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Infinite series containing quotients of central binomial coefficients

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Abstract: By making use of the Wallis' integral formulae and integration by parts, two classes of infinite series are evaluated, in closed form, in terms of π and Riemann zeta function.
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1 Introduction and motivation

The central binomial coefficient
$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$
for $n \geq 0$ is a fundamental concept in combinatorics, probability and statistics, and many areas of mathematics. Their generating function is given by
$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$
Further properties and applications of central binomial coefficient have extensively been studied in the literature (see Gould [4], Lehmer [6] and Zucker [7], for example).

The infinite series identities for central binomial coefficients have been studied for a long time. Denoting as usual by $\mathbb{Z}^+$ the set of natural numbers, the aim of this paper is to examine, for an integer number $\lambda \in \mathbb{Z}^+$, the following two infinite series involving central binomial coefficients:

$$\mathrm{U}_\lambda = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+\lambda)\binom{2n+2\lambda}{n+\lambda}} \quad \text{and} \quad \mathrm{V}_\lambda = \sum_{n=1}^{\infty} \frac{\binom{2n+2\lambda}{n+\lambda}}{n^2\binom{2n}{n}}.$$

Throughout the paper, we shall utilize Wallis' integral formulae (cf. Bhandari [1]):

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1} x dx = \frac{2^{2m-1}}{m\binom{2m}{m}}, \tag{1}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2m} x dx = \frac{\binom{2m}{m}\pi}{2^{2m+1}}; \tag{2}$$

and two known trigonometric formulae (see Gradshteyn [5, §1.32]):

$$\sin^{2m} x = \frac{1}{2^{2m-1}} \left\{ \sum_{k=0}^{m-1} (-1)^{m+k} \binom{2m}{k} \cos(2m-2k)x + \frac{1}{2}\binom{2m}{m} \right\}, \tag{3}$$

$$\sin^{2m+1} x = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^{m+k} \binom{2m+1}{k} \sin(2m-2k+1)x. \tag{4}$$

2 Evaluation of U_λ

In this section, we shall examine the series U_λ. A general summation theorem will be proved and several explicit formulae will be presented as consequences.

According to the Wallis' integral formula (1), U_λ can be rewritten as

$$\begin{aligned}
\mathrm{U}_\lambda &= \frac{1}{2^{2\lambda-1}} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}(2n+1)} \frac{2^{2n+2\lambda}}{(n+\lambda)\binom{2n+2\lambda}{n+\lambda-1}} \\
&= \frac{1}{2^{2\lambda-1}} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}(2n+1)} \int_0^{\frac{\pi}{2}} \sin^{2n+2\lambda-1} x dx.
\end{aligned}$$

Making use of the following identity (cf. Chu [2] and Lehmer [6]):

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)} t^{2n} = \frac{\arcsin 2t}{2t},$$

the above expression can be evaluated as

$$\begin{aligned}
\mathrm{U}_\lambda &= \frac{1}{2^{2\lambda-1}} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}(2n+1)} \int_0^{\frac{\pi}{2}} \sin^{2n+2\lambda-1} x dx \\
&= \frac{1}{2^{2\lambda-1}} \int_0^{\frac{\pi}{2}} \sin^{2\lambda-2} x \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)} \frac{\sin^{2n+1} x}{2^{2n}} dx \\
&= \frac{1}{2^{2\lambda-1}} \int_0^{\frac{\pi}{2}} x \sin^{2\lambda-2} x dx.
\end{aligned} \tag{5}$$

It is obvious that U_λ is convergent only when $\lambda \geq 1$.

According to (3), we can further proceed with

$$
\begin{aligned}
\mathrm{U}_\lambda &= \frac{1}{2^{2\lambda-1}}\int_0^{\frac{\pi}{2}} x\sin^{2\lambda-2}x dx\\
&= \int_0^{\frac{\pi}{2}}\frac{x}{2^{4\lambda-4}}\left\{\sum_{k=0}^{\lambda-2}(-1)^{\lambda+k-1}\binom{2\lambda-2}{k}\cos(2\lambda-2k-2)x+\frac{1}{2}\binom{2\lambda-2}{\lambda-1}\right\}dx\\
&= \frac{\pi^2}{2^{4\lambda}}\binom{2\lambda-2}{\lambda-1}-\sum_{k=0}^{\lambda-2}\frac{(-1)^{\lambda+k}}{2^{4\lambda-4}}\binom{2\lambda-2}{k}\int_0^{\frac{\pi}{2}}x\cos(2\lambda-2k-2)xdx.
\end{aligned}
$$

By means of integration by parts, the rightmost integral can be explicitly calculated:

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x\cos(2\lambda-2k-2)xdx &= \frac{1}{2\lambda-2k-2}\int_0^{\frac{\pi}{2}} xd\sin(2\lambda-2k-2)x\\
&= -\frac{1}{2\lambda-2k-2}\int_0^{\frac{\pi}{2}}\sin(2\lambda-2k-2)xdx\\
&= \frac{1}{(2\lambda-2k-2)^2}\cos(2\lambda-2k-2)x\Big|_0^{\frac{\pi}{2}}\\
&= \frac{1}{(2\lambda-2k-2)^2}\left\{(-1)^{\lambda-k-1}-1\right\}.
\end{aligned}
$$

Remark 1. *The integral in* (5) *can also be evaluated as*

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x\sin^{2\lambda-2}xdx &= \int_0^{\pi} x\sin^{2\lambda-2}xdx-\int_{\frac{\pi}{2}}^{\pi} x\sin^{2\lambda-2}xdx\\
&= \int_0^{\pi} x\sin^{2\lambda-2}xdx-\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}+x\right)\sin^{2\lambda-2}\left(\frac{\pi}{2}+x\right)dx\\
&= \int_0^{\pi} x\sin^{2\lambda-2}xdx-\int_0^{\frac{\pi}{2}} x\cos^{2\lambda-2}xdx-\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^{2\lambda-2}xdx.
\end{aligned}
$$

The above three integrals are known: the first two of these can be found in Gradshteyn and Ryzhik's book [5, §3.821.1 and §3.821.3]; while the third one is a Wallis' integral.

Finally, we establish, after substitution, the following summation formula.

Theorem 2. *For $\lambda\in\mathbb{Z}^+$, the following equality holds:*

$$
\begin{aligned}
\mathrm{U}_\lambda &= \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)(n+\lambda)\binom{2n+2\lambda}{n+\lambda}}\\
&= \frac{\pi^2}{2^{4\lambda}}\binom{2\lambda-2}{\lambda-1}+\frac{1}{2^{4\lambda-2}}\sum_{k=0}^{\lambda-2}\frac{1+(-1)^{\lambda+k}}{(\lambda-k-1)^2}\binom{2\lambda-2}{k}.
\end{aligned}
$$

For other small $\lambda\in\mathbb{Z}^+$, the following interesting formulae are recorded as examples.

Corollary 3. *For $\lambda \in \mathbb{Z}^+$, there hold explicit formulae:*

$$\mathrm{U}_1 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+1)\binom{2n+2}{n+1}} = \frac{\pi^2}{16},$$

$$\mathrm{U}_2 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+2)\binom{2n+4}{n+2}} = \frac{1}{32} + \frac{\pi^2}{128},$$

$$\mathrm{U}_3 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+3)\binom{2n+6}{n+3}} = \frac{1}{128} + \frac{3\pi^2}{2048},$$

$$\mathrm{U}_4 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+4)\binom{2n+8}{n+4}} = \frac{17}{9216} + \frac{5\pi^2}{16384},$$

$$\mathrm{U}_5 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+5)\binom{2n+10}{n+5}} = \frac{1}{2304} + \frac{35\pi^2}{524288},$$

$$\mathrm{U}_6 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)(n+6)\binom{2n+12}{n+6}} = \frac{21}{204800} + \frac{63\pi^2}{4194304}.$$

3 Evaluation of V_λ

This section will be devoted to another series V_λ. A general summation theorem will be shown that includes several explicit formulae as special cases.

Taking into account Wallis' formula (2), we have

$$\mathrm{V}_\lambda = 2^{2\lambda} \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}} \frac{\binom{2n+2\lambda}{n+\lambda}}{2^{2n+2\lambda}} = \frac{2^{2\lambda+1}}{\pi} \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}} \int_0^{\frac{\pi}{2}} \sin^{2n+2\lambda} x dx.$$

In view of a known identity (cf. Chu [2] and Edwards [3])

$$\sum_{n=1}^{\infty} \frac{(2t)^{2n}}{n^2\binom{2n}{n}} = 2(\arcsin t)^2,$$

we can deduce the following expression:

$$\begin{aligned}
\mathrm{V}_\lambda &= \frac{2^{2\lambda+1}}{\pi} \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}} \int_0^{\frac{\pi}{2}} \sin^{2n+2\lambda} x dx \\
&= \frac{2^{2\lambda+1}}{\pi} \int_0^{\frac{\pi}{2}} \sin^{2\lambda} x \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}} \sin^{2n} x dx \\
&= \frac{2^{2\lambda+2}}{\pi} \int_0^{\frac{\pi}{2}} x^2 \sin^{2\lambda} x dx.
\end{aligned}$$

By applying (4) and the integration by parts, we can reformulate

$$\begin{aligned}
\mathrm{V}_\lambda &= \frac{2^{2\lambda+2}}{\pi} \int_0^{\frac{\pi}{2}} x^2 \sin^{2\lambda} x dx \\
&= \frac{8}{\pi} \int_0^{\frac{\pi}{2}} x^2 \left\{ \sum_{k=0}^{\lambda-1} (-1)^{\lambda+k} \binom{2\lambda}{k} \cos(2\lambda-2k)x + \frac{1}{2}\binom{2\lambda}{\lambda} \right\} dx \\
&= \frac{\pi^2}{6}\binom{2\lambda}{\lambda} + \frac{8}{\pi} \sum_{k=0}^{\lambda-1} (-1)^{\lambda+k} \binom{2\lambda}{k} \int_0^{\frac{\pi}{2}} x^2 \cos(2\lambda-2k) x dx.
\end{aligned}$$

Evaluating the last integral

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2\cos(2\lambda-2k)x dx =&\frac{1}{2\lambda-2k}\int_0^{\frac{\pi}{2}} x^2 d\sin(2\lambda-2k)x\\ =&\frac{2}{(2\lambda-2k)^2}\int_0^{\frac{\pi}{2}} x d\cos(2\lambda-2k)x\\ =&\frac{(-1)^{\lambda-k}\pi}{(2\lambda-2k)^2}-\frac{2}{(2\lambda-2k)^2}\int_0^{\frac{\pi}{2}}\cos(2\lambda-2k)x dx\\ =&\frac{(-1)^{\lambda-k}\pi}{(2\lambda-2k)^2},\end{aligned}$$

we arrive at the general formula in the following theorem.

Theorem 4. *For $\lambda \in \mathbb{Z}^+$, the following equality holds:*

$$\mathrm{V}_\lambda = \sum_{n=1}^{\infty}\frac{\binom{2n+2\lambda}{n+\lambda}}{n^2\binom{2n}{n}} = \frac{\pi^2}{6}\binom{2\lambda}{\lambda} + \sum_{k=0}^{\lambda-1}\frac{2}{(\lambda-k)^2}\binom{2\lambda}{k}.$$

For $1 \le \lambda \le 5$, the corresponding closed formulae are displayed as follows.

Corollary 5. *For $\lambda \in \mathbb{Z}^+$, there hold explicit formulae:*

$$\begin{aligned}\mathrm{V}_1 &= \sum_{n=1}^{\infty}\frac{\binom{2n+2}{n+1}}{n^2\binom{2n}{n}} = 2+\frac{\pi^2}{3},\\ \mathrm{V}_2 &= \sum_{n=1}^{\infty}\frac{\binom{2n+4}{n+2}}{n^2\binom{2n}{n}} = \frac{17}{2}+\pi^2,\\ \mathrm{V}_3 &= \sum_{n=1}^{\infty}\frac{\binom{2n+6}{n+3}}{n^2\binom{2n}{n}} = \frac{299}{9}+\frac{10\pi^2}{3},\\ \mathrm{V}_4 &= \sum_{n=1}^{\infty}\frac{\binom{2n+8}{n+4}}{n^2\binom{2n}{n}} = \frac{9209}{72}+\frac{35\pi^2}{3},\\ \mathrm{V}_5 &= \sum_{n=1}^{\infty}\frac{\binom{2n+10}{n+5}}{n^2\binom{2n}{n}} = \frac{49133}{100}+42\pi^2.\end{aligned}$$

There are two exceptional cases that are worth mentioning. First, for $\lambda = 0$, the corresponding formula is well known:

$$\mathrm{V}_0 = \zeta(2) = \sum_{n=0}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

When $\lambda = -1$, we can evaluate V_{-1} as

$$\mathrm{V}_{-1} = \sum_{n=1}^{\infty}\frac{\binom{2n-2}{n-1}}{n^2\binom{2n}{n}} = \sum_{n=1}^{\infty}\frac{1}{2n(2n-1)} = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right) = \ln 2,$$

since

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \sum_{n=1}^{\infty}\left(\frac{x^{2n-1}}{2n-1}-\frac{x^{2n}}{2n}\right), \quad -1 < x \le 1,$$

and

$$\binom{2n-2}{n-1} = \frac{n}{2(2n-1)}\binom{2n}{n}.$$

4 Concluding comments

By utilizing the formulae (1) and (2), as well as the generalized binomial coefficient, which is defined as

$$\binom{n}{m+\frac{1}{2}} \quad \text{with} \quad n, m \in \mathbb{Z},$$

the following identity can be derived

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \frac{2^{2n+1}}{(n+1)\binom{2n+2}{n+1}} = \frac{\binom{2n+1}{n+\frac{1}{2}}\pi}{2^{2n+2}}.$$

It brings about

$$\binom{2n+1}{n+\frac{1}{2}} = \frac{4}{\pi}\frac{2^{4n}}{(2n+1)\binom{2n}{n}}.$$

In this case, U_λ converges for λ a half integer with $\lambda > 0$, while V_λ converges for λ a half integer with $\lambda > -1$, since

$$\mathrm{U}_\lambda = \frac{1}{2^{2\lambda-1}}\int_0^{\frac{\pi}{2}} x \sin^{2\lambda-2} x dx \quad \text{and} \quad \mathrm{V}_\lambda = \frac{2^{2\lambda+2}}{\pi}\int_0^{\frac{\pi}{2}} x^2 \sin^{2\lambda} x dx.$$

When the absolute value of λ is small, some interesting results can be obtained. For example, letting $\lambda = \frac{1}{2}$ and $\lambda = \frac{3}{2}$ in U_λ, we have the following results:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(2n+1)2^{4n}} = \frac{4G}{\pi} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{n\binom{2n}{n}^2}{(2n-1)^2 2^{4n}} = \frac{1}{\pi},$$

where G is Catalan's constant, since (cf. Gradshteyn [5, §3.747.2])

$$\int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx = 2G \quad \text{and} \quad \int_0^{\frac{\pi}{2}} x \sin x dx = 1.$$

Letting $\lambda = -\frac{1}{2}$ in V_λ, the following identity can be established

$$\mathrm{V}_{-\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{\binom{2n-1}{n-\frac{1}{2}}}{n^2\binom{2n}{n}} = \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{2^{4n}}{n^3\binom{2n}{n}^2} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx.$$

Taking into account the following Fourier series (cf. Bhandari [1])

$$\ln \tan t = -2\sum_{k=0}^{\infty} \frac{\cos(2k+1)2t}{2k+1},$$

we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx &= \int_0^{\frac{\pi}{2}} x^2 d \ln \tan \frac{x}{2} = -2\int_0^{\frac{\pi}{2}} x \ln \tan \frac{x}{2} dx \\ &= 4\int_0^{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{x \cos(2k+1)x}{2k+1} dx. \end{aligned}$$

Recalling integration by parts, it is obtained that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx &= 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \int_0^{\frac{\pi}{2}} x d\sin(2k+1)x \\ &= 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \int_0^{\frac{\pi}{2}} \sin(2k+1)x dx \\ &= 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \\ &= 2\pi G - \frac{7}{2}\zeta(3),\end{aligned}$$

where $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ is Catalan's constant and $\zeta(3) = \sum_{k=0}^{\infty} \frac{1}{k^3}$ is Riemann zeta function. This leads to the following interesting identity:

$$\sum_{n=1}^{\infty} \frac{2^{4n}}{n^3 \binom{2n}{n}^2} = 8\pi G - 14\zeta(3).$$

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