

# On a new congruence in the Catalan triangle

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**Abstract:** For  $0 \leq k \leq n$ , the number  $C(n, k)$  represents the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, k)$ , using steps  $(1, 0)$  and  $(0, 1)$ , that never rise above the main diagonal  $y = x$ . The Fuss–Catalan number of order three  $C_n^{(3)}$  represents the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(2n, n)$ , using steps  $(1, 0)$  and  $(0, 1)$ , that do not rise above the line  $y = \frac{x}{2}$ . The generalized Schröder number  $\text{Schr}(n, m, 2)$  of order two represents the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, m)$ , using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , that never go below the line  $y = 2x$ . We present a new alternating convolution formula for the numbers  $C(2n, k)$  multiplied by a power of a binomial coefficient. Using a new class of binomial sums that we call  $M$  sums, we prove that this sum is divisible by  $C_n^{(3)}$  and by the central binomial coefficient  $\binom{2n}{n}$ . We do this by examining the numbers  $T(n, j) = \frac{1}{2n+1} \binom{2n+j}{j} \binom{2n+1}{n+j+1}$ , for which we present a new combinatorial interpretation, connecting them to the generalized Schröder numbers of order two.

**Keywords:** Catalan triangle, Fuss–Catalan number of order three, Catalan number, Central binomial coefficient,  $M$  sum, Schröder number of order two, Lattice path, Induction principle.

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## 1 Introduction

Let  $C_n = \frac{1}{n+1} \binom{2n}{n}$  denote the  $n$ -th Catalan number. Catalan numbers form the famous sequence [17] with the most applications in combinatorics, after binomial coefficients. For example,  $C_n$  is



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the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, n)$ , using steps  $(1, 0)$  and  $(0, 1)$ , that never rise above the main diagonal  $y = x$ .

This sequence starts with:  $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$ , etc., and can be found as sequence A000108 in [16] (see also [2, Introduction]).

Let  $n$  and  $k$  be non-negative integers such that  $k \leq n$ . The number  $C(n, k)$  represents the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, k)$ , using steps  $(1, 0)$  and  $(0, 1)$ , that never rise above the main diagonal  $y = x$ . The numbers  $C(n, k)$  therefore form the Catalan triangle, where the Catalan numbers appear on the main diagonal:

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 2 & & & & \\ 1 & 3 & 5 & 5 & & & \\ 1 & 4 & 9 & 14 & 14 & & \\ 1 & 5 & 14 & 28 & 42 & 42 & \\ 1 & 6 & 20 & 48 & 90 & 132 & 132 \\ \vdots & & & & & & \end{array}$$

Furthermore, the following two recurrences hold for  $C(n, k)$ :

$$\begin{aligned} C(n+1, k) &= C(n+1, k-1) + C(n, k), \quad 1 < k < n+1 \text{ and} \\ C(n+1, n+1) &= C(n+1, n). \end{aligned}$$

It is well known that  $\sum_{k=0}^n C(n, k) = C_{n+1}$ . It follows that the sum of elements of every row of the Catalan triangle is again a Catalan number.

An explicit expression for  $C(n, k)$  is

$$C(n, k) = \frac{n-k+1}{n+1} \binom{n+k}{n}. \quad (1)$$

The number  $C(n, k)$  is always an integer due to the fact that

$$C(n, k) = \binom{n+k}{n} - \binom{n+k}{n+1}. \quad (2)$$

The numbers  $C(n, k)$  [6, Section 15, p. 347] can also be defined in terms of binary words.

Other identities exist with the numbers  $C(n, k)$  (see, for example, [7, Theorems 1.1, 1.2; p. 2]). Note that there are some other triangles that are also called Catalan triangles. For example, there is another Catalan triangle [15] introduced by Shapiro (see also [6, Section 14, p. 333] and [8]).

Interestingly, one can construct the Catalan triangle by using a small change in the rules for constructing Pascal's like triangles (see the book [1]).

Let  $C_n^{(3)} = \frac{1}{2n+1} \binom{3n}{n}$  denote the Fuss–Catalan number of order three. The number  $C_n^{(3)}$  counts all lattice paths from the point  $(0, 0)$  to the point  $(2n, n)$ , using steps  $(1, 0)$  and  $(0, 1)$ , that do not rise above the line  $y = \frac{x}{2}$  (see also [2, Eq. (2.2), p. 5]). This sequence starts with:  $C_0^{(3)} = 1, C_1^{(3)} = 1, C_2^{(3)} = 3, C_3^{(3)} = 12, C_4^{(3)} = 55$ , etc., and it can be found as sequence A001764 in [16].

The central binomial coefficient  $\binom{2n}{n}$  represents the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, n)$ , using steps  $(1, 0)$  and  $(0, 1)$ . This sequence starts with 1, 2, 6, 20, 70, etc., and it can be found as sequence A000984 in [16].

In this article, we present a connection between the numbers  $C(2n, k)$  and  $C_n^{(3)}$ . Let  $n$  be a non-negative integer, and let  $m$  be a natural number. Let us consider the following alternating convolution for the numbers  $C(2n, k)$ :

$$S(2n, m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m C(2n, k) \cdot C(2n, 2n - k). \quad (3)$$

We present a new alternating convolution formula in the Catalan triangle as our first main result.

**Theorem 1.** *For any  $n \in \mathbb{N}_0$ , we have*

$$S(2n, 1) = (-1)^n \cdot C_n^{(3)} \cdot C_n \cdot (2n^2 + n + 1). \quad (4)$$

Our other two main results are the following theorems.

**Theorem 2.** *The sum  $S(2n, m)$  is divisible by  $C_n^{(3)}$  for any non-negative integer  $n$  and for any natural number  $m$ .*

**Theorem 3.** *The sum  $S(2n, m)$  is divisible by  $\binom{2n}{n}$  for any non-negative integer  $n$  and for any natural number  $m$ .*

For the proofs of Theorems 1, 2, and 3, we use a new class of binomial sums that we call  $M$  sums.  $M$  sums represent the sums associated with certain  $S$  sums which are multiparametric generalizations  $S(n, m; a_1, a_2, \dots, a_l)$  of sums of powers of binomial coefficients multiplied by an integer-valued function  $F(n, k; a_1, a_2, \dots, a_l)$ . More precisely, the  $M$  sums depend on three main parameters  $n, j, t$ , and on the function  $F$ . The  $M$  sums satisfy the main recurrence and can be used to derive formulas for  $S$  sums by setting  $j = 0$ . Furthermore, by using  $M$  sums, one can prove various results of the divisibility of a certain sum  $S$  by the central binomial coefficient  $\binom{2n}{n}$ , by the half of the the super-Catalan number  $T(n, m) = \frac{\binom{2n}{n} \binom{2m}{m}}{\binom{n+m}{n}}$ , or by the binomial coefficient  $\binom{a+n}{a}$ .

In 1998 Calkin [4, Thm. 1] proved that the following sum  $S_1(2n, m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m$  is divisible by the central binomial coefficient  $\binom{2n}{n}$  for any non-negative integer  $n$  and for any natural number  $m$ . For  $1 \leq m \leq 3$ , the sum  $S_1(2n, m)$  has a closed form. Especially, by the original Dixon formula,  $S_1(2n, 3) = (-1)^n \binom{2n}{n} \binom{3n}{n}$ . By using asymptotic methods, Bruijn [3] has been proved that no closed form exists for  $S_1(2n, m)$  for  $m \geq 4$ . However, there exist formulas [5, Eq. (5.12), Eq. (5.13)] for  $S_1(2n, 4)$  and  $S_1(2n, 5)$ . These formulas are derived by using Rogers–Ramanujan sums.

The  $M$  sums give [9, Chapter 5, p. 50] an elementary proof of the Calkin result. By using  $M$  sums, one can easily derive [9, Eq. (26), p. 52] the original Dixon formula. Furthermore, by using  $M$  sums, one can derive formulas for  $S_1(2n, m)$  for  $m \geq 4$ .

Let us consider the  $M(2n, j, t)$  sums associated to sums  $S(2n, m)$  from Equation (3). For  $t = 0$ , numbers  $T(n, j) = \frac{1}{2n+1} \binom{2n+j}{j} \binom{2n+1}{n+j+1}$  appear in such  $M$  sums. The numbers  $T(n, j)$  are integers for all non-negative integers  $j$  such that  $j \leq n$ . Note that  $T(n, 0) = C_n$  and  $T(n, n) = C_n^{(3)}$ .

Let us recall that, for  $m \geq 2n$ , the generalized Schröder numbers  $\text{Schr}(n, m, 2)$  of order two represent the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, m)$ , using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , that never go below the line  $y = 2x$ . Furthermore, let  $\text{Schr}(n, m, j, 2)$  denote the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, m)$ , using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , that never go below the line  $y = 2x$ , and with exactly  $j$  steps of the form  $(1, 0)$ .

Our fourth main result is a new combinatorial interpretation of  $T(n, j)$ .

**Theorem 4.** *Let  $n$  be a natural number, and let  $j$  be a non-negative integer such that  $j \leq n$ . The number  $T(n, j) = \frac{1}{2n+1} \binom{2n+j}{j} \binom{2n+1}{n+j+1}$  represents the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, 2n)$ , using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , that never go below the line  $y = 2x$  and with exactly  $j$  steps of the form  $(1, 0)$ .*

Finally, we shall prove a more general result than Theorem 4.

**Theorem 5.** *Let  $n$  be a natural number, and let  $j$  be a non-negative integer such that  $j \leq n$ . Let  $m$  be a natural number such that  $m \geq 2n$ . Then*

$$\text{Schr}(n, m, j, 2) = \frac{m - 2n + 1}{n} \binom{n}{j} \binom{m + j}{n - 1}. \quad (5)$$

## 2 A new class of binomial sums

In this section we define a crucial tool in our research, namely the  $M$  sums. We start with the following definition.

**Definition 6.** Let  $S(n, m, a) = \sum_{k=0}^n \binom{n}{k}^m \cdot F(n, k, a)$ , where  $F(n, k, a)$  is an integer-valued function and the number  $a$  is also an integer. Then  $M$  sums for the sum  $S(n, m, a)$  are given as follows:

$$M_S(n, j, t; a) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \binom{n}{j+v}^t F(n, j+v, a), \quad (6)$$

where  $j$  and  $t$  are non-negative integers such that  $j \leq \lfloor \frac{n}{2} \rfloor$ .

Obviously,

$$S(n, m, a) = M_S(n, 0, m-1; a). \quad (7)$$

It is known that the  $M$  sums [12, Eq. (8), p. 3] satisfy the following main recurrence:

$$M_S(n, j, t+1; a) = \binom{n}{j} \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{u} M_S(n, j+u, t; a). \quad (8)$$

In one particular situation, Equation (8) implies a simple consequence which is important for us.

Let  $n$  and  $a$  be fixed non-negative integers. Let  $t_0$  be a non-negative integer, and let  $j$  be an arbitrary integer in the range  $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$ . Suppose that  $q = q(n, a)$  is a positive integer which divides  $M_S(n, j, t_0; a)$  sums for all  $j$  in the given range. We want  $q$  to be as large as possible. Then, by using Eqns. (7), (8), and the induction principle, it can be shown that  $q$  divides  $S(n, t+1, a)$  for all  $t$  such that  $t \geq t_0$ .

Furthermore, let  $S(n, m, a) = \sum_{k=0}^n \binom{n}{k}^m F(n, k, a)$  be a sum from Definition 6, and let  $P(n, m, a) = \sum_{k=0}^n \binom{n}{k}^m \binom{a+k}{a} \binom{a+n-k}{a} F(n, k, a)$ .

Then, the following equation is true [12, Thm. 3, p. 3]:

$$M_P(n, j, 0; a) = \binom{a+j}{a} \sum_{l=0}^a \binom{n-j+l}{l} \binom{n-j}{a-l} M_S(n, j+a-l, 0; a). \quad (9)$$

We use the following agreement. If the sum  $S(n, m, a)$  does not depend on  $a$ , we shall write  $S(n, m)$  instead of  $S(n, m, a)$ , and  $F(n, k)$  instead of  $F(n, k, a)$ . Similarly, in that case, we shall write  $M_S(n, j, 0)$  instead of  $M_S(n, j, 0; a)$ .

Let  $S(2n, m)$  be our main sum from Equation (3).

We shall prove the following lemma.

**Lemma 7.**

$$M_S(2n, j, 0) = (-1)^n \cdot C_n^{(3)} \cdot \frac{\binom{2n+j}{j} \cdot \binom{2n+1}{n+j+1}}{2n+1} \cdot (2n^2 + n + 1 - j(n-1)). \quad (10)$$

Let  $Q(2n, m, a)$  denote the sum  $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{a+k}{a} \binom{a+2n-k}{a}$ . It is known that  $Q(2n, m, a)$  is divisible by  $\text{lcm}(\binom{a+n}{a}, \binom{2n}{n})$  for any non-negative integer  $n$  and for any natural number  $m$  (see [9, Thm. 12, p. 55] and [12, Remark 4, p. 860]). Furthermore, by using Equation (9), the following equations can be seen to hold:

$$M_Q(2n, j, 0; a) = (-1)^n \binom{a+n}{a} \binom{a+j}{j} \binom{a}{n-j}, \quad (11)$$

$$M_Q(2n, j, 1; a) = (-1)^n \binom{2n}{n} \sum_{u=0}^{n-j} \binom{n}{j+u} \binom{j+u}{u} \binom{a+j+u}{j+u} \binom{a+n}{2n-j-u}. \quad (12)$$

There are several applications of  $M$  sums for proving various congruences (see, for example, [12, Background], [9], and [10]).

Recently,  $M$  sums have been used to prove a new theorem [11, Thm. 1 and Thm. 2, p. 5] from the combinatorial number theory. The first part of this theorem consists of two equations [13, Thm. 1.2, Eq. (1.8) and Eq. (1.9), p. 60] that are published recently.

It is well known that the largest exponent of two that divides  $\binom{2n}{n}$  is equal to number of ones in the binary representation of  $n$ . By using a new theorem from the number theory [11, Thm. 1, p. 5], we can confirm this result.

Central Delannoy numbers  $D_n$  count all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, n)$  by using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . Such paths are often called royal paths. Large Schröder numbers  $S_n$  count all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, n)$  by using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$  such that never go above the main diagonal  $y = x$ . Finally, little Schröder numbers  $s_n$  count all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, n)$  by using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$  such that never go above the main diagonal  $y = x$  and on the main diagonal steps of the form  $(1, 1)$  are forbidden.

By using a new theorem from the number theory, we can calculate the largest exponent of three that divides central Delannoy numbers [11, Thm. 5, p. 6]. Furthermore, by using a variation of the new theorem from the number theory, we can calculate the largest exponent of three that divides little and big Schröder numbers [11, Thm. 6, p. 6]. Note that these results are completely new.

By using  $M$  sums, one can easily derive the less-known third formula for central Delannoy numbers. The first two formulas for central Delannoy numbers can be found in [13, Eq. (1.4) and Eq. (1.5), p. 59]. The third formula for central Delannoy numbers can be derived from [11, Eq. (41), p. 8] by setting  $j := 0$ ,  $a := 2$ , and  $b := 1$ .

### 3 Auxiliary results

We shall also prove the following auxiliary propositions.

**Proposition 8.**  *$T(n, j)$  is an integer for any non-negative integers  $n$  and  $j$  such that  $j \leq n$ . Furthermore, if  $n$  is a natural number, then*

$$T(n, j) = \frac{1}{n} \binom{n}{j} \binom{2n+j}{n-1}.$$

**Proposition 9.** *Let  $n$  be a non-negative integer. Then  $\frac{2\binom{3n}{n}}{(n+1)(2n+1)}$  is an integer.*

**Proposition 10.** *Let  $n$  and  $t$  be non-negative integers. Then  $t \cdot \binom{2n+t}{t}$  is divisible by  $2n+1$ .*

**Proposition 11.** *Let  $n$  and  $t$  be non-negative integers. Then  $\binom{3n}{n+t} \cdot \binom{2n+t}{2n}$  is divisible by  $2n+1$ .*

**Proposition 12.** *Let  $n$  and  $t$  be non-negative integers. Then  $\binom{3n+1}{n+t+1} \cdot \binom{2n+t}{2n}$  is divisible by  $2n+1$ .*

The rest of the paper is structured as follows. In Section 4, we give a proof of Lemma 7. In Section 5, we give proofs of Theorems 1 and 2 and the proof of Proposition 8. In Section 6, we prove Theorem 3, as well as Propositions 9, 10, 11, and 12. Finally, in Section 7, we give proofs of Thms. 5 and 4.

### 4 Proof of Lemma 7

Let  $S(2n, m)$  denote the sum from Equation (3). Obviously,  $S(2n, m)$  corresponds to Definition 6, where  $F(2n, k) = (-1)^k \cdot C(2n, k) \cdot C(2n, 2n-k)$ .

By setting  $t = 0$  in Equation (6), we obtain that:

$$M_S(2n, j, 0) = \binom{2n-j}{j} \sum_{v=0}^{2n-2j} \binom{2n-2j}{v} F(2n, j+v), \quad (13)$$

where  $j$  and  $t$  are non-negative integers such that  $j \leq n$ .

By Equation (1), we can rewrite  $F(2n, k)$  as

$$F(2n, k) = \frac{1}{(2n+1)^2} \cdot F_1(2n, k, 1), \quad (14)$$

where  $F_1(2n, k, 1)$  is equal to

$$(-1)^k \cdot (2n-k+1)(k+1) \cdot \binom{2n+k}{2n} \cdot \binom{4n-k}{2n}. \quad (15)$$

Now, let  $S_1(2n, m, 1)$  denote the following sum  $\sum_{k=0}^{2n} \binom{2n}{k}^m F_1(2n, k, 1)$ .

By Eqs. (13) and (14), we obtain that

$$M_S(2n, j, 0) = \frac{1}{(2n+1)^2} \cdot M_{S_1}(2n, j, 0; 1). \quad (16)$$

More precisely, by Equation (15), the sum  $S_1(2n, m, 1)$  can be written as:

$$S_1(2n, m, 1) = \sum_{k=0}^{2n} \binom{2n}{k}^m \binom{1+k}{1} \cdot \binom{1+2n-k}{1} \cdot (-1)^k \cdot \binom{2n+k}{2n} \cdot \binom{4n-k}{2n}. \quad (17)$$

By setting  $P := S_1$ ,  $a = 1$ , and  $F_2(2n, k, 1) := (-1)^k \cdot \binom{2n+k}{2n} \cdot \binom{4n-k}{2n}$  in Equation (9), we obtain that:

$$M_{S_1}(2n, j, 0; 1) = \binom{1+j}{1} \sum_{l=0}^1 \binom{2n-j+l}{l} \binom{2n-j}{1-l} M_{S_2}(2n, j+1-l, 0; 1), \quad (18)$$

where

$$S_2(2n, m, 1) = \sum_{k=0}^{2n} \binom{2n}{k}^m (-1)^k \cdot \binom{2n+k}{2n} \cdot \binom{4n-k}{2n}. \quad (19)$$

Note that:

$$S_2(2n, m, 1) = Q(2n, m, 2n). \quad (20)$$

Hence, by Equation (5), it follows

$$M_{S_2}(2n, j, 0; 1) = M_Q(2n, j, 0; 2n). \quad (21)$$

Therefore, by Equation (18), it follows that  $M_{S_1}(2n, j, 0; 1)$  is equal to

$$(1+j) \sum_{l=0}^1 \binom{2n-j+l}{l} \binom{2n-j}{1-l} M_Q(2n, j+1-l, 0; 2n). \quad (22)$$

The first summand in Equation (22), for  $l = 0$ , is equal to

$$(1+j) \cdot (2n-j) \cdot M_Q(2n, j+1, 0; 2n). \quad (23)$$

By setting  $a := 2n$  and  $j := j+1$  in Equation (11), it follows that

$$M_Q(2n, j+1, 0; 2n) = (-1)^n \binom{3n}{2n} \binom{2n+j+1}{j+1} \binom{2n}{n-j-1}. \quad (24)$$

Similarly, the second summand in Equation (22), for  $l = 1$ , is equal to

$$(1+j) \cdot (2n-j+1) \cdot M_Q(2n, j, 0; 2n). \quad (25)$$

By setting  $a := 2n$  in Equation (11), we obtain that

$$M_Q(2n, j, 0; 2n) = (-1)^n \binom{3n}{2n} \binom{2n+j}{j} \binom{2n}{n-j}. \quad (26)$$

Finally, by Equations (22), (23), (24), and (25), we obtain that  $M_{S_1}(2n, j, 0; 1)$  is equal to

$$(1+j)(-1)^n \binom{3n}{2n} \left( (2n-j) \binom{2n+j+1}{j+1} \binom{2n}{n-j-1} + (2n-j+1) \binom{2n+j}{j} \binom{2n}{n-j} \right). \quad (27)$$

We shall use two well known binomial formulae:

$$\binom{n+j+1}{j+1} = \frac{n+j+1}{j+1} \binom{n+j}{j}, \quad (28)$$

$$\binom{2n}{n-j-1} = \frac{n-j}{n+1+j} \cdot \binom{2n}{n-j}. \quad (29)$$

We have that  $(2n-j) \binom{2n+j+1}{j+1} \binom{2n}{n-j-1}$  is equal to:

$$(2n-j) \cdot \frac{2n+j+1}{j+1} \cdot \frac{n-j}{n+1+j} \cdot \binom{2n+j}{j} \cdot \binom{2n}{n-j}. \quad (30)$$

After cancellation of  $1+j$ , it follows that  $M_{S_1}(2n, j, 0; 1)$  is equal to

$$(-1)^n \binom{3n}{2n} \binom{2n+j}{j} \cdot \binom{2n}{n-j} ((2n-j) \cdot (2n+j+1) \cdot \frac{n-j}{n+1+j} + (2n-j+1)(j+1)). \quad (31)$$

It can be shown, after some computations, that  $(2n-j) \cdot (2n+j+1) \cdot \frac{n-j}{n+1+j} + (2n-j+1)(j+1)$  is equal to

$$\frac{4n^3 + 2n^2(2-j) + n(j+3) + (j+1)}{n+j+1}. \quad (32)$$

Therefore,  $M_{S_1}(2n, j, 0; 1)$  is equal to

$$(-1)^n \binom{3n}{2n} \binom{2n+j}{j} \cdot \binom{2n}{n-j} \cdot \frac{4n^3 + 2n^2(2-j) + n(j+3) + (j+1)}{n+j+1}. \quad (33)$$

By Equation (16), it follows that  $M_S(2n, j, 0)$  is equal to

$$\frac{1}{(2n+1)^2} \cdot (-1)^n \binom{3n}{2n} \binom{2n+j}{j} \cdot \binom{2n}{n-j} \cdot \frac{4n^3 + 2n^2(2-j) + n(j+3) + (j+1)}{n+j+1}. \quad (34)$$

Note that polynomial  $4n^3 + 2n^2(2-j) + n(j+3) + (j+1)$  is divisible by  $2n+1$ , since

$$4n^3 + 2n^2(2-j) + n(j+3) + (j+1) = (2n^2 + n(1-j) + (j+1)) \cdot (2n+1). \quad (35)$$

By Equation (35), the sum  $M_S(2n, j, 0)$  from Equation (34) becomes

$$(-1)^n \frac{1}{2n+1} \cdot \binom{3n}{2n} \binom{2n+j}{j} \cdot \binom{2n}{n+j} \cdot \frac{1}{n+j+1} \cdot (2n^2 + n(1-j) + (j+1)). \quad (36)$$

It is readily verified that [6, Eq. (1.1), p. 5]:

$$\binom{2n}{n+j} \cdot \frac{1}{n+j+1} = \frac{1}{2n+1} \cdot \binom{2n+1}{n+j+1}. \quad (37)$$

By using Equation (37), the sum  $M_S(2n, j, 0)$  from Equation (36) becomes

$$(-1)^n \cdot \frac{1}{2n+1} \cdot \binom{3n}{2n} \cdot \frac{\binom{2n+j}{j} \cdot \binom{2n+1}{n+j+1}}{2n+1} \cdot (2n^2 + n + 1 - j(n-1)). \quad (38)$$

By setting  $C_n^{(3)} = \frac{1}{2n+1} \cdot \binom{3n}{2n}$ , we obtain that

$$M_S(2n, j, 0) = (-1)^n \cdot C_n^{(3)} \cdot \frac{\binom{2n+j}{j} \cdot \binom{2n+1}{n+j+1}}{2n+1} \cdot (2n^2 + n + 1 - j(n-1)). \quad (39)$$

The last equation above proves Equation (10). This completes the proof of Lemma 7.  $\square$

## 5 Proofs of Theorem 1, Proposition 8, and Theorem 2

We give a proof of Theorem 1.

Let us recall that  $T(n, j)$  is equal to  $\frac{\binom{2n+j}{j} \cdot \binom{2n+1}{n+j+1}}{2n+1}$ .

### 5.1 Proof of Theorem 1

Obviously,  $T(n, 0) = C_n$ . By setting  $j := 0$  in Equation (10), we obtain that

$$M_S(2n, 0, 0) = (-1)^n \cdot C_n^{(3)} \cdot C_n \cdot (2n^2 + n + 1). \quad (40)$$

By setting  $m := 1$  in Equation (7), it follows that:

$$S(2n, 1) = M_S(2n, 0, 0). \quad (41)$$

Finally, by Equations (40) and (41), it follows that

$$S(2n, 1) = (-1)^n \cdot C_n^{(3)} \cdot C_n \cdot (2n^2 + n + 1).$$

The last equation above is exactly Equation (4). This completes the proof of Theorem 1.  $\square$

Next, we give a proof of Proposition 8.

### 5.2 Proof of Proposition 8

Clearly,  $T(0, 0) = 1$ . Now, let us assume that  $n$  is a natural number.

It is readily verified [6, Eq. (1.1), p. 5] that

$$\binom{2n+1}{n+j+1} = \frac{2n+1}{n+j+1} \binom{2n}{n+j}. \quad (42)$$

We have

$$\begin{aligned} T(n, j) &= \binom{2n+j}{2n} \cdot \binom{2n}{n+j} \cdot \frac{1}{n+j+1} \quad (\text{by using Equation (42)}) \\ &= \binom{2n+j}{n+j} \cdot \binom{n}{j} \cdot \frac{1}{n+j+1} \quad (\text{by using [6, Equation (1.4), p. 5]}) \\ &= \binom{2n+j}{n+j} \cdot \frac{1}{n+j+1} \cdot \binom{n}{j} \\ &= \frac{1}{n} \cdot \binom{2n+j}{n+j+1} \cdot \binom{n}{j} \quad (\text{since } n \text{ is a natural number}) \\ &= \frac{1}{n} \cdot \binom{2n+j}{n-1} \cdot \binom{n}{j}. \end{aligned} \quad (43)$$

By Equation (43), it follows that

$$n \cdot \binom{2n+j}{j} \cdot \binom{2n+1}{n+j+1} = (2n+1) \cdot \binom{2n+j}{n-1} \cdot \binom{n}{j}. \quad (44)$$

Note that the integers  $n$  and  $2n+1$  are relatively prime. Therefore, it follows from Equation (44) that  $n$  must divide  $\binom{2n+j}{n-1} \cdot \binom{n}{j}$ , and  $2n+1$  must divide  $\binom{2n+j}{j} \cdot \binom{2n+1}{n+j+1}$ . This completes the proof of Proposition 8.  $\square$

Now, we are ready to prove Theorem 2.

### 5.3 Proof of Theorem 2

By using Proposition 8, we know that  $T(n, j)$  is an integer for any non-negative integer  $n$  and any  $j$  such that  $j \leq n$ . By Equation (10),  $C_n^{(3)} = \frac{1}{2n+1} \cdot \binom{3n}{2n}$  divides  $M_S(2n, j, 0)$  for any non-negative  $j$  such that  $j \leq n$ . By Equation (7),  $C_n^{(3)}$  divides the sum  $M_S(2n, j, t)$  for any non-negative integer  $t$ . By Equation (6), the sum  $S(2n, m)$  is divisible by  $C_n^{(3)}$  for any non-negative integer  $n$ , and for any natural number  $m$ . This completes the proof of Theorem 2.  $\square$

## 6 Proof of Theorem 3

Our proof of Theorem 3 consists of two parts. In the first part, we prove that the sum  $S(2n, 1)$  is divisible by  $\binom{2n}{n}$  for any non-negative integer  $n$ . In the second part, we prove that  $S(2n, m)$  is divisible by  $\binom{2n}{n}$  for any non-negative integer  $n$  and for any natural number  $m$  such that  $m \geq 2$ .

We begin with the proof of Proposition 9.

### 6.1 Proof of Proposition 9

Let us prove that  $\frac{2\binom{3n}{n}}{n+1}$  is an integer.

We have

$$\begin{aligned} \binom{3n}{n+1} \cdot (n+1) &= \binom{3n}{n+1} \cdot \binom{n+1}{n}, \\ &= \binom{3n}{n} \cdot \binom{2n}{1}, & (\text{By Newton's identity [6, Eq. (1.4), p. 5]}) \\ &= \binom{3n}{n} \cdot (2n). \end{aligned} \tag{45}$$

Since  $n+1$  and  $n$  are relatively prime integers, it follows from Equation (45), that  $n+1$  must divide  $2\binom{3n}{n}$ .

Next, from the definition of Fuss–Catalan number  $C_n^{(3)}$  of order three, we know that  $2n+1$  must divide the binomial coefficient  $\binom{3n}{n}$ . Obviously, then  $2n+1$  must also divide  $2\binom{3n}{n}$ .

Finally, due to the fact that  $n+1$  and  $2n+1$  are relatively prime, it follows that  $(n+1)(2n+1)$  must divide  $2\binom{3n}{n}$ . Therefore,  $\frac{2\binom{3n}{n}}{(n+1)(2n+1)}$  must be an integer. This completes the proof of Proposition 9.  $\square$

### 6.2 Proof of the First Part of Theorem 3

By Equation (4), it follows that

$$S(2n, 1) = (-1)^n \cdot \frac{2n^2 + n + 1}{(n+1)(2n+1)} \cdot \binom{3n}{n} \cdot \binom{2n}{n}. \tag{46}$$

Let us prove that  $\frac{2n^2+n+1}{(n+1)(2n+1)} \cdot \binom{3n}{n}$  is an integer for any non-negative integer  $n$ .

It is readily verified that

$$\begin{aligned} \frac{2n^2 + n + 1}{(n+1)(2n+1)} \cdot \binom{3n}{n} &= \frac{2n^2}{(n+1)(2n+1)} \cdot \binom{3n}{n} + C_n^{(3)}, \\ &= n^2 \cdot \frac{2}{(n+1)(2n+1)} \cdot \binom{3n}{n} + C_n^{(3)}. \end{aligned} \quad (47)$$

By Proposition 9,  $\frac{2n^2}{(n+1)(2n+1)} \cdot \binom{3n}{n}$  must be an integer. By using Equation (47) and the fact that the Fuss–Catalan number of order three  $C_n^{(3)}$  is an integer, it follows that  $\frac{2n^2+n+1}{(n+1)(2n+1)} \cdot \binom{3n}{n}$  is an integer.

By using Equation (47) and the fact that  $\frac{2n^2+n+1}{(n+1)(2n+1)} \cdot \binom{3n}{n}$  is an integer, it follows that  $S(2n, 1)$  is divisible by the central binomial coefficient  $\binom{2n}{n}$ . This completes the first part of the proof of Theorem 3.  $\square$

Now, we give proofs of the remaining propositions.

### 6.3 Proof of Proposition 10

For  $t = 0$ , Proposition 10 is true. Let us therefore assume that  $t$  is a natural number.

By using [6, Eq. (1.4), p. 5], it follows

$$\begin{aligned} \binom{2n+t}{t} \cdot t &= \binom{2n+t}{t} \cdot \binom{t}{t-1}, \\ &= \binom{2n+t}{t-1} \cdot \binom{2n+1}{1}. \end{aligned} \quad (48)$$

By Equation (48), it follows that

$$\binom{2n+t}{t} \cdot t = (2n+1) \cdot \binom{2n+t}{t-1}. \quad (49)$$

Finally, Equation (49) completes the proof of Proposition 10.  $\square$

### 6.4 Proof of Proposition 11

Let us consider the integer  $(3n+1) \binom{3n}{n+t} \cdot \binom{2n+t}{2n}$ .

We have

$$\begin{aligned} (3n+1) \binom{3n}{n+t} \cdot \binom{2n+t}{2n} &= \binom{3n+1}{3n} \cdot \binom{3n}{n+t} \cdot \binom{2n+t}{2n}, \\ &= \binom{3n+1}{n+t} \cdot \binom{2n+1-t}{2n-t} \cdot \binom{2n+t}{2n}, \\ &= \binom{3n+1}{n+t} \cdot (2n+1-t) \cdot \binom{2n+t}{2n}. \end{aligned} \quad (50)$$

By Equation (50) and the symmetry of binomial coefficients, it follows that

$$(3n+1) \binom{3n}{n+t} \cdot \binom{2n+t}{2n} = (2n+1) \cdot \binom{3n+1}{n+t} \cdot \binom{2n+t}{2n} - \binom{3n+1}{n+t} \cdot t \cdot \binom{2n+t}{t}. \quad (51)$$

By using Equation (49), Equation (51) becomes

$$(3n+1) \binom{3n}{n+t} \cdot \binom{2n+t}{2n} = (2n+1) \cdot \binom{3n+1}{n+t} \left( \binom{2n+t}{2n} - \binom{2n+t}{t-1} \right). \quad (52)$$

Due to the fact that  $2n+1$  and  $3n+1$  are relatively prime, it follows from Equation (52) that  $2n+1$  must divide  $\binom{3n}{n+t} \cdot \binom{2n+t}{2n}$ . This completes the proof of Proposition 11.  $\square$

## 6.5 Proof of Proposition 12

Let us consider the integer  $(3n+2) \binom{3n+1}{n+t+1} \cdot \binom{2n+t}{2n}$ .

We shall use the well known formula [6, Eq. (1.4), p. 5]. We have

$$\begin{aligned} (3n+2) \binom{3n+1}{n+t+1} \cdot \binom{2n+t}{2n} &= \binom{3n+2}{3n+1} \cdot \binom{3n+1}{n+t+1} \cdot \binom{2n+t}{2n}, \\ &= \binom{3n+2}{n+t+1} \cdot \binom{2n+1-t}{2n-t} \cdot \binom{2n+t}{2n}, \\ &= \binom{3n+2}{n+t+1} \cdot (2n+1-t) \cdot \binom{2n+t}{2n}. \end{aligned} \quad (53)$$

By using Equation (49) and the symmetry of binomial coefficients, Equation (53) becomes

$$(3n+2) \binom{3n+1}{n+t+1} \cdot \binom{2n+t}{2n} = (2n+1) \cdot \binom{3n+2}{n+t+1} \left( \binom{2n+t}{2n} - \binom{2n+t}{t-1} \right). \quad (54)$$

Due to the fact that integers  $2n+1$  and  $3n+2$  are relatively prime, it follows from Equation (54) that  $2n+1$  must divide  $\binom{3n+1}{n+t+1} \cdot \binom{2n+t}{2n}$ . This completes the proof of Proposition 12.  $\square$

**Remark 13.** By using Propositions 11 and 12, it also follows that  $\binom{3n}{n+t+1} \cdot \binom{2n+t}{2n}$  is divisible by  $2n+1$ . One can use Pascal's formula [6, Theorem 1.1, p. 5]:

$$\binom{3n+1}{n+t+1} = \binom{3n}{n+t+1} + \binom{3n}{n+t}. \quad (55)$$

## 6.6 Proof of the Second Part of Theorem 3

We shall prove that the sum  $M_S(2n, j, 1)$  is divisible by  $\binom{2n}{n}$  for any non-negative integers  $n$  and any  $j$  such that  $j \leq n$ , where  $S(2n, m)$  is our main sum from Equation (3).

By setting  $t := 0$  in Equation (8), it follows that

$$M_S(2n, j, 1) = \binom{2n}{j} \sum_{u=0}^{n-j} \binom{2n-j}{u} \cdot M_S(2n, j+u, 0). \quad (56)$$

By setting  $j := j+u$  in Equation (10), it follows that  $M_S(2n, j+u, 0)$  is equal to

$$(-1)^n \binom{3n}{n} \cdot \frac{1}{2n+1} \cdot \frac{\binom{2n+j+u}{j} \cdot \binom{2n+1}{n+j+u+1}}{2n+1} \cdot (2n^2 + n + 1 - (j+u)(n-1)). \quad (57)$$

By using Equation (42), we obtain that

$$\frac{\binom{2n+1}{n+j+u+1}}{2n+1} = \frac{\binom{2n}{n+j+u}}{n+j+u+1}. \quad (58)$$

By using Eqs. (57) and (58), it follows that the summand  $\binom{2n}{j} \binom{2n-j}{u} M_S(2n, j+u, 0)$  from Equation (56) is equal to

$$\frac{(-1)^n}{2n+1} \binom{3n}{2n} \binom{2n}{j} \binom{2n-j}{u} \frac{1}{n+j+u+1} \binom{2n+j+u}{j+u} \binom{2n}{n+j+u} \cdot (2n^2 + n + 1 - (j+u)(n-1)). \quad (59)$$

By using [6, Eq. (1.4), p. 5], it is readily verified that

$$\binom{3n}{2n} \binom{2n}{j} \binom{2n-j}{u} = \binom{3n}{n+j+u} \binom{n+j+u}{n} \binom{j+u}{u}. \quad (60)$$

Similarly, by using [6, Eq. (1.4), p. 5], it follows that

$$\binom{2n}{n+j+u} \binom{n+j+u}{n} = \binom{2n}{n} \cdot \binom{n}{j+u}. \quad (61)$$

By using Eqs. (60), (61), and the substitution  $t = j+u$ , the inner summand of Equation (59) becomes:

$$\frac{(-1)^n}{2n+1} \binom{3n}{n+t} \binom{2n+t}{t} \cdot \binom{2n}{n} \binom{n}{t} \binom{t}{j} \frac{1}{n+t+1} (2n^2 + n + 1 - t(n-1)). \quad (62)$$

Let us prove that the number

$$N(n, j, t) = \frac{(-1)^n}{2n+1} \binom{3n}{n+t} \binom{2n+t}{t} \binom{n}{t} \binom{t}{j} \frac{1}{n+t+1} (2n^2 + n + 1 - t(n-1)) \quad (63)$$

is an integer for any non-negative integers  $n, j$ , and  $t$  such that  $j \leq t \leq n$ .

Note that

$$\frac{2n^2 + n + 1 - t(n-1)}{n+t+1} = \frac{n(2n-t)}{n+t+1} + 1. \quad (64)$$

It is readily verified that

$$\binom{3n}{n+t} \cdot \frac{2n-t}{n+t+1} = \binom{3n}{n+t+1}. \quad (65)$$

By using Eqs. (64) and (65), it follows that

$$N(n, j, t) = N_1(n, j, t) + N_2(n, j, t), \quad (66)$$

where

$$N_1(n, j, t) = (-1)^n \frac{\binom{3n}{n+t} \binom{2n+t}{2n}}{2n+1} \binom{n}{t} \binom{t}{j}, \quad (67)$$

$$N_2(n, j, t) = n(-1)^n \frac{\binom{3n}{n+t+1} \binom{2n+t}{2n}}{2n+1} \binom{n}{t} \binom{t}{j}. \quad (68)$$

By Proposition 11,  $N_1(n, j, t)$  is an integer. By Remark 13,  $N_2(n, j, t)$  is also an integer. Finally, by Equation (66), it follows that  $N(n, j, t)$  is also an integer. Furthermore, by using Eqs. (56) and (63), it follows that

$$M_S(2n, j, 1) = \binom{2n}{n} \sum_{u=0}^{n-j} N(n, j, t), \quad (69)$$

where  $t = j + u$ .

Since  $N(n, j, t)$  is an integer, it follows from Equation (69) that  $M_S(2n, j, 1)$  is divisible by the central binomial coefficient  $\binom{2n}{n}$  for any non-negative integers  $n$  and  $j$  such that  $j \leq n$ .

By using Equation (8) and the induction principle, it can be shown that the sum  $M_S(2n, j, t)$  is divisible by  $\binom{2n}{n}$  for any non-negative integers  $n$  and  $j$  such that  $j \leq n$ , and for any natural number  $t$ .

By using Equation (7), it follows that the sum  $S(2n, m)$  is divisible by  $\binom{2n}{n}$  for any non-negative integer  $n$  and for any natural number  $m$  such that  $m \geq 2$ . This proves the second part of the proof of Theorem 3.

## 7 Proof of Theorem 5

We shall use the induction principle on the pairs  $(n, m)$ , where  $m \geq 2n$ . We say that  $(n_1, m_1) < (n_2, m_2)$  if  $n_1 < n_2$  or  $n_1 = n_2$  and  $m_1 < m_2$ .

For  $n = 1$  and for any natural number  $m \geq 2$ , it is readily verified that  $\text{Schr}(1, m, j, 2) = m - 1$ , where  $j = 0$  or  $j = 1$ . Therefore, Equation (5) is true for all pairs  $(1, m)$  where  $m \geq 2$ . Note that  $\text{Schr}(n, m, j, 2) = 0$  if  $m < 2n$  and  $\text{Schr}(n, m, j, 2) = 0$  if  $j > n$  or  $j < 0$ . Thus Equation (5) also holds whenever  $m = 2n - 1$ .

Let us suppose that  $n$  is a fixed natural number greater than 1, and let  $m$  be an arbitrary natural number such that  $m \geq 2n$ .

Now, we use a substitution  $m = 2n + p$ , where  $p$  is a non-negative integer. We have that  $\text{Schr}(n, 2n + p, j, 2)$  is equal to

$$\text{Schr}(n - 1, 2n + p, j - 1, 2) + \text{Schr}(n - 1, 2n + p - 1, j, 2) + \text{Schr}(n, 2n + p - 1, j, 2), \quad (70)$$

where  $0 \leq j \leq n$ .

Let us suppose that Equation (5) is true for all pairs  $(n_1, m_1)$  such that  $(n_1, m_1) < (n, 2n + p)$ .

Therefore, by using the induction hypothesis, we know that:

$$\text{Schr}(n - 1, 2n + p, j - 1, 2) = \frac{p + 3}{n - 1} \binom{n - 1}{j - 1} \binom{2n + p + j - 1}{n - 2}, \quad (71)$$

$$\text{Schr}(n - 1, 2n + p - 1, j, 2) = \frac{p + 2}{n - 1} \binom{n - 1}{j} \binom{2n + p + j - 1}{n - 2}, \quad (72)$$

$$\text{Schr}(n, 2n + p - 1, j, 2) = \frac{p}{n} \binom{n}{j} \binom{2n + p + j - 1}{n - 1}. \quad (73)$$

It can be shown that by adding Eqs. (71) and (72), we obtain that

$$\text{Schr}(n-1, 2n+p, j-1, 2) + \text{Schr}(n-1, 2n+p-1, j, 2) = \frac{1}{n} \binom{n}{j} \binom{2n+p+j-1}{n-2} \frac{2n+pn+j}{n-1}. \quad (74)$$

Furthermore, by adding Eqs. (74) and (73), we obtain that Equation (70) becomes

$$\frac{1}{n} \binom{n}{j} \binom{2n+p+j-1}{n-2} \frac{2n+pn+j}{n-1} + \frac{p}{n} \binom{n}{j} \binom{2n+p+j-1}{n-1}. \quad (75)$$

After some calculations, it can be shown that Equation (75) becomes

$$\frac{p+1}{n} \binom{n}{j} \binom{2n+p+j}{n-1}. \quad (76)$$

By using the induction principle, it follows that Equation (5) is true for all natural numbers  $n$  and  $m$  such that  $m \geq 2n$ . This completes the proof of Theorem 5.  $\square$

**Remark 14.** Theorem 4 follows from Theorem 5 by setting  $m = 2n$  in Equation (5) and by using Proposition 8.

**Remark 15.** Let  $n$ ,  $m$ , and  $l$  be natural numbers such that  $m \geq ln$ . Let the number  $\text{Schr}(n, m, j, l)$  denote the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, m)$ , using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , that never go below the line  $y = lx$  with exactly  $j$  steps of the form  $(1, 0)$ . By using a similar idea to the one in the proof of Theorem 5, it can be shown that

$$\text{Schr}(n, m, j, l) = \frac{m - ln + 1}{n} \binom{n}{j} \binom{m+j}{n-1}. \quad (77)$$

Furthermore, let the number  $\text{Schr}(n, m, l)$  denote the number of all lattice paths in the plane from the point  $(0, 0)$  to the point  $(n, m)$ , using steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , that never go below the line  $y = lx$ . By using a combinatorial interpretation for the numbers  $\text{Schr}(n, m, l, j)$  in Equation (77), we obtain a new combinatorial proof of the following known [14, Theorem 2.9, p. 6] formula

$$\text{Schr}(n, m, l) = \frac{m - ln + 1}{n} \sum_{j=0}^n \binom{n}{j} \binom{m+j}{n-1}. \quad (78)$$

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