hybrid numbers. In addition, these numbers have been widely utilized in science, design, and hypothetical physical science. These sequences have many applications in different fields, like linear algebra, kinematics, number theory, and geometry. Hybrid numbers with various sequences have earned a lot of interest in recent years [14, 21, 24, 25].

A significant aspect of this study is that it extends the rapidly growing body of research on quaternionic sequences. Earlier works have examined Fibonacci quaternions [2, 10, 12, 18, 26], Pell and Pell–Lucas quaternions [3, 28], and more recently split variants such as split Fibonacci quaternions and split k-Fibonacci and k-Lucas quaternions [1, 22]. Similarly, hybrid numbers have been studied in the context of Jacobsthal, Mersenne, and other classical sequences [5, 19, 24]. To the best of our knowledge, however, no prior work has addressed split Narayana and Narayana–Lucas hybrid quaternions. The present study fills this gap by introducing these novel constructs, integrating the Narayana and Narayana–Lucas sequences into the quaternionic hybrid framework. By deriving recurrence relations, Binet formulas, generating functions, and exponential generating functions, we provide explicit identities analogous to those known for Fibonacci and Pell–Lucas quaternions, thereby enriching the catalogue of algebraic structures in this domain. In doing so, this work not only bridges a gap in the literature but also opens avenues for future research where Narayana sequences, hybrid numbers, and quaternionic algebra intersect.

2 Preliminaries

In order to address the problem in hand related to the construction of split Narayana and Narayana–Lucas hybrid quaternions, here in this preliminary section we are providing certain definitions which will serve as a foundational framework for the subsequent analysis and interpretation of the results.

Definition 2.1. [23] The Narayana sequence N_n and Narayana–Lucas sequence U_n are defined, as follows:

$$N_{n+3} = N_{n+2} + N_n, N_0 = 0, N_1 = 1, N_2 = 1, \tag{2}$$

$$U_{n+3} = U_{n+2} + U_n, U_0 = 3, U_1 = 1, U_2 = 1, \tag{3}$$

for $n \geq 3$, respectively.

Definition 2.2. [13] The Narayana hybrid sequence NH_n and Narayana–Lucas hybrid sequence UH_n are defined as follows:

$$NH_n = N_n + \iota N_{n+1} + \epsilon N_{n+2} + hN_{n+3}, \tag{4}$$

$$UH_n = U_n + \iota U_{n+1} + \epsilon U_{n+2} + hU_{n+3}, \tag{5}$$

where ι, ϵ, h are operators such that $\iota^2 = -1, \epsilon^2 = 0, h^2 = 1, \iota h = -h\iota = \epsilon + \iota$.

To establish a foundation for understanding the concept of split Narayana and Narayana–Lucas quaternions, we will commence by providing definitions for two essential quaternionic structures: Narayana quaternions and Narayana–Lucas quaternions. The ensuing definitions serve as a precursor to the introduction of split Narayana and Narayana–Lucas quaternions.

Definition 2.3. Let Z_n and T_n be the Narayana quaternions and Narayana–Lucas quaternions, respectively, which can be defined as:

$$Z_n = N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}, \tag{6}$$

$$T_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}, \tag{7}$$

respectively, where $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$

3 Split Narayana and Narayana–Lucas quaternions

In this section, we present the concepts of split Narayana quaternions and split Narayana–Lucas quaternions by employing the preliminary definitions as outlined in the preceding Section 2. We substantiate our discussion with theorems and provide valuable identities pertaining to these newly formed split Narayana and Narayana–Lucas quaternions.

Definition 3.1. Let Z_n and T_n be the split Narayana quaternions and split Narayana–Lucas quaternions, respectively, which can be defined as:

$$Z_n = N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}, \tag{8}$$

$$T_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}, \tag{9}$$

where N_n denotes the n-th Narayana numbers and U_n denotes the n-th Narayana–Lucas numbers and i, j, k denote the split quaternionic units which satisfy the non commutative multiplication rules:

$$i^2 = -1, j^2 = k^2 = 1, ij = -ji = k, jk = -kj = -i, ki = -ik = j. \tag{10}$$

The scalar and vector parts of split Narayana quaternions Z_n are defined as $SZ_n = N_n$ and $\vec{V}Z_n = iN_{n+1} + jN_{n+2} + kN_{n+3}$, respectively. If $SZ_n = 0$, then Z_n is a pure split Narayana quaternion.

Let us consider the two split Narayana quaternions Z_n and K_n as:

$$Z_n = N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}, \tag{11}$$

$$K_n = X_n + iX_{n+1} + jX_{n+2} + kX_{n+3}. \tag{12}$$

The addition, subtraction and multiplication of split quaternions can be defined as:

$$(Z_n \mp K_n) = (N_n \mp X_n) + i(N_{n+1} \mp X_{n+1}) + j(N_{n+2} \mp X_{n+2}) + k(N_{n+3} \mp X_{n+3}), \tag{13}$$

$$(Z_n.K_n) = (N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}).(X_n + iX_{n+1} + jX_{n+2} + kX_{n+3}). \tag{14}$$

The conjugate of split Narayana quaternions Z_n and split Narayana–Lucas quaternions T_n can be defined as:

$$\overline{Z_n} = N_n - iN_{n+1} - jN_{n+2} - kN_{n+3}, \tag{15}$$

$$\overline{T_n} = U_n - iU_{n+1} - jU_{n+2} - kU_{n+3}. \tag{16}$$

The norm of split Narayana quaternions can be defined as:

$$\mathcal{N}Z_{q'} = Z_{q'} \cdot \overline{Z_{q'}} = N^2_{q'} + N^2_{q'+1} - N^2_{q'+2} - N^2_{q'+3}. \tag{17}$$

Lemma 3.1. *Let Z_n and T_n be the split Narayana quaternions and split Narayana–Lucas quaternions, respectively, then the recurrence relations for these quaternions can be defined as:*

$$Z_n = Z_{n-1} + Z_{n-3},$$
$$T_n = T_{n-1} + T_{n-3}.$$

Proof. From Equation (8), we have

$$\begin{aligned} Z_{n-1}+Z_{n-3} &= N_{n-1}+iN_n+jN_{n+1}+kN_{n+2}+(N_{n-3}+iN_{n-2}+jN_{n-1}+kN_n), \\ &= (N_{n-1}+N_{n-3})+i(N_n+N_{n-2})+j(N_{n+1}+N_{n-1})+k(N_{n+2}+N_n), \\ Z_{n-1}+Z_{n-3} &= Z_n. \end{aligned}$$

Similarly from Equation (9), we obtained that

$$T_n = T_{n-1}+T_{n-3}. \qquad \square$$

Thus, we can state the following theorems by taking into account the aforementioned results related to split quaternions.

Theorem 3.1. *For $n \geq 1$, the following relations hold between Narayana quaternions and split Narayana quaternions as:*

$$\begin{aligned} Z_{n-1}+Z_{n+1} &= Z_{n+2}, \\ Z_n - iZ_{n+1} - jZ_{n+2} - kZ_{n+3} &= -5N_{n+1} - 3N_{n-1} - 2N_n. \end{aligned}$$

Proof. From Equations (8) and (10), we get

$$\begin{aligned} Z_{n-1}+Z_{n+1} &= (N_{n-1}+iN_n+jN_{n+1}+kN_{n+2})+(N_{n+1}+iN_{n+2}+jN_{n+3}+kN_{n+4}), \\ &= (N_{n-1}+N_{n+1})+i(N_n+N_{n+2})+j(N_{n+1}+N_{n+3})+k(N_{n+2}+N_{n+4}), \\ &= N_{n+2}+iN_{n+3}+jN_{n+4}+kN_{n+5}, \end{aligned}$$

which shows that

$$Z_{n-1}+Z_{n+1} = Z_{n+2}. \tag{18}$$

From Equations (2), (8) and (10), we get

$$\begin{aligned} Z_n - iZ_{n+1} - jZ_{n+2} - kZ_{n+3} &= (N_n+iN_{n+1}+jN_{n+2}+kN_{n+3}) - i(N_{n+1}+iN_{n+2} \\ &\quad +jN_{n+3}+kN_{n+4}) - j(N_{n+2}+iN_{n+3}+jN_{n+4} \\ &\quad +kN_{n+5}) - k(N_{n+3}+iN_{n+4}+jN_{n+5}+kN_{n+6}), \\ &= N_n+N_{n+2}-N_{n+4}-N_{n+6}, \end{aligned}$$

which shows that

$$Z_n - iZ_{n+1} - jZ_{n+2} - kZ_{n+3} = -5N_{n+1} - 3N_{n-1} - 2N_n. \tag{19}$$

This completes the proof. $\square$

Theorem 3.2. *Let Z_n and T_n be split Narayana quaternions and split Narayana–Lucas quaternions, respectively, then the following relations hold:*

$$\begin{aligned} Z_n + 3Z_{n-2} &= T_n, \\ 3Z_{n+1} - 2Z_n &= T_n. \end{aligned}$$

Proof. From Equation (8) and (10), it follows that

$$\begin{aligned} Z_n + 3Z_{n-2} &= N_n + iN_{n+1} + jN_{n+2} + kN_{n+3} + 3(N_{n-2} + iN_{n-1} + jN_n + kN_{n+1}), \\ &= (N_n + 3N_{n-2}) + i(N_{n+1} + 3N_{n-1}) + j(N_{n+2} + 3N_n) + k(N_{n+3} + 3N_{n+1}), \end{aligned}$$

by using the identity of Narayana numbers $U_n = N_n + 3N_{n-2}$ (see [23]) in the above equation, we obtain

$$\begin{aligned} Z_n + 3Z_{n-2} &= U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}, \\ Z_n + 3Z_{n-2} &= T_n, \ n \geq 2. \end{aligned} \tag{20}$$

In a similar manner, we can see that

$$\begin{aligned} 3Z_{n+1} - 2Z_n &= 3(N_{n+1} + iN_{n+2} + jN_{n+3} + kN_{n+4}) - 2(N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}), \\ &= (3N_{n+1} - 2N_n) + i(3N_{n+2} - 2N_{n+1}) + j(3N_{n+3} - 2N_{n+2}) + k(3N_{n+4} - 2N_{n+3}), \end{aligned}$$

and by utilising the identity $U_n = 3N_{n+1} - 2N_n$ (see [23]) in the above equation, this equation reduces to the following form

$$\begin{aligned} 3Z_{n+1} - 2Z_n &= U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}, \\ 3Z_{n+1} - 2Z_n &= T_n. \end{aligned} \tag{21}$$

This completes the proof. □

Theorem 3.3. *Let Z_n and $\overline{Z_n}$ be split Narayana quaternions and conjugate of Z_n, respectively, then the following relations hold:*

$$\begin{aligned} Z_n\overline{Z_n} &= N_n^2 + N_{n+1}^2 - N_{n+2}^2 - N_{n+3}^2, \\ Z_n + \overline{Z_n} &= 2N_n, \\ Z_n^2 &= 2Z_nN_n - Z_n\overline{Z_n}. \end{aligned}$$

Proof. From Equation (8),(10) and (15), we have

$$\begin{aligned} Z_n\overline{Z_n} &= (N_n + iN_{n+1} + jN_{n+2} + kN_{n+3})(N_n - iN_{n+1} - jN_{n+2} - kN_{n+3}), \\ Z_n\overline{Z_n} &= N_n^2 + N_{n+1}^2 - N_{n+2}^2 - N_{n+3}^2. \end{aligned} \tag{22}$$

In a similar manner, we can see that

$$\begin{aligned} Z_n + \overline{Z_n} &= (N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}) + (N_n - iN_{n+1} - jN_{n+2} - kN_{n+3}), \\ Z_n + \overline{Z_n} &= 2N_n. \end{aligned} \tag{23}$$

From Equation (23), it follows that

$$Z_n^2 = Z_nZ_n = Z_n(2N_n - \overline{Z_n}) = 2Z_nN_n - Z_n\overline{Z_n}. \tag{24}$$

This completes the proof. □

The characteristics equation of the relation (18) can be written as:

$$t^3 - t^2 - 1 = 0. \tag{25}$$

If μ, ν and λ are the roots of this characteristic Equation (25), then they can be written as:

$$\mu = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$
$$\nu = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$
$$\lambda = \frac{1}{3} + w^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

where

$$w = \frac{-1 + \iota\sqrt{3}}{2} = \exp(\frac{2\pi\iota}{3}).$$

For the split Narayana quaternions Z_n and Z_{n-2}, we get

$$\begin{aligned}\mu Z_n + Z_{n-2} &= \mu(N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}) + (N_{n-2} + iN_{n-1} + jN_n + kN_{n+1}),\\ &= (\mu N_n + N_{n-2}) + i(\mu N_{n+1} + N_{n-1}) + j(\mu N_{n+2} + N_n) + k(\mu N_{n+3} + N_{n+1}),\end{aligned}$$

where $n \geq 0$. Putting the relation $\mu^n = \mu N_n + N_{n-2}$ in the above equation, we get

$$\begin{aligned}\mu Z_n + Z_{n-2} &= \mu^n + i\mu^{n+1} + j\mu^{n+2} + k\mu^{n+3}\\ &= (1 + i\mu^1 + j\mu^2 + k\mu^3)\mu^n,\\ \mu Z_n + Z_{n-2} &= \mu^{'}\mu^n, \end{aligned} \tag{26}$$

where $\mu^{'} = (1 + i\mu + j\mu^2 + k\mu^3)$.

In a similar manner, by considering the relation $\nu^n = \nu N_n + N_{n-2}$, in the same way as Equation (26), we get

$$\nu Z_n + Z_{n-2} = \nu^{'}\nu^n, \tag{27}$$

where $\nu^{'} = (1 + i\nu + j\nu^2 + k\nu^3)$.

In a similar manner, by considering the relation $\lambda^n = \lambda N_n + N_{n-2}$, in the same way as Equation (26), we get

$$\lambda Z_n + Z_{n-2} = \lambda^{'}\lambda^n, \tag{28}$$

where $\lambda^{'} = (1 + i\lambda + j\lambda^2 + k\lambda^3)$.

Theorem 3.4. *Let Z_n and T_n be the split Narayana quaternions and split Narayana–Lucas quaternions, respectively, then the Binet formulas can be defined as:*

$$\begin{aligned}Z_n &= \frac{\mu^{'}\mu^{n+1}}{(\mu - \nu)(\mu - \lambda)} + \frac{\nu^{'}\nu^{n+1}}{(\nu - \mu)(\nu - \lambda)} + \frac{\lambda^{'}\lambda^{n+1}}{(\lambda - \mu)(\lambda - \nu)},\\ T_n &= \mu^{'}\mu^n + \nu^{'}\nu^n + \lambda^{'}\lambda^n.\end{aligned}$$

Proof. Let Z_n be the split Narayana quaternions. From Equation (8), we have

$$Z_n = N_n + iN_{n+1} + jN_{n+2} + kN_{n+3}.$$

By using the Binet formulas for the Narayana sequence [23] and the definition of split Narayana quaternions from Equation (8) we have,

$$\begin{aligned}
Z_n &= \left(\frac{\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right) \\
&\quad + i\left(\frac{\mu^{n+2}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{n+2}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{n+2}}{(\lambda-\mu)(\lambda-\nu)}\right) \\
&\quad + j\left(\frac{\mu^{n+3}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{n+3}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\mu)(\lambda-\nu)}\right) \\
&\quad + k\left(\frac{\mu^{n+4}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{n+4}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{n+4}}{(\lambda-\mu)(\lambda-\nu)}\right), \\
&= \frac{\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}(1 + i\mu + j\mu^2 + k\mu^3) + \frac{\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}(1 + i\nu + j\nu^2 + k\nu^3) \\
&\quad + \frac{\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}(1 + i\lambda + j\lambda^2 + k\lambda^3),
\end{aligned}$$

Hence,

$$Z_n = \frac{\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}\mu^{'} + \frac{\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}\nu^{'} + \frac{\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\lambda^{'}. \tag{29}$$

Moreover, from Equation (9) we have

$$T_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}.$$

By using the Binet formulas for the Narayana–Lucas sequence [23] and the definition of split Narayana–Lucas quaternions from Equation (9) we have,

$$\begin{aligned}
T_n &= \mu^n + \nu^n + \lambda^n + i(\mu^{n+1} + \nu^{n+1} + \lambda^{n+1}) + j(\mu^{n+2} + \nu^{n+2} + \lambda^{n+2}) \\
&\quad + k(\mu^{n+3} + \nu^{n+3} + \lambda^{n+3}), \\
&= \mu^n(1 + i\mu + j\mu^2 + k\mu^3) + \nu^n(1 + i\mu + j\mu^2 + k\mu^3) + \lambda^n(1 + i\mu + j\mu^2 + k\mu^3), \\
T_n &= \mu^{'}\mu^n + \nu^{'}\nu^n + \lambda^{'}\lambda^n. \qquad (30)
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let Z_n and T_n be the split Narayana quaternions and split Narayana–Lucas quaternions, respectively, then the generating functions can be defined as:*

$$\begin{aligned}
G(t) &= \sum_{n=0}^{\infty} Z_n t^n = \frac{Z_0 + t(Z_1 - Z_0) + t^2(Z_2 - Z_1)}{1 - t - t^3}, \\
G(t) &= \sum_{n=0}^{\infty} T_n t^n = \frac{T_0 + t(T_1 - T_0) + t^2(T_2 - T_1)}{1 - t - t^3}.
\end{aligned}$$

Proof. Let us consider the following formal power series to be the generating function for the split Narayana quaternions as:

$$G(t) = \sum_{n=0}^{\infty} Z_n t^n = Z_0 + Z_1 t + Z_2 t^2 + \cdots$$

Then we have

$$\begin{aligned} tG(t) &= Z_0(r)t + Z_1(r)t^2 + Z_2 t^3 + \cdots , \\ t^3 G(t) &= Z_0 t^3 + Z_1 t^4 + Z_2 t^5 + \cdots \end{aligned}$$

Therefore, we get

$$\begin{aligned} G(t) - tG(t) - t^3 G(t) &= (Z_0 + Z_1 t + Z_2 t^2 + \cdots) - (Z_0 t + Z_1 t^2 + Z_2 t^3 + \cdots) \\ &\quad - (Z_0 t^3 + Z_1 t^4 + Z_2 t^5 + \cdots), \\ G(t)(1 - t - t^3) &= Z_0 + (Z_1 - Z_0)t + (Z_2 - Z_1)t^2, \\ G(t) = \sum_{n=0}^{\infty} Z_n t^n &= \frac{Z_0 + t(Z_1 - Z_0) + t^2(Z_2 - Z_1)}{(1 - t - t^3)}. \end{aligned} \tag{31}$$

Thus the proof is completed for the split Narayana quaternions.

In a similar manner, we can also prove the following generating function for split Narayana–Lucas quaternions as:

$$G(t) = \sum_{n=0}^{\infty} T_n t^n = \frac{T_0 + t(T_1 - T_0) + t^2(T_2 - T_1)}{1 - t - t^3}. \tag{32}$$

This completes the proof. □

Theorem 3.6. *Let Z_n and T_n be the split Narayana quaternions and split Narayana–Lucas quaternions, respectively, then the exponential generating functions can be defined as:*

$$\begin{aligned} &\sum_{n=0}^{\infty} Z_n \frac{t^n}{n!} = \frac{\mu' \mu}{(\mu - \nu)(\mu - \lambda)} e^{\mu t} + \frac{\nu' \nu}{(\nu - \mu)(\nu - \lambda)} e^{\nu t} + \frac{\lambda' \lambda}{(\lambda - \mu)(\lambda - \nu)} e^{\lambda t}, \\ &\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \mu' \mu e^{\mu t} + \nu' \nu e^{\nu t} + \lambda' \lambda e^{\lambda t}. \end{aligned}$$

Proof. By using the Binet formulas for the split Narayana quaternions (which we already defined in (29)) and the definition of exponential generating function we obtained as:

$$\begin{aligned} \sum_{n=0}^{\infty} Z_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \Big[\frac{\mu' \mu^{n+1}}{(\mu - \nu)(\mu - \lambda)} + \frac{\nu' \nu^{n+1}}{(\nu - \mu)(\nu - \lambda)} + \frac{\lambda' \lambda^{n+1}}{(\lambda - \mu)(\lambda - \nu)} \Big] \frac{t^n}{n!}, \\ &= \frac{\mu' \mu}{(\mu - \nu)(\mu - \lambda)} \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} + \frac{\nu' \nu}{(\nu - \mu)(\nu - \lambda)} \sum_{n=0}^{\infty} \frac{(\nu t)^n}{n!} + \frac{\lambda' \lambda}{(\lambda - \mu)(\lambda - \nu)} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}, \\ \sum_{n=0}^{\infty} Z_n \frac{t^n}{n!} &= \frac{\mu' \mu}{(\mu - \nu)(\mu - \lambda)} e^{\mu t} + \frac{\nu' \nu^n}{(\nu - \mu)(\nu - \lambda)} e^{\nu t} + \frac{\lambda' \lambda^n}{(\lambda - \mu)(\lambda - \nu)} e^{\lambda t}. \end{aligned} \tag{33}$$

Thus the proof is completed for the exponential generating function for the split Narayana quaternions.

In a similar manner, we can also prove the following exponential generating function for split Narayana–Lucas quaternions as:

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \mu^{'}\mu e^{\mu t} + \nu^{'}\nu e^{\nu t} + \lambda^{'}\lambda e^{\lambda t}. \tag{34}$$

This completes the proof. □

3.1 Numerical simulation for split Narayana quaternions

To better illustrate or visualize the theoretical findings, we present a numerical simulation of the newly introduced split Narayana quaternions. Table 1 reports the numerical values for Narayana sequence and the norm of split Narayana quaternions $\mathcal{N}(Z_n)$ for indices $0 \leq n \leq 20$. Then Figure 1 is plotted with the help of Table 1.

Table 1. Numerical values for Narayana sequence and norm of split Narayana quaternion $\mathcal{N}(Z_n)$.

n	N_n	N_{n+1}	N_{n+2}	N_{n+3}	$\mathcal{N}(Z_n)$
0	0	1	1	1	-1
1	1	1	1	2	-3
2	1	1	2	3	-11
3	1	2	3	4	-20
4	2	3	4	6	-39
5	3	4	6	9	-92
6	4	6	9	13	-198
7	6	9	13	19	-413
8	9	13	19	28	-895
9	13	19	28	41	-1935
10	19	28	41	60	-4136
11	28	41	60	88	-8879
12	41	60	88	129	-19104
13	60	88	129	189	-41018
14	88	129	189	277	-88065
15	129	189	277	406	-189203
16	189	277	406	595	-406411
17	277	406	595	872	-872844
18	406	595	872	1278	-1874807
19	595	872	1278	1873	-4024004
20	872	1278	1873	2745	-8649486

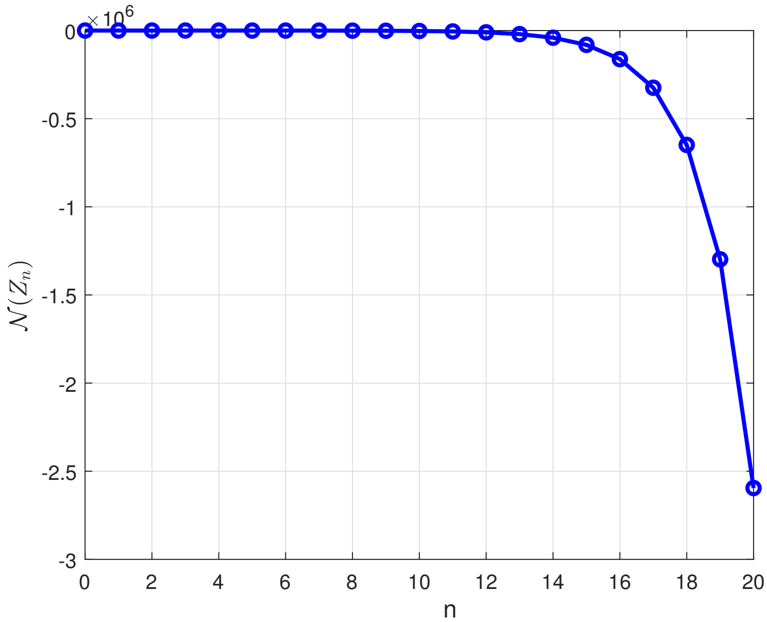

Figure 1. Norm of split Narayana quaternion $\mathcal{N}(Z_n)$ vs n.

The results reveal that all norms are negative, which decrease steadily as n increases. Starting with $\mathcal{N}(Z_0) = -1$, the norm decreases gradually, reaching $\mathcal{N}(Z_{10}) = -4136$, and then drops sharply at higher indices, attaining $\mathcal{N}(Z_{20}) = -8649486$. This sharp decrease indicates that the quadratic form defined by the norm, $\mathcal{N}(Z_n) = N_n^2 + N_{n+1}^2 - N_{n+2}^2 - N_{n+3}^2$, is significantly dominated by the negative terms as n grows. In other words, the components $\{N_{n+2}, N_{n+3}\}$ grow faster than the components $\{N_n, N_{n+1}\}$, causing the overall norm to shrink. In geometrical interpretation, a sharp decrease suggests that the quaternion's trajectory in its four-dimensional space moves more intensively toward a light-like region. As the norm becomes extremely small (or extremely negative), it may lead to instability in encryption, signal modeling, or physical interpretation where these split quaternions are applied.

4 Split Narayana and Narayana–Lucas hybrid quaternions

In this section, we introduce the concept of hybrid numbers within the context of split Narayana quaternions and split Narayana–Lucas quaternions (as discussed in Section 3). Additionally, we have presented certain theorems along with their proofs, and identities associated with these newly introduced hybrid quaternions.

Definition 4.1. *Let ZH_n be the n-th split Narayana hybrid quaternions can be defined as:*

$$\begin{aligned} ZH_n &= NH_n + iNH_{n+1} + jNH_{n+2} + kNH_{n+3}, \\ &= (N_n + \iota N_{n+1} + \epsilon N_{n+2} + hN_{n+3}) + i(N_{n+1} + \iota N_{n+2} + \epsilon N_{n+3} + hN_{n+4}) \\ &\quad + j(N_{n+2} + \iota N_{n+3} + \epsilon N_{n+4} + hN_{n+5}) + k(N_{n+3} + \iota N_{n+4} + \epsilon N_{n+5} + hN_{n+6}), \end{aligned}$$

where ι, ϵ, h are hybrid units and i, j, k are split quaternionic units.

$$ZH_n = Z_n + \iota Z_{n+1} + \epsilon Z_{n+2} + hZ_{n+3}. \tag{35}$$

Definition 4.2. *Let TH_n be the n-th split Narayana–Lucas hybrid quaternions can be defined as:*

$$\begin{aligned}TH_n &= UH_n + iUH_{n+1} + jUH_{n+2} + kUH_{n+3},\\ &= (U_n + \iota U_{n+1} + \epsilon U_{n+2} + hU_{n+3}) + i(U_{n+1} + \iota U_{n+2} + \epsilon U_{n+3} + hU_{n+4})\\ &\quad + j(U_{n+2} + \iota U_{n+3} + \epsilon U_{n+4} + hU_{n+5}) + k(U_{n+3} + \iota U_{n+4} + \epsilon U_{n+5} + hU_{n+6}),\end{aligned}$$

where ι, ϵ, h are hybrid units and i, j, k are split quaternionic units.

$$TH_n = T_n + \iota T_{n+1} + \epsilon T_{n+2} + hT_{n+3}. \tag{36}$$

Lemma 4.1. *Let ZH_n and TH_n be the split Narayana hybrid quaternions and split Narayana–Lucas hybrid quaternions, respectively, then the recurrence relation for these quaternions can be defined as:*

$$\begin{aligned}ZH_n &= ZH_{n-1} + ZH_{n-3},\\ TH_n &= TH_{n-1} + TH_{n-3}.\end{aligned}$$

Proof. From Equation (35), we have

$$\begin{aligned}ZH_{n-1} + ZH_{n-3} &= Z_{n-1} + \iota Z_n + \epsilon Z_{n+1} + hZ_{n+2} + (Z_{n-3} + \iota Z_{n-2} + \epsilon Z_{n-1} + hZ_n),\\ &= (Z_{n-1} + Z_{n-3}) + \iota(Z_n + Z_{n-2}) + \epsilon(Z_{n+1} + Z_{n-1}) + h(Z_{n+2} + Z_n),\\ ZH_{n-1} + ZH_{n-3} &= ZH_n.\end{aligned}$$

Similarly, from Equation (36), we obtain

$$TH_n = TH_{n-1} + TH_{n-3}.$$ □

Theorem 4.1. *Let ZH_n and TH_n be the split Narayana hybrid quaternions and split Narayana–Lucas hybrid quaternions, respectively, then the Binet formulas for these quaternions can be defined as:*

$$\begin{aligned}ZH_n &= \frac{\mu^{'}\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}e^{\mu} + \frac{\nu^{'}\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}e^{\nu} + \frac{\lambda^{'}\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}e^{\lambda},\\ TH_n &= \mu^{'}\mu^n e^{\mu} + \nu^{'}\nu^n e^{\nu} + \lambda^{'}\lambda^n e^{\lambda}.\end{aligned}$$

Proof. Let ZH_n be the split Narayana hybrid quaternions. From Equation (35), we have

$$ZH_n = Z_n + \iota Z_{n+1} + \epsilon Z_{n+2} + hZ_{n+3}.$$

Therefore, utilising above relation in Equation (29), we have

$$\begin{aligned}ZH_n = &\left(\frac{\mu^{'}\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{'}\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{'}\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right)\\ &+\iota\left(\frac{\mu^{'}\mu^{n+2}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{'}\nu^{n+2}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{'}\lambda^{n+2}}{(\lambda-\mu)(\lambda-\nu)}\right)\\ &+\epsilon\left(\frac{\mu^{'}\mu^{n+3}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{'}\nu^{n+3}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{'}\lambda^{n+3}}{(\lambda-\mu)(\lambda-\nu)}\right)\\ &+h\left(\frac{\mu^{'}\mu^{n+4}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^{'}\nu^{n+4}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^{'}\lambda^{n+4}}{(\lambda-\mu)(\lambda-\nu)}\right),\end{aligned}$$

$$= \frac{\mu^{'}\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}(1+\iota\mu+\epsilon\mu^2+h\mu^3)+\frac{\nu^{'}\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}(1+\iota\nu+\epsilon\nu^2+h\nu^3)$$
$$+\frac{\lambda^{'}\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}(1+\iota\lambda+\epsilon\lambda^2+h\lambda^3),$$

Hence,

$$ZH_n = \frac{\mu^{'}\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}e^{\mu}+\frac{\nu^{'}\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}e^{\nu}+\frac{\lambda^{'}\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}e^{\lambda}, \tag{37}$$

where $e^{\mu}=(1+\iota\mu+\epsilon\mu^2+h\mu^3), e^{\nu}=(1+\iota\nu+\epsilon\nu^2+h\nu^3), e^{\lambda}=(1+\iota\lambda+\epsilon\lambda^2+h\lambda^3)$. Moreover, from Equation (36), we have

$$TH_n = T_n+\iota T_{n+1}+\epsilon T_{n+2}+hT_{n+3}.$$

Therefore, utilising the above relation in Equation (30), we have

$$TH_n = (\mu^{'}\mu^n+\nu^{'}\nu^n+\lambda^{'}\lambda^n)+\iota(\mu^{'}\mu^{n+1}+\nu^{'}\nu^{n+1}+\lambda^{'}\lambda^{n+1})+\epsilon(\mu^{'}\mu^{n+2}+\nu^{n+2}+\lambda^{'}\lambda^{n+2})$$
$$+h(\mu^{'}\mu^{n+3}+\nu^{'}\nu^{n+3}+\lambda^{n+3}),$$
$$=\mu^{'}\mu^n(1+\iota\mu+\epsilon\mu^2+h\mu^3)+\nu^{'}\nu^n(1+\iota\nu+\epsilon\nu^2+h\nu^3)+\lambda^n(1+\iota\lambda+\epsilon\lambda^2+h\lambda^3),$$
$$TH_n = \mu^{'}\mu^n e^{\mu}+\nu^{'}\nu^n e^{\nu}+\lambda^{'}\lambda^n e^{\lambda}. \tag{38}$$

This completes the proof. □

Theorem 4.2. *Let ZH_n and TH_n be the split Narayana hybrid quaternions and split Narayana–Lucas hybrid quaternions, respectively, then the generating functions for these quaternions can be defined as:*

$$G(t)=\sum_{n=0}^{\infty} ZH_n t^n = \frac{ZH_0+t(ZH_1-ZH_0)+t^2(ZH_2-ZH_1)}{1-t-t^3},$$
$$G(t)=\sum_{n=0}^{\infty} TH_n t^n = \frac{TH_0+t(TH_1-TH_0)+t^2(TH_2-TH_1)}{1-t-t^3}.$$

Proof. Let us consider the following formal power series to be the generating function for the split Narayana hybrid quaternions as:

$$G(t)=\sum_{n=0}^{\infty} ZH_n t^n = ZH_0+ZH_1t+ZH_2t^2+\cdots$$

Then, we have

$$tG(t)=ZH_0t+ZH_1t^2+ZH_2t^3+\cdots,$$
$$t^3G(t)=ZH_0t^3+ZH_1t^4+ZH_2t^5+\cdots$$

Therefore, we get

$$\begin{aligned}
G(t)-tG(t)-t^3G(t) &= (ZH_0+ZH_1t+ZH_2t^2+\cdots)-(ZH_0t+ZH_1t^2+ZH_2t^3+\cdots)\\
&\quad -(ZH_0t^3+ZH_1t^4+ZH_2t^5+\cdots),\\
G(t)(1-t-t^3) &= ZH_0+t(ZH_1-ZH_0)+t^2(ZH_2-ZH_1),\\
G(t) &= \frac{ZH_0+t(ZH_1-ZH_0)+t^2(ZH_2-ZH_1)}{(1-t-t^3)},\\
G(t) &= \sum_{m=0}^{\infty} ZH_n t^n = \frac{ZH_0+t(ZH_1-ZH_0)+t^2(ZH_2-ZH_1)}{(1-t-t^3)}. \qquad (39)
\end{aligned}$$

Thus the proof is completed for the generating functions for split Narayana hybrid quaternions.

In a similar manner, we can also prove the following generating function for split Narayana–Lucas hybrid quaternions as:

$$G(t)=\sum_{n=0}^{\infty} TH_n t^n = \frac{TH_0+t(TH_1-TH_0)+t^2(TH_2-TH_1)}{1-t-t^3}. \qquad (40)$$

This completes the proof. □

Theorem 4.3. *Let ZH_n and TH_n be the split Narayana hybrid quaternions and split Narayana–Lucas hybrid quaternions, respectively, then the exponential generating functions for these quaternions can be defined as:*

$$\sum_{n=0}^{\infty} ZH_n\frac{t^n}{n!} = \frac{\mu'\mu}{(\mu-\nu)(\mu-\lambda)}e^{\mu}e^{\mu t}+\frac{\nu'\nu}{(\nu-\mu)(\nu-\lambda)}e^{\nu}e^{\nu t}+\frac{\lambda'\lambda}{(\lambda-\mu)(\lambda-\nu)}e^{\lambda}e^{\lambda t},$$
$$\sum_{n=0}^{\infty} TH_n\frac{t^n}{n!} = \mu' e^{\mu}e^{\mu t}+\nu' e^{\nu}e^{\nu t}+\lambda' e^{\lambda}e^{\lambda t}.$$

Proof. By using the Binet formulas for the split Narayana and Narayana– Lucas hybrid quaternions from Theorem 4.1, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} ZH_n\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\Big[\frac{\mu'\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}(1+\iota\mu+\epsilon\mu^2+h\mu^3)+\frac{\nu'\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}(1+\iota\nu+\epsilon\nu^2+h\nu^3)\\
&\quad +\frac{\lambda'\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}(1+\iota\lambda+\epsilon\lambda^2+h\lambda^3)\Big]\frac{t^n}{n!},\\
&= \frac{\mu'\mu}{(\mu-\nu)(\mu-\lambda)}(1+\iota\mu+\epsilon\mu^2+h\mu^3)\sum_{n=0}^{\infty}\frac{(\mu t)^n}{n!}+\frac{\nu'\nu}{(\nu-\mu)(\nu-\lambda)}(1+\iota\nu+\epsilon\nu^2+h\nu^3),
\end{aligned}$$

$$\sum_{n=0}^{\infty}\frac{(\nu t)^n}{n!}+\frac{\lambda'\lambda}{(\lambda-\mu)(\lambda-\nu)}(1+\iota\lambda+\epsilon\lambda^2+h\lambda^3)\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!},$$

$$\sum_{n=0}^{\infty} ZH_n\frac{t^n}{n!} = \frac{\mu'\mu}{(\mu-\nu)(\mu-\lambda)}e^{\mu}e^{\mu t}+\frac{\nu'\nu}{(\nu-\mu)(\nu-\lambda)}e^{\nu}e^{\nu t}+\frac{\lambda'\lambda}{(\lambda-\mu)(\lambda-\nu)}e^{\lambda}e^{\lambda t}. \qquad (41)$$

Thus the proof is completed for the exponential generating function for the split Narayana hybrid quaternions.

In a similar manner, we can also prove the exponential generating function for split Narayana–Lucas hybrid quaternions, we have

$$\sum_{n=0}^{\infty} TH_n \frac{t^n}{n!} = \mu^{'} e^{\mu} e^{\mu t} + \nu^{'} e^{\nu} e^{\nu t} + \lambda^{'} e^{\lambda} e^{\lambda t}. \tag{42}$$

This completes the proof. □

Theorem 4.4. *Let m and n be any positive integers and $m \geq n$, then the following relations hold as:*

$$ZH_m TH_n + TH_m ZH_n = \frac{2e^{2\lambda}(\mu-\nu)\lambda'^2\lambda^{2m+n+1} + \nu\lambda^{'}\mu^{'}(\lambda-\mu)\left(-e^{\lambda+\mu}\right)(\mu^m\lambda^n + \lambda^m\mu^n)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}$$
$$+ \frac{-2e^{2\mu}(\lambda-\nu)\mu' 2\mu^{2m+n+1} - \lambda\mu^{'}\nu^{'}(\mu-\nu)e^{\mu+\nu}(\nu^m\mu^n + \mu^m\nu^n) + 2e^{2\nu}(\lambda-\mu)\nu'^2\nu^{2m+n+1}}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)},$$

$$ZH_m TH_n - TH_m ZH_n = \frac{\lambda^{'}\mu^{'}\left(-e^{\lambda+\mu}\right)(2\lambda\mu - \lambda\nu - \mu\nu)(\mu^m\lambda^n - \lambda^m\mu^n)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}$$
$$+ \frac{\lambda^{'}\nu^{'}e^{\lambda+\nu}(-\lambda\mu + 2\lambda\nu - \mu\nu)(\nu^m\lambda^n - \lambda^m\nu^n) - \mu^{'}\nu^{'}e^{\mu+\nu}(2\mu\nu - \lambda(\mu+\nu))(\nu^m\mu^n - \mu^m\nu^n)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.$$

Proof. By using the Binet formulas from Theorem 4.1, we obtain

$$ZH_m TH_n + TH_m ZH_n$$
$$= \left(\frac{e^{\nu}\left(\nu^{'}\nu^{m+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^{\mu}\left(\mu^{'}\mu^{m+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^{\lambda}\left(\lambda^{'}\lambda^{m+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right)\left(\nu^{'}e^{\nu}\nu^n + \mu^{'}e^{\mu}\mu^n + \lambda^{'}e^{\lambda}\lambda^n\right)$$
$$+ \left(\nu^{'}e^{\nu}\nu^m + \mu^{'}e^{\mu}\mu^m + \lambda^{'}e^{\lambda}\lambda^m\right)\left(\frac{e^{\nu}\left(\nu^{'}\nu^{n+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^{\mu}\left(\mu^{'}\mu^{n+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^{\lambda}\left(\lambda^{'}\lambda^{n+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right),$$

$$ZH_m TH_n + TH_m ZH_n = \frac{2e^{2\lambda}(\mu-\nu)\lambda'^2\lambda^{2m+n+1} + \nu\lambda^{'}\mu^{'}(\lambda-\mu)\left(-e^{\lambda+\mu}\right)(\mu^m\lambda^n + \lambda^m\mu^n)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}$$
$$+ \frac{-2e^{2\mu}(\lambda-\nu)\mu' 2\mu^{2m+n+1} - \lambda\mu^{'}\nu^{'}(\mu-\nu)e^{\mu+\nu}(\nu^m\mu^n + \mu^m\nu^n) + 2e^{2\nu}(\lambda-\mu)\nu'^2\nu^{2m+n+1}}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.$$

Similarly, we have

$$ZH_m TH_n - TH_m ZH_n$$
$$= \left(\frac{e^{\nu}\left(\nu^{'}\nu^{m+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^{\mu}\left(\mu^{'}\mu^{m+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^{\lambda}\left(\lambda^{'}\lambda^{m+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right)\left(\nu^{'}e^{\nu}\nu^n + \mu^{'}e^{\mu}\mu^n + \lambda^{'}e^{\lambda}\lambda^n\right)$$
$$- \left(\nu^{'}e^{\nu}\nu^m + \mu^{'}e^{\mu}\mu^m + \lambda^{'}e^{\lambda}\lambda^m\right)\left(\frac{e^{\nu}\left(\nu^{'}\nu^{n+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^{\mu}\left(\mu^{'}\mu^{n+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^{\lambda}\left(\lambda^{'}\lambda^{n+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right),$$

$$ZH_m TH_n - TH_m ZH_n = \frac{\lambda^{'}\mu^{'}\left(-e^{\lambda+\mu}\right)(2\lambda\mu - \lambda\nu - \mu\nu)(\mu^m\lambda^n - \lambda^m\mu^n)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}$$
$$+ \frac{\lambda^{'}\nu^{'}e^{\lambda+\nu}(-\lambda\mu + 2\lambda\nu - \mu\nu)(\nu^m\lambda^n - \lambda^m\nu^n) - \mu^{'}\nu^{'}e^{\mu+\nu}(2\mu\nu - \lambda(\mu+\nu))(\nu^m\mu^n - \mu^m\nu^n)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.$$

This completes the proof. □

Theorem 4.5. *(Catalan's identities) Let ZH_n and TH_n be the split Narayana and Narayana–Lucas hybrid quaternions, respectively. Therefore, for $n \geq 1$, we have*

$$\begin{aligned}
ZH_{n+r}ZH_{n-r} - (ZH_n)^2 &= \frac{\lambda^{-r}\mu^{-r}\nu^{-r}((\mu-\lambda)(\lambda-\nu)\lambda^r\mu'\mu^{n+1}\nu'\nu^{n+1}e^{\mu+\nu}\left(\mu^r-\nu^r\right)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+ \frac{\lambda^{-r}\mu^{-r}\nu^{-r}((\lambda-\mu)(\mu-\nu)\mu^r\lambda'\lambda^{n+1}\nu'\nu^{n+1}e^{\lambda+\nu}(\lambda^r-\nu^r)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+ \frac{\lambda^{-r}\mu^{-r}\nu^{-r}((\lambda-\nu)(\mu-\nu)\nu^r\lambda'\lambda^{n+1}\mu'\mu^{n+1}\left(-e^{\lambda+\mu}\right)(\lambda^r-\mu^r)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2},
\end{aligned}$$

$$\begin{aligned}
TH_{n+r}TH_{n-r} - (TH_n)^2 &= \lambda^{-r}\mu^{-r}\nu^{-r}(\lambda^r\mu'\mu^n\nu'\nu^n e^{\mu+\nu}\left(\mu^r-\nu^r\right)^2 + \mu^r\lambda'\lambda^n\nu'\nu^n e^{\lambda+\nu}\left(\lambda^r-\nu^r\right)^2 \\
&+ \nu^r\lambda'\lambda^n\mu'\mu^n e^{\lambda+\mu}\left(\lambda^r-\mu^r\right)^2).
\end{aligned}$$

Proof. By using the Binet formulas from Theorem 4.1, we obtain

$$\begin{aligned}
&ZH_{n+r}ZH_{n-r} - (ZH_n)^2 \\
&= \left(\frac{e^\nu\left(\nu'\nu^{n+r+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^\mu\left(\mu'\mu^{n+r+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^\lambda\left(\lambda'\lambda^{n+r+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right) \\
&\quad\cdot\left(\frac{e^\nu\left(\nu'\nu^{n-r+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^\mu\left(\mu'\mu^{n-r+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^\lambda\left(\lambda'\lambda^{n-r+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right) \\
&\quad-\left(\frac{e^\nu\left(\nu'\nu^{n+1}\right)}{(\nu-\lambda)(\nu-\mu)} + \frac{e^\mu\left(\mu'\mu^{n+1}\right)}{(\mu-\lambda)(\mu-\nu)} + \frac{e^\lambda\left(\lambda'\lambda^{n+1}\right)}{(\lambda-\mu)(\lambda-\nu)}\right)^2
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2\lambda'\lambda^{n+1}\mu'\mu^{n+1}e^{\lambda+\mu}}{(\lambda-\mu)(\mu-\lambda)(\lambda-\nu)(\mu-\nu)} - \frac{2\lambda'\lambda^{n+1}\nu'\nu^{n+1}e^{\lambda+\nu}}{(\lambda-\mu)(\lambda-\nu)(\nu-\lambda)(\nu-\mu)} - \frac{2\mu'\mu^{n+1}\nu'\nu^{n+1}e^{\mu+\nu}}{(\mu-\lambda)(\nu-\lambda)(\mu-\nu)(\nu-\mu)} \\
&+ \frac{e^{\lambda+\mu}\lambda'\lambda^{n+r+1}\mu'\mu^{n-r+1}}{(\lambda-\mu)(\mu-\lambda)(\lambda-\nu)(\mu-\nu)} + \frac{e^{\lambda+\mu}\lambda'\lambda^{n-r+1}\mu'\mu^{n+r+1}}{(\lambda-\mu)(\mu-\lambda)(\lambda-\nu)(\mu-\nu)} \\
&+ \frac{e^{\lambda+\nu}\lambda'\lambda^{n+r+1}\nu'\nu^{n-r+1}}{(\lambda-\mu)(\lambda-\nu)(\nu-\lambda)(\nu-\mu)} + \frac{e^{\lambda+\nu}\lambda'\lambda^{n-r+1}\nu'\nu^{n+r+1}}{(\lambda-\mu)(\lambda-\nu)(\nu-\lambda)(\nu-\mu)} \\
&+ \frac{e^{\mu+\nu}\mu'\mu^{n-r+1}\nu'\nu^{n+r+1}}{(\mu-\lambda)(\nu-\lambda)(\mu-\nu)(\nu-\mu)} + \frac{e^{\mu+\nu}\mu'\mu^{n+r+1}\nu'\nu^{n-r+1}}{(\mu-\lambda)(\nu-\lambda)(\mu-\nu)(\nu-\mu)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
ZH_{n+r}ZH_{n-r} - (ZH_n)^2 &= \frac{\lambda^{-r}\mu^{-r}\nu^{-r}((\mu-\lambda)(\lambda-\nu)\lambda^r\mu'\mu^{n+1}\nu'\nu^{n+1}e^{\mu+\nu}\left(\mu^r-\nu^r\right)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+ \frac{\lambda^{-r}\mu^{-r}\nu^{-r}((\lambda-\mu)(\mu-\nu)\mu^r\lambda'\lambda^{n+1}\nu'\nu^{n+1}e^{\lambda+\nu}(\lambda^r-\nu^r)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+ \frac{\lambda^{-r}\mu^{-r}\nu^{-r}((\lambda-\nu)(\mu-\nu)\nu^r\lambda'\lambda^{n+1}\mu'\mu^{n+1}\left(-e^{\lambda+\mu}\right)(\lambda^r-\mu^r)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2},
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&TH_{n+r}TH_{n-r}-(TH_n)^2\\
&=\left(\nu' e^{\nu}\nu^{n+r}+\mu' e^{\mu}\mu^{n+r}+\lambda' e^{\lambda}\lambda^{n+r}\right)\left(\nu' e^{\nu}\nu^{n-r}+\mu' e^{\mu}\mu^{n-r}+\lambda' e^{\lambda}\lambda^{n-r}\right)\\
&\quad-\left(\nu' e^{\nu}\nu^{n}+\mu' e^{\mu}\mu^{n}+\lambda' e^{\lambda}\lambda^{n}\right)^2,\\
&=-2\lambda'\lambda^n\mu'\mu^n e^{\lambda+\mu}-2\lambda'\lambda^n\nu'\nu^n e^{\lambda+\nu}-2\mu'\mu^n\nu'\nu^n e^{\mu+\nu}\\
&\quad+e^{\lambda+\mu}\lambda'\lambda^{n+r}\mu'\mu^{n-r}+e^{\lambda+\mu}\lambda'\lambda^{n-r}\mu'\mu^{n+r}\\
&\quad+e^{\lambda+\nu}\lambda'\lambda^{n+r}\nu'\nu^{n-r}+e^{\lambda+\nu}\lambda'\lambda^{n-r}\nu'\nu^{n+r}\\
&\quad+e^{\mu+\nu}\mu'\mu^{n-r}\nu'\nu^{n+r}+e^{\mu+\nu}\mu'\mu^{n+r}\nu'\nu^{n-r},\\
&=\lambda^{-r}\mu^{-r}\nu^{-r}\left(\lambda^r\mu'\mu^n\nu'\nu^n e^{\mu+\nu}(\mu^r-\nu^r)^2+\mu^r\lambda'\lambda^n\nu'\nu^n e^{\lambda+\nu}(\lambda^r-\nu^r)^2+\nu^r\lambda'\lambda^n\mu'\mu^n e^{\lambda+\mu}(\lambda^r-\mu^r)^2\right).
\end{aligned}
$$

This completes the proof. □

Theorem 4.6. *(Cassini's identities) Let ZH_n and TH_n be the split Narayana and Narayana–Lucas hybrid quaternions, respectively. Therefore, for $n \geq 1$, we have*

$$
\begin{aligned}
ZH_{n+1}ZH_{n-1}-(ZH_n)^2 &= \frac{\lambda^{-1}\mu^{-1}\nu^{-1}((\mu-\lambda)(\lambda-\nu)\lambda\mu'\mu^{n+1}\nu'\nu^{n+1}e^{\mu+\nu}(\mu-\nu)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2}\\
&+\frac{\lambda^{-1}\mu^{-1}\nu^{-1}((\lambda-\mu)(\mu-\nu)\mu\lambda'\lambda^{n+1}\nu'\nu^{n+1}e^{\lambda+\nu}(\lambda-\nu)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2}\\
&+\frac{\lambda^{-1}\mu^{-1}\nu^{-1}((\lambda-\nu)(\mu-\nu)\nu\lambda'\lambda^{n+1}\mu'\mu^{n+1}\left(-e^{\lambda+\mu}\right)(\lambda-\mu)^2)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2},\\
TH_{n+1}TH_{n-1}-(TH_n)^2 &= \lambda^{-1}\mu^{-1}\nu^{-1}\Big(\lambda\mu'\mu^n\nu'\nu^n e^{\mu+\nu}(\mu-\nu)^2+\mu\lambda'\lambda^n\nu'\nu^n e^{\lambda+\nu}(\lambda-\nu)^2\\
&\qquad+\nu\lambda'\lambda^n\mu'\mu^n e^{\lambda+\mu}(\lambda-\mu)^2\Big).
\end{aligned}
$$

Proof. By substituting $r=1$ in Theorem 4.5, this theorem can be easily proved. □

4.1 Numerical simulation for split Narayana–Lucas hybrid quaternions

To better illustrate or visualize the theoretical findings, we present a numerical simulation of the newly introduced split Narayana–Lucas hybrid quaternions. Table 2 reports the numerical values for Narayana–Lucas hybrid sequence and norm of split Narayana–Lucas hybrid quaternions $\mathcal{N}(TH_n)$ for indices $0 \leq n \leq 20$. Then Figure 2 is plotted with the help of Table 2. The norm of a split Narayana–Lucas hybrid quaternion is computed using the following relation:

$$
\begin{aligned}
\mathcal{N}(TH_n) &= U_n^2+2U_{n+1}^2-2U_{n+1}U_{n+2}-2U_{n+2}U_{n+3}-3U_{n+3}^2+2U_{n+3}U_{n+4}-2U_{n+4}^2\\
&\quad+2U_{n+4}U_{n+5}+U_{n+5}^2+U_{n+6}^2.
\end{aligned}
$$

Table 2. Numerical values of Narayana–Lucas hybrid sequence and norm of split Narayana–Lucas hybrid quaternions $\mathcal{N}(TH_n)$.

n	UH_n	UH_{n+1}	UH_{n+2}	UH_{n+3}	$\mathcal{N}(TH_n)$
0	3	1	1	1	139
1	1	1	1	2	313
2	1	1	2	3	711
3	1	2	3	4	1468
4	2	3	4	6	3139
5	3	4	6	9	6673
6	4	6	9	13	14164
7	6	9	13	19	30079
8	9	13	19	28	63712
9	13	19	28	41	134923
10	19	28	41	60	285274
11	28	41	60	88	603607
12	41	60	88	129	1277425
13	60	88	129	189	2702122
14	88	129	189	277	5716903
15	129	189	277	406	12099034
16	189	277	406	595	25604293
17	277	406	595	872	54197467
18	406	595	872	1278	114756334
19	595	872	1278	1873	242958451
20	872	1278	1873	2745	648741012

The simulation results reveal a rapid growth in $\mathcal{N}(TH_n)$ as n increases. Starting with $\mathcal{N}(TH_0) = 139$, the norm rises steadily, surpassing one million at $n = 12$ and ultimately reaching $\mathcal{N}(TH_{20}) = 6.487 \times 10^8$. According to the classification provided by Özdemir [17], the nature of hybrid numbers and quaternions depends on the sign and growth of their norm. From Figure 2, the geometrical interpretation of the norm of split Narayana–Lucas hybrid quaternions remains strictly positive for all computed values of n, the corresponding hybrid quaternions are elliptic in nature. As the norm increases, the split Narayana–Lucas hybrid quaternions exhibit stable and elliptic behavior, which may lead to enhanced performance in image encryption, improved convergence in hybrid differential systems, and greater stability in quantum like modeling.

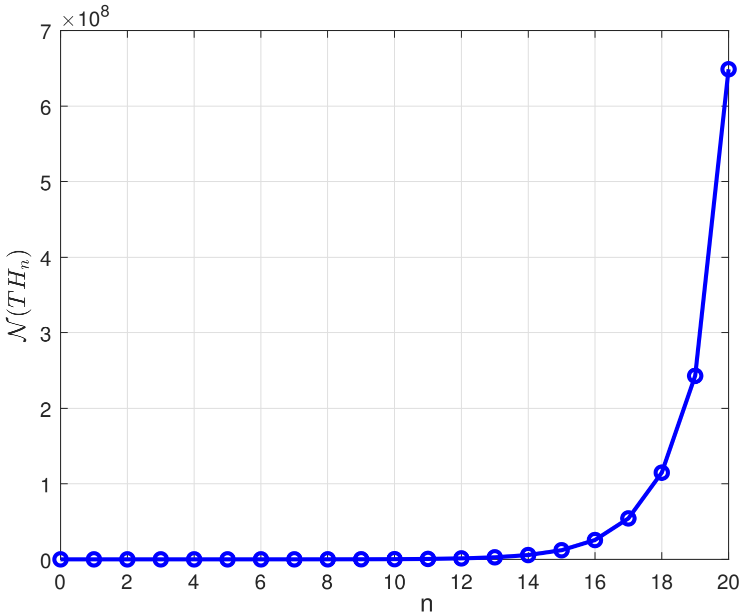

Figure 2. Norm of split Narayana–Lucas hybrid quaternion $\mathcal{N}(TH_n)$ vs. n.

5 Conclusion

In this study, we associate the concept of hybrid numbers in the context of split Narayana quaternions as well as split Narayana–Lucas quaternions. We further explore their interrelationships and derive Binet formulas, generating functions, exponential generating functions, and various well-known identities respectively related to these newly introduced quaternions. Furthermore, we provide a numerical simulation to illustrate the behaviour and implications of both split Narayana quaternions and split Narayana–Lucas hybrid quaternions.

The novelty of this work lies in its potential to formulate hybrid wavelets, which can be applied to solve both linear and nonlinear hybrid differential equations. The Caputo–Katugampola derivative, Caputo–Fabrizio derivative, generalized Caputo fractional derivative, and several other fractional derivative operators are significantly growing as fundamental tools in contemporary fractional calculus. Earlier research has already applied these operators to quaternions, showing that they work well for studying quaternionic systems. In the same way, our work expects the extension of these concepts to quaternionic hybrid constructions, where the exploration of such connections may provide interesting possibilities for future research. Besides this, the present work may contribute well to the development of designing the new cryptographic protocols for providing a robust security mechanism in real-world applications.

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