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Theta function identities involving fourth power

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Abstract: On page 241 of his Second Notebook, Ramanujan recorded one of his theta function identity, which involves the ratio of the fourth power of theta functions with respect to $\psi(q)$. In this article, we give a new proof for this theta function identity. Also, we give a new proof of another identity with respect to $\varphi(q)$ established by B. C. Berndt and we establish two new theta function identities analogous to Ramanujan's theta function identities.

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1 Introduction

As usual for any complex number a and q with $|q| < 1$, we define

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, n \text{ is an integer.}$$

Ramanujan defined his theta function

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1.$$

By Jacobi's triple product identity, we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Further, Ramanujan consider following three interesting special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty.$$

He also defined

$$\chi(q) = (-q;q^2)_\infty.$$

On page 241, of his second notebook [3], Ramanujan recorded the following interesting theta function identity:

$$1 + \frac{\chi^9(-q^3)}{q\chi^3(-q)} = \frac{\psi^4(q)}{q\psi^4(q^3)}. \tag{1}$$

For more details of the above, one may refer to [2]. The following similar theta function identity with respect to $\varphi(q)$ can be found in [2, p. 347]:

$$1 - 8q\frac{\chi^3(-q)}{\chi^9(-q^3)} = \frac{\varphi^4(-q)}{\varphi^4(-q^3)}. \tag{2}$$

The main objective of this paper is to give a simple and alternating proof of (1) and (2). Further, we prove the following two theta function identities:

$$\frac{\varphi^4(-q)}{\varphi^4(-q^5)} - 1 = 8q\frac{\chi^3(-q)}{\chi^7(-q^5)}\left[\frac{\chi^2(-q^5)}{\chi^2(-q)} - 2q\frac{1}{\chi(-q)\chi^3(-q^5)}\right] \tag{3}$$

and

$$\frac{\psi^4(q^2)}{q^4\psi^4(q^{10})} - 1 = \frac{\chi^7(-q^{10})}{q^4\chi^3(-q^2)}\left[-\chi(-q^2)\chi^3(-q^{10}) + 2\frac{f_4}{f_{20}^3}\right]. \tag{4}$$

.

The identities (3) and (4) seem to be new.

2 Main results

In this section, we prove Identities (1)–(4). In our proof, we require the following theta function identity due to Ramanujan from Entry 25(vii) of Chapter 16 of [2, p. 40]:

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{5}$$

A proof of the above can be found in [2, p. 40–41].

Proof of (2): From Entry 8(ii) in Chapter 17 of [3, p. 209], we have

$$\frac{1}{8}\left(1 - \varphi^4(-q)\right) = \sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} - 2\sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2},$$

which implies that

$$\varphi^4(-q) = 1 - 8\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} + 16\sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2}. \tag{6}$$

We have

$$\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} = \sum_{n=0}^{\infty} \frac{q^{3n+1}}{(1+q^{3n+1})^2} + \sum_{n=0}^{\infty} \frac{q^{3n+2}}{(1+q^{3n+2})^2} + \sum_{n=1}^{\infty} \frac{q^{3n}}{(1+q^{3n})^2}$$

and

$$\sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} = \sum_{n=0}^{\infty} \frac{q^{6n+2}}{(1+q^{6n+2})^2} + \sum_{n=0}^{\infty} \frac{q^{6n+4}}{(1+q^{6n+4})^2} + \sum_{n=1}^{\infty} \frac{q^{6n}}{(1+q^{6n})^2}.$$

Using the above two equations in (6), we find that

$$\begin{aligned}\varphi^4(-q) - \varphi^4(-q^3) = &-8\left(\sum_{n=0}^{\infty} \frac{q^{3n+1}}{(1+q^{3n+1})^2} + \sum_{n=0}^{\infty} \frac{q^{3n+2}}{(1+q^{3n+2})^2}\right)\\ &+ 16\left(\sum_{n=0}^{\infty} \frac{q^{6n+2}}{(1+q^{6n+2})^2} + \sum_{n=0}^{\infty} \frac{q^{6n+4}}{(1+q^{6n+4})^2}\right),\end{aligned}$$

which implies that

$$\begin{aligned}\varphi^4(-q) - \varphi^4(-q^3) &= -8\left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n+1}}{(1+q^{3n+1})^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{3n-1}}{(1+q^{3n-1})^2}\right)\\ &= -8\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n+1}}{(1+q^{3n+1})^2} \qquad (7)\\ &= -8\left(\sum_{n=-\infty}^{\infty} \frac{q^{6n+1}}{(1+q^{6n+1})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{6n+4}}{(1+q^{6n+4})^2}\right).\end{aligned}$$

Bailey's summation formula [1] is given by

$$\sum_{n=-\infty}^{\infty} \frac{aq^n}{(1-aq^n)^2} - \sum_{n=-\infty}^{\infty} \frac{bq^n}{(1-bq^n)^2} = af^6(-q)\frac{f(-ab, -\frac{q}{ab})f(-\frac{b}{a}, -\frac{aq}{b})}{f^2(-a, -\frac{q}{a})f^2(-b, -\frac{q}{b})}. \tag{8}$$

For a proof of the above, one may refer to [4]. Replacing q by q^6 and then substituting $a = -q$ and $b = -q^4$ in Bailey's summation formula, we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{6n+1}}{(1+q^{6n+1})^2} - \sum_{n=-\infty}^{\infty} \frac{q^{6n+4}}{(1+q^{6n+4})^2} = qf_6^6 \frac{f(-q,-q^5)f(-q^3,-q^3)}{f^2(q,q^5)f^2(q^2,q^4)}.$$

Using the above in (7), we have

$$\begin{aligned}
\varphi^4(-q) - \varphi^4(-q^3) &= -8qf_6^6 \frac{f(-q,-q^5)f(-q^3,-q^3)}{f^2(q,q^5)f^2(q^2,q^4)} \\
&= -8q \frac{(q;q^2)_\infty (q^3;q^6)_\infty (-q^3;q^3)_\infty^2}{(-q;q^2)_\infty^2} \\
&= -8qf_6^4 \frac{\chi^3(-q)}{\chi(-q^3)}.
\end{aligned}$$

On dividing the above equation throughout by $\varphi^4(-q^3)$, we obtain the required result. □

Proof of (1): From (5), we have

$$16q\psi^4(q^2) - 16q^3\psi^4(q^6) = [\varphi^4(q) - \varphi^4(q^3)] - [\varphi^4(-q) - \varphi^4(-q^3)].$$

From (7), it follows that

$$16q\psi^4(q^2) - 16q^3\psi^4(q^6) = 8\bigg(\sum_{n=-\infty}^{\infty} \frac{q^{6n+1}}{(1-q^{6n+1})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{6n+1}}{(1+q^{6n+1})^2} \bigg).$$

Changing q to q^6 and then substituting $a = q$, and $b = -q$ in Bailey's summation formula (8), we find that

$$2q\psi^4(q^2) - 2q^3\psi^4(q^6) = qf_6^6 \frac{f(q^2,q^4)f(1,q^6)}{f^2(-q,-q^5)f^2(q,q^5)},$$

which implies that

$$\begin{aligned}
\psi^4(q^2) - q^2\psi^4(q^6) &= f_6^6 \frac{(-q^2;q^2)_\infty (q^6;q^{12})_\infty^2}{(q^2;q^4)_\infty^2 (q^6;q^6)_\infty^2} \\
&= f_6^4 \frac{\chi^2(-q^6)}{\chi^3(-q^2)}.
\end{aligned}$$

On dividing the above equation throughout by $q^2\psi^4(q^6)$ and changing q to $q^{\frac{1}{2}}$, we obtain the required result. □

Proof of (3): We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} = \sum_{n=0}^{\infty} \frac{q^{5n+1}}{(1+q^{5n+1})^2} + \sum_{n=0}^{\infty} \frac{q^{5n+2}}{(1+q^{5n+2})^2} + \sum_{n=0}^{\infty} \frac{q^{5n+3}}{(1+q^{5n+3})^2} \\
+ \sum_{n=0}^{\infty} \frac{q^{5n+4}}{(1+q^{5n+4})^2} + \sum_{n=1}^{\infty} \frac{q^{5n}}{(1+q^{5n})^2}.
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} = \sum_{n=0}^{\infty} \frac{q^{10n+2}}{(1+q^{10n+2})^2} + \sum_{n=0}^{\infty} \frac{q^{10n+4}}{(1+q^{10n+4})^2} + \sum_{n=0}^{\infty} \frac{q^{10n+6}}{(1+q^{10n+6})^2} \\
+ \sum_{n=0}^{\infty} \frac{q^{10n+8}}{(1+q^{10n+8})^2} + \sum_{n=1}^{\infty} \frac{q^{10n}}{(1+q^{10n})^2}.
\end{aligned}$$

Using the above two equations in (6), we obtain

$$\begin{aligned}\varphi^4(-q)-\varphi^4(-q^5)=&-8\sum_{n=-\infty}^{\infty}\frac{q^{5n+1}}{(1+q^{5n+1})^2}-8\sum_{n=-\infty}^{\infty}\frac{q^{5n+2}}{(1+q^{5n+2})^2}\\&+16\sum_{n=-\infty}^{\infty}\frac{q^{10n+1}}{(1+q^{10n+1})^2}+16\sum_{n=-\infty}^{\infty}\frac{q^{10n+4}}{(1+q^{10n+4})^2},\end{aligned}$$

which implies

$$\begin{aligned}\varphi^4(-q)-\varphi^4(-q^5)=&-8\sum_{n=-\infty}^{\infty}\frac{q^{10n+1}}{(1+q^{10n+1})^2}+8\sum_{n=-\infty}^{\infty}\frac{q^{10n+2}}{(1+q^{10n+2})^2}\\&-8\sum_{n=-\infty}^{\infty}\frac{q^{10n+3}}{(1+q^{10n+3})^2}+8\sum_{n=-\infty}^{\infty}\frac{q^{10n+4}}{(1+q^{10n+4})^2}.\end{aligned}\tag{9}$$

Changing q to q^{10} and then substituting $(a,b)=(-q,-q^2)$ and $(-q^3,-q^4)$ in Bailey's summation formula (8), respectively, we find that

$$\sum_{n=-\infty}^{\infty}\frac{q^{10n+1}}{(1+q^{10n+1})^2}-\sum_{n=-\infty}^{\infty}\frac{q^{10n+2}}{(1+q^{10n+2})^2}=qf_{10}^6\frac{f(-q,-q^9)f(-q^3,-q^7)}{f^2(q,q^9)f^2(q^2,q^8)}$$

and

$$\sum_{n=-\infty}^{\infty}\frac{q^{10n+3}}{(1+q^{10n+3})^2}-\sum_{n=-\infty}^{\infty}\frac{q^{10n+4}}{(1+q^{10n+4})^2}=q^3f_{10}^6\frac{f(-q,-q^9)f(-q^3,-q^7)}{f^2(q^3,q^7)f^2(q^4,q^6)}.$$

Employing the above two equations in (9), we obtain

$$\begin{aligned}\varphi^4(-q)-\varphi^4(-q^5)&=8qf_{10}^6\frac{f(-q,-q^9)f(-q^3,-q^7)}{f^2(q,q^9)f^2(q^2,q^8)}+8q^3f_{10}^6\frac{f(-q,-q^9)f(-q^3,-q^7)}{f^2(q^3,q^7)f^2(q^4,q^6)}\\&=8qf_{10}^6\frac{(q;q^2)_\infty(q^{10};q^{10})_\infty^2}{(q^5;q^{10})_\infty}\times\\&\qquad\left[\frac{f^2(q^3,q^7)f^2(q^4,q^6)+q^2f^2(q,q^9)f^2(q^2,q^8)}{f^2(q,q^9)f^2(q^2,q^8)f^2(q^3,q^7)f^2(q^4,q^6)}\right]\\&=8q\frac{\chi^3(-q)}{\chi^3(-q^5)}\left[f^2(q^3,q^7)f^2(q^4,q^6)+q^2f^2(q,q^9)f^2(q^2,q^8)\right].\end{aligned}\tag{10}$$

From Entry 29 of Chapter 16 of Ramanujan's notebook [3], we have

$$f(a,b)f(c,d)=f(ac,bd)f(ad,bc)+af\left(\frac{b}{c},\frac{c}{b}abcd\right)f\left(\frac{b}{d},\frac{d}{b}abcd\right).$$

Setting $a=q$, $b=q^4$, $c=q^2$, and $d=q^3$ in the above equation and then squaring the both side of the resulting identity, we obtain

$$\begin{aligned}&f^2(q^3,q^7)f^2(q^4,q^6)+q^2f^2(q,q^9)f^2(q^2,q^8)\\&\qquad=f^2(q^2,q^3)f^2(q,q^4)-2qf(q^3,q^7)f(q^4,q^6)f(q^2,q^8)f(q,q^9).\end{aligned}$$

Using the above equation in (10), we find that

$$\varphi^4(-q)-\varphi^4(-q^5)=8q\frac{\chi^3(-q)}{\chi^3(-q^5)}\left[\frac{(-q;q)_\infty^2}{(-q^5;q^5)_\infty^2}f_5^4-2q\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}f_{10}^4\right].$$

On dividing the above equation throughout by $\varphi^4(-q^5)$, we obtain the required result. □

Proof of (4): From (3), we have

$$16q\psi^4(q^2) - 16q^5\psi^4(q^{10}) = [\varphi^4(q) - \varphi^4(q^5)] - [\varphi^4(-q) - \varphi^4(-q^5)].$$

Employing (9) in the above equation, we obtain

$$\begin{aligned} 16q\psi^4(q^2) - 16q^5\psi^4(q^{10}) = 8\Bigg(&\sum_{n=-\infty}^{\infty} \frac{q^{10n+1}}{(1-q^{10n+1})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{10n+1}}{(1+q^{10n+1})^2} \\ &+ \sum_{n=-\infty}^{\infty} \frac{q^{10n+1}}{(1-q^{10n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{10n+3}}{(1+q^{10n+3})^2} \Bigg). \end{aligned} \tag{11}$$

Changing q to q^{10} and then substituting $(a, b) = (q, -q^3)$ and $(q^3, -q)$ in Bailey's summation formula (8), respectively, we find that

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n+1}}{(1-q^{10n+1})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{10n+3}}{(1+q^{10n+3})^2} = qf_{10}^6 \frac{f(q^4, q^6)f(q^2, q^8)}{f^2(-q, -q^9)f^2(q^3, q^7)}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{q^{10n+3}}{(1-q^{10n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{q^{10n+1}}{(1+q^{10n+1})^2} = qf_{10}^6 \frac{f(q^4, q^6)f(q^2, q^8)}{f^2(q, q^9)f^2(-q^3, -q^7)}.$$

Employing above two equations in (11), we obtain

$$\begin{aligned} 2q\psi^4(q^2) - 2q^5\psi^4(q^{10}) &= qf_{10}^6 f(q^4, q^6)f(q^2, q^8)\times \\ &\quad \left[\frac{f^2(-q^3, -q^7)f^2(q, q^9) + f^2(-q, -q^9)f^2(q^3, q^7)}{f^2(-q, -q^9)f^2(-q^3, -q^7)f^2(q, q^9)f^2(q^3, q^7)}\right] \\ &= q\frac{\chi^3(-q^{10})}{\chi^3(-q^2)} \left[f^2(-q^3, -q^7)f^2(q, q^9) + f^2(-q, -q^9)f^2(q^3, q^7)\right]. \end{aligned} \tag{12}$$

Setting $a = -q^3$, $b = -q^7$, $c = q$, and $d = q^3$ in Entry 29 of Chapter 16 of Ramanujan's notebook [3], we find that

$$f^2(-q^3, -q^7)f^2(q, q^9) + f^2(-q, -q^9)f^2(q^3, q^7) = -2\frac{(q; q^2)_\infty(-q; q^2)_\infty}{(q^5; q^{10})_\infty(-q^5; q^{10})_\infty} f_{10}^4 + 4f_4f_{20}.$$

Using the above in (12), we obtain

$$2q\psi^4(q^2) - 2q^5\psi^4(q^{10}) = q\frac{\chi^3(-q^{10})}{\chi^3(-q^2)} \left[-2\frac{\chi(-q^2)}{\chi(-q^{10})} f_{10}^4 + 4f_4f_{20}\right].$$

On dividing the above equation throughout by $q^5\psi^4(q^{10})$, we obtain the required result. □

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