Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 3, 658–666
DOI: 10.7546/nntdm.2025.31.3.658-666

On special integer linear combinations of terms of rational cycles for the generalized $3x + 1$ problem

Yagub N. Aliyev

School of IT and Engineering, ADA University
Ahmadbey Aghaoglu 61, AZ1008 Baku, Azerbaijan
e-mail: yaliyev@ada.edu.az

Received: 2 December 2024 **Revised:** 26 September 2025
Accepted: 26 September 2025 **Online First:** 29 September 2025

Abstract: In the paper, some special linear combinations of the terms of rational cycles of generalized Collatz sequences are studied. It is proved that for specific choice of the coefficients these linear combinations are integers. The discussed results are demonstrated on some examples. In some particular cases the obtained results can be used to explain some patterns of digits in p-adic representations of the terms of the rational cycles.
Keywords: $3x + 1$ problem, Collatz conjecture, Rational cycles, Integer linear combinations.
2020 Mathematics Subject Classification: 11B75, 11A63, 11D88.

1 Introduction

Collatz conjecture or the $3x + 1$ problem claims that for any positive integer x_0, the recursive sequence defined for $n \geq 0$ by $x_{n+1} = S(x_n) = \frac{3x_n + 1}{2}$ if x_n is odd and $x_{n+1} = T(x_n) = \frac{x_n}{2}$ if x_n is even, there is a positive integer N such that $x_N = 1$ [12]. It is known that this holds true for almost all x_0 in the sense of some density over the set of positive integers. For the works in this direction, see [5,9,22], which culminated in the very important work of T. Tao [21]. Another approach would be to find all cycles of this recursive sequence. It is conjectured that there are only finitely many such cycles. The only known cycle is the one generated by $x_0 = 1$. There are more cycles if x_0 is allowed to be zero or a negative integer but it is conjectured that their number is also finite (The Finite Cycles Conjecture [11,12]). The only known non-positive integer cycles are the ones generated by $x_0 = 0$, $x_0 = -1$, $x_0 = -5$, and $x_0 = -17$. If a sequence of functions consisted of several S and T is given, then one can speak about rational cycles [1]. There is a rational number x_0 such that if the functions S and T are applied in the given order, then the final result is again x_0. Rational cycles generated by such x_0 have some interesting properties [3,11]. Many generalizations of Collatz conjecture were considered by replacing functions S and T by more general $S_k(x) = \frac{p_i x + k}{q}$. One can find such generalizations in [7, 14, 17, 18]. In [4, 13] similar generalizations are used to prove some results related to undecidability properties. In [10,20] these generalizations are considered in the context of 2-adic numbers and q based numeral systems. Weaker versions of the $3x + 1$ problem attracted some attention recently and were discussed in problem-solving columns of various journals [8, 15, 16, 19]. A. O. Gelfond [6] and A. G. Kurosh considered functions similar to $S_k(x)$ above in the context of the separation of the set of the positive integers into classes of numbers connected via compositions of such functions.

In this paper we focus on properties of the terms of rational cycles x_i, which show that these rational numbers are similar to the integers. A linear combination with integer coefficients of two integers is obviously always an integer. In the paper it is proved that there are special integer linear combinations of rational numbers x_i and x_{i+b}, for all i. In general, it is not surprising that there are integer linear combinations of rational numbers. One can, for example, take the coefficients of the linear combination as the denominators of the rational numbers. The surprising fact in the following results is that the coefficiens of these linear combinations are either fixed or involve product of several consecutive p_i, while i changes. Two worked out examples demonstarting the results on particular cases are given. The results of the current paper were first discovered experimentally by observing patterns of digits in p-adic representations of these rational numbers x_i and given at the end of the paper as applications.

2 Notations and lemmas

Consider composition $P = B_0 \circ B_1 \circ \cdots \circ B_{n-1}$ of functions $B_i(x) = \frac{p_i x + k_i}{q}$, where $n > 1$, k_i are integers, p_i, q are non-zero integers such that $(p_i, q) = 1$ for $i = 0, 1, \ldots, n-1$. When it is necessary to extend the index i, beyond the interval $[0, n-1]$, we suppose that $B_i = B_j$ if $i \equiv j \pmod{n}$. Consider equation $B_0 \circ B_1 \circ \cdots \circ B_{n-1}(x) = x$, which can also be written as

$$\cfrac{p_0 \cfrac{p_1 \cfrac{\vdots}{q} + k_1}{q} + k_0}{q} = x, \qquad \text{with innermost level } \cfrac{p_{n-2}\frac{p_{n-1}x+k_{n-1}}{q} + k_{n-2}}{q}.$$

Note that its solution x_0 is a rational number (cf. [11, Formulas 1.2 and 1.3]):

$$x_0 = \frac{p_0p_1\cdots p_{n-2}k_{n-1} + p_0p_1\cdots p_{n-3}k_{n-2}q + \cdots + p_0k_1q^{n-2} + k_0q^{n-1}}{q^n - p_0p_1\cdots p_{n-1}}.$$

Similarly, consider equations $B_i \circ B_{i+1} \circ \cdots \circ B_{i+n-1}(x) = x$ for $i = 0, 1, \ldots, n-1$. Note that their solutions x_i are also rational numbers:

$$x_i = \frac{p_ip_{i+1}\cdots p_{i+n-2}k_{i-1} + p_ip_{i+1}\cdots p_{i+n-3}k_{i-2}q + \cdots + p_ik_{i+1}q^{n-2} + k_iq^{n-1}}{q^n - p_0p_1\cdots p_{n-1}},$$

where all the indices are taken modulo n. In the following, it is assumed that $x_i = x_j$ if $i \equiv j \pmod n$. Consider also numbers $U_i = \frac{q^i}{q^n - p_0p_1\cdots p_{n-1}}$, for $i = 0, 1, \ldots, n$. Note that $U_n = p_0p_1\cdots p_{n-1}U_0 + 1$. Note also that

$$x_i = p_ip_{i+1}\cdots p_{i+n-2}k_{i-1}U_0 + p_ip_{i+1}\cdots p_{i+n-3}k_{i-2}U_1 + \cdots + p_ik_{i+1}U_{n-2} + k_iU_{n-1}.$$

Note that there are infinitely many pairs of non-zero integers α, β and integers b, such that $0 < b < n$ and $\alpha U_0 + \beta U_b$ is an integer or equivalently, $q^n - p_0p_1\cdots p_{n-1}|\alpha + \beta q^b$. Indeed, one can take, for example, $\alpha = k$, $\beta = -kq^{\phi(|q^n - p_0p_1\cdots p_{n-1}|)-1}$, and $b = 1$, where $k = 1, 2, \ldots$ and ϕ is Euler's totient function. If α and β are fixed, then one can ask if such b exists. The answer to this question depends on the choice of α and β . For example, if $q = 2$, $n = 4$, $p_0 = 3$, $p_1 = p_2 = p_3 = 1$, then $q^n - p_0p_1p_2p_3 = 13$. If $\alpha = 1, \beta = 1$, then such b ($0 < b < 4$) does not exist. But if $\alpha = 9, \beta = 1$ then $b = 2$ satisfies the condition.

Lemma 2.1. *If $\alpha U_0 + \beta U_b$ is an integer, then $p_0p_1\cdots p_{n-1}\beta U_0 + \alpha U_{n-b}$ is also an integer.*

Proof. Our claim is equivalent to prove that if $q^n - p_0p_1\cdots p_{n-1}|\alpha + \beta q^b$, then

$$q^n - p_0p_1\cdots p_{n-1}|p_0p_1\cdots p_{n-1}\beta + \alpha q^{n-b}.$$

Indeed, since $(q, q^n - p_0p_1\cdots p_{n-1}) = 1$ and

$$q^b(p_0p_1\cdots p_{n-1}\beta + \alpha q^{n-b}) = p_0p_1\cdots p_{n-1}\beta q^b + \alpha q^n,$$

it is sufficient to show that

$$q^n - p_0p_1\cdots p_{n-1}|p_0p_1\cdots p_{n-1}\beta q^b + \alpha q^n,$$

which follows from

$$p_0p_1\cdots p_{n-1}\beta q^b + \alpha q^n = p_0p_1\cdots p_{n-1}(\alpha + \beta q^b) + \alpha(q^n - p_0p_1\cdots p_{n-1}). \qquad \square$$

Lemma 2.2. *If $\alpha U_0 + \beta U_b$ is an integer, then for $i = 0, 1, 2, \ldots$ the numbers $\alpha U_i + \beta U_{i+b}$ and $p_0p_1\cdots p_{n-1}\beta U_i + \alpha U_{n+i-b}$ are also integers.*

Proof. By multiplying $\alpha U_0 + \beta U_b$ and $p_0p_1\cdots p_{n-1}\beta U_0 + \alpha U_{n-b}$, which are both integers, by integer q^i, we obtain that both of the numbers $\alpha U_i + \beta U_{i+b}$ and $p_0p_1\cdots p_{n-1}\beta U_i + \alpha U_{n+i-b}$ are integers. $\square$

3 Main results

Theorem 3.1. *If $\alpha U_0 + \beta U_b$ is an integer, then for any i, satisfying $0 \le i < i+b < n$, the number $\alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b}$ is also an integer.*

Proof. Note that we can write

$$\begin{aligned}
&\alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b} \\
&= \alpha(p_i p_{i+1} \cdots p_{i+n-2} k_{i-1} U_0 + p_i p_{i+1} \cdots p_{i+n-3} k_{i-2} U_1 + \cdots + p_i k_{i+1} U_{n-2} + k_i U_{n-1}) \\
&\quad + \beta p_i p_{i+1} \cdots p_{i+b-1}(p_{i+b} p_{i+b+1} \cdots p_{i+b+n-2} k_{i+b-1} U_0 \\
&\quad + p_{i+b} p_{i+b+1} \cdots p_{i+b+n-3} k_{i+b-2} U_1 + \cdots + p_{i+b} k_{i+b+1} U_{n-2} + k_{i+b} U_{n-1}) \\
&= k_0 M_0 + k_1 M_1 + \cdots + k_{n-1} M_{n-1},
\end{aligned}$$

where

$$M_j = \begin{cases} p_i p_{i+1} \cdots p_{j-1}(\alpha U_{n+i-1-j} + \beta p_0 p_1 \cdots p_{n-1} U_{i+b-1-j}), & \text{if } i \le j < i+b, \\ p_i p_{i+1} \cdots p_{n+j-1}(\alpha U_{i-1-j} + \beta U_{i+b-1-j}), & \text{if } 0 \le j < i, \\ p_i p_{i+1} \cdots p_{j-1}(\alpha U_{n+i-1-j} + \beta U_{n+i+b-1-j}), & \text{if } i+b \le j < n. \end{cases}$$

By Lemma 2.1 and Lemma 2.2, all M_j, for $j = 0, 1, \ldots n-1$, are integers and therefore the claim is true. □

Corollary 3.1. *If $\alpha U_0 + \beta U_b$ is an integer, then for any i, the number $\alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b}$ is also an integer.*

Proof. Since it was assumed that $x_i = x_j$ if $i \equiv j \pmod{n}$, without loss of generality, we can suppose that $0 \le i < n$. The case $i + b < n$ was considered in Theorem 3.1. So, we can suppose that $i + b \ge n$. Since $0 < b < n$, we have $0 \le i + b - n < i$, and therefore $x_{i+b} = x_{i+b-n}$. Consequently,

$$\alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b} = \alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b-n}.$$

Multiply this number by $p_{i+b-n} p_{i+b-n+1} \cdots p_{i-1}$, which is relatively prime to $q^n - p_0 p_1 \cdots p_{n-1}$, and therefore can not change the property of being or not being an integer for the number $\alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b}$, we obtain

$$\beta p_0 p_1 \cdots p_{n-1} x_{i+b-n} + \alpha p_{i+b-n} p_{i+b-n+1} \cdots p_{i-1} x_i,$$

which is an integer by Lemma 2.1 and Theorem 3.1. □

Remark 1. *The above results are trivially true for the cases when $b = 0$ and $b = n$. Indeed, for the case when $b = 0$, if $(\alpha + \beta) U_0$ is an integer, then $(q^n - p_0 p_1 \cdots p_{n-1}) | (\alpha + \beta)$, and therefore $(\alpha + \beta) x_i$ is also an integer for $i = 0, 1, \ldots, n-1$. For the case when $b = n$, if $\alpha U_0 + \beta U_n = \alpha U_0 + \beta p_0 p_1 \cdots p_{n-1} U_0 + \beta$ is an integer, then $(\alpha + \beta p_0 p_1 \cdots p_{n-1}) U_0$ is also an integer (β is an integer), and therefore $(q^n - p_0 p_1 \cdots p_{n-1}) | (\alpha + p_0 p_1 \ldots p_{n-1} \beta)$. Consequently $(\alpha + \beta p_0 p_1 \cdots p_{n-1}) x_i$ is an integer for $i = 0, 1, \ldots, n-1$.*

Remark 2. *If we denote the denominator of x_i by d_i, then it is possible to prove that $d_i = d_{i+1}$ by using the fact that all integers d_i are coprime with q and with any of the p_j, since they divide $q^n - p_0 p_1 \cdots p_{n-1}$ (cf. [11, p. 39]). In the following two corollaries this is proved using the main results of the current paper.*

Corollary 3.2. *If one of the numbers x_j $(j \in \{0, 1, \ldots, n-1\})$ is a fraction with denominator d in its simplest form, then all of x_i for $i = 0, 1, \ldots, n-1$, are like fractions with the same denominator d.*

Proof. It was mentioned earlier that one can always take $\alpha = 1$, $\beta = -q^{\phi(|q^n - p_0 p_1 \cdots p_{n-1}|)-1}$, and $b = 1$. Then by the main result, the number $x_i + \beta p_i x_{i+1}$ is an integer for $i = 0, 1, \ldots, n-1$. Note that d is a divisor of $q^n - p_0 p_1 \cdots p_{n-1}$, and therefore d is relatively prime to β and all p_i for $i = 0, 1, \ldots, n-1$. Since x_j is a fraction with denominator d, the other numbers x_{j-1}, x_{j-2}, $x_{j-3}, \ldots$ are all like fractions with the same denominator d in their simplest forms. □

Corollary 3.3. *If one of the numbers x_j $(j \in \{0, 1, \ldots, n-1\})$ is an integer, then all of x_i for $i = 0, 1, \ldots, n-1$, are integers.*

4 Examples

Let us take $q = 3$. Consider composition of functions $P = B_0 \circ B_1 \circ B_2 \circ B_3$, where $B_0(x) = \frac{-5x-2}{3}$, $B_1(x) = \frac{2x+1}{3}$, $B_2(x) = \frac{7x+6}{3}$, $B_3(x) = \frac{-x+3}{3}$. Here $n = 4$, $p_0 = -5$, $k_0 = -2$, $p_1 = 2$, $k_1 = 1$, $p_2 = 7$, $k_2 = 6$, $p_3 = -1$, $k_3 = 3$, and $q^n - p_0 p_1 p_2 p_3 = 3^4 - (-5) \cdot 2 \cdot 7 \cdot (-1) = 11$.

The solution of equation $B_0 \circ B_1 \circ B_2 \circ B_3(x) = x$ is the number $x_0 = -69/11$. Note that $x_0 = x_4$. We can also find the other numbers $x_1 = x_5 = 37/11$, $x_2 = 50/11$, $x_3 = 12/11$, by solving the equations $B_1 \circ B_2 \circ B_3 \circ B_0(x) = x$, $B_2 \circ B_3 \circ B_0 \circ B_1(x) = x$, $B_3 \circ B_0 \circ B_1 \circ B_2(x) = x$, respectively. We also find the numbers $U_i = 2^i/11$ $(i = 0, 1, 2, 3, 4)$. Note that $4U_0 + 2U_2 = 2$ is an integer, which is equivalent to say that $11|(4 + 2 \cdot 3^2)$. So, we can take $\alpha = 4$, $\beta = 2$, and $b = 2$. We observe that $4x_i + 2p_i p_{i+1} x_{i+2}$ is an integer for each of $i = 0, 1, 2, 4$. Indeed,

$$\begin{aligned}
4x_0 + 2p_0p_1x_2 &= 4 \cdot (-69/11) + 2 \cdot (-5) \cdot 2 \cdot (50/11) &&= -116, \\
4x_1 + 2p_1p_2x_3 &= 4 \cdot (37/11) + 2 \cdot 2 \cdot 7 \cdot (12/11) &&= 44, \\
4x_2 + 2p_2p_3x_4 &= 4 \cdot (50/11) + 2 \cdot 7 \cdot (-1) \cdot (-69/11) &&= 106, \\
4x_3 + 2p_3p_4x_5 &= 4 \cdot (12/11) + 2 \cdot (-1) \cdot (-5) \cdot (37/11) &&= 38,
\end{aligned}$$

are all integers.

Now, note that $-5U_0 - 13U_1 = -4$ is also an integer, which is equivalent to say that $11|-44 = -5 - 13 \cdot 3^1$. This means that we can take $\alpha = -5$, $\beta = -13$, and $b = 1$. So, $-5x_i - 13p_i x_{i+1}$ should be an integer for each of $i = 0, 1, 2, 3$. Indeed,

$$\begin{aligned}
-5x_0 - 13p_0x_1 &= -5 \cdot (-69/11) - 13 \cdot (-5) \cdot (37/11) &&= 250, \\
-5x_1 - 13p_1x_2 &= -5 \cdot (37/11) - 13 \cdot 2 \cdot (50/11) &&= -135, \\
-5x_2 - 13p_2x_3 &= -5 \cdot (50/11) - 13 \cdot 7 \cdot (12/11) &&= -122, \\
-5x_3 - 13p_3x_4 &= -5 \cdot (12/11) - 13 \cdot (-1) \cdot (-69/11) &&= -87,
\end{aligned}$$

are all integers. These observations are in perfect agreement with the main results of the current paper.

5 Applications

If $p_i \in \{1, p\}$, where p is a nonzero integer and $(p, q) = 1$, then there are two type of functions $S_k(x) = \frac{px+k}{q}$ and $T_k(x) = \frac{x+k}{q}$. Let us denote by m the number of S functions in P. Then $U_i = \frac{q^i}{q^n - p^m}$, for $i = 0, 1, \ldots, n$. Denote by $\sigma(i, j)$ the number of S functions in the fragment $B_i B_{i+1} \cdots B_{j-1}$ of P. In particular, $\sigma(i, i) = 0$, because it corresponds to the empty fragment of P. Let x_i be the solution of the equation $B_i \circ B_{i+1} \circ B_{i+2} \circ \cdots \circ B_{i+n-1}(x) = x$, where all the indices are taken modulo n. Take $\alpha = p^l$ for some non-negative integer l, and $\beta = -1$. For this special case the main result of the current paper can be written as $p^l x_i - p^{\sigma(i,i+b)} x_{i+b} \in \mathbb{Z}$. This can be visualized by writing x_i as p-adic numbers in a table and noting that p-adic digits at the corresponding place values of $p^l x_i$ and $p^{\sigma(i,i+b)} x_{i+b}$ are identical, except for finitely many digits at lower place values.

Let us demonstrate this with an example. Let $q = 2, p = 11, P = B_0 \circ B_1 \circ B_2 \circ B_3 \circ B_4 \circ B_5 \circ B_6$, where $B_0(x) = B_1(x) = B_2(x) = B_3(x) = B_5(x) = T_0(x)$, $B_4(x) = S_5(x)$, and $B_6(x) = S_3(x)$. Here $n = 7$, $m = 2$, $q^n - p^m = 2^7 - 11^2 = 7$ and $U_i = 2^i/7$ $(i = 0, 1, 2, \ldots)$. Note that

$$U_0 - U_3 = -1, \;\; 11U_0 - U_2 = 1, \;\; 11^2 U_0 - U_1 = 17, \;\; 11^3 U_0 - U_0 = 190,$$

are all integers, which is equivalent to say that

$$7|(1 - 2^3), \;\; 7|(11 - 2^2), \;\; 7|(11^2 - 2^1), \;\; 7|(11^3 - 2^0),$$

respectively. By the main result of the current paper we can say that

$$x_i - 11^{\sigma(i,i+3)} x_{i+3}, \;\; 11x_i - 11^{\sigma(i,i+2)} x_{i+2}, \;\; 11^2 x_i - 11^{\sigma(i,i+1)} x_{i+1}, \;\; 11^3 x_i - x_i,$$

are also integers for $i = 0, 1, 2, \ldots$. The functions B_i, the numbers x_i, their 11-adic representations (letter A below means digit 10), and the patterns formed by the digits can be seen in Table 1.

Table 1. Illustrative computation of x_i and associated sequences.

$x_0 = 53/7$	$= \cdots$	7	9	4	7	9	4	8	6			$S_3 = B_6$
$x_6 = 302/7$	$= \cdots$	9	4	7	9	4	7	9	8	7		$T_0 = B_5$
$x_5 = 151/7$	$= \cdots$	4	7	9	4	7	9	4	9	9		$S_5 = B_4$
$x_4 = 848/7$	$= \cdots$	7	9	4	7	9	4	7	A	4	8	$T_0 = B_3$
$x_3 = 424/7$	$= \cdots$	9	4	7	9	4	7	9	5	2	4	$T_0 = B_2$
$x_2 = 212/7$	$= \cdots$	4	7	9	4	7	9	4	8	1	2	$T_0 = B_1$
$x_1 = 106/7$	$= \cdots$	7	9	4	7	9	4	7	9	6	1	$T_0 = B_0$
$x_0 = 53/7$	$= \cdots$	9	4	7	9	4	7	9	4	8	6	

More examples of such patterns and applications to the original $3x + 1$ problem are given in [1] and the references therein. In particular, the main results of the current paper can be used to explain the appearance of the same patterns of digits in rational cycles corresponding to compositions P with common number of S and T functions (m and $n - m$). These rational cycles also share common numbers U_i, whose digits enjoy the same patterns. If the functions in composition P are permuted, then these patterns of digits can disappear and reappear again depending on whether x_i are integers or not. Better understanding of these patterns can be useful for determination of all integer cycles. Note that these patterns are not limited to the periodic digits of each x_i individually. The patterns that interest us most involve two different numbers x_i and x_{i+b}, and hold true for any choice of i, while l and b are fixed. In a certain sense, the pair of numbers l and b is an invariant for the table formed by the digits of x_i as in the above table. Note that the digits of each number x_i corresponding to some S function and all the numbers above it need to be shifted to the left by 1 digit to make these patterns visible. In the above calculations, this shift was implemented using function $\sigma(i, i+b)$.

The Finite Cycles Conjecture mentioned at the beginning of this paper claims that the only integer cycles for $3x+1$ problem are the ones generated by $x_0 = 0$, $x_0 = -1$, $x_0 = 1$, $x_0 = -5$, and $x_0 = -17$. These numbers correspond to compositions $P_1 = T$, $P_2 = S$, $P_3 = T \circ S$, $P_4 = T \circ S \circ S$, and $P_5 = T \circ T \circ T \circ S \circ S \circ S \circ T \circ S \circ S \circ S \circ S$, where $T = T_0$ and $S = S_1$. For these compositions $q = 2$, $p_i = 1$ or 3, and the numbers in Table 2 are calculated.

Table 2. Values of x_0, associated P_i, and corresponding expressions $q^n - p_0 p_1 \cdots p_{n-1}$.

x_0	P	n	$q^n - p_0 p_1 \cdots p_{n-1}$
0	P_1	1	$2^1 - 1 = 1$
-1	P_2	1	$2^1 - 3 = -1$
1	P_3	2	$2^2 - 3 = 1$
-5	P_4	3	$2^3 - 3 \cdot 3 = -1$
-17	P_5	11	$2^{11} - 3^7 = -139$

These compositions with integer x_0 also show that the main results of the current paper, namely Theorem 3.1 and Corollary 3.1 can not be written as "if and only if" statements. Indeed, if the numbers x_i ($i \in \{0, 1, \ldots, n-1\}$) are integers, then $\alpha x_i + \beta p_i p_{i+1} \cdots p_{i+b-1} x_{i+b}$ is an integer for any choice of integers α, β, b, which is not the case for $\alpha U_0 + \beta U_b$. Nevertheless, Lemma 2.1 can be written as an "if and only if" statement.

The composition P_5 is different from the others in the sense that $q^n - p_0 p_1 \cdots p_{n-1} \neq \pm 1$ but x_0 is still an integer. The compositions P_i^k ($i = 1, 2, \ldots, 5$; $k = 1, 2, \ldots$), defined recursively by $P_i^1 = P_i$ and $P_i^{k+1} = P_i^k \circ P_i$ ($i = 1, 2, \ldots, 5$; $k = 1, 2, \ldots$) also have integer x_0, and $q^n - p_0 p_1 \cdots p_{n-1} \neq \pm 1$ for $i \geq 2$. It would be interesting to determine all such compositions with integer x_0 or prove that all other compositions correspond to non-integer rational x_0. It would also be interesting to generalize the results above by taking p_i, k_i, and q as Gaussian integers, and then x_i as Gaussian rationals.

Note that the main results of the current paper can also be proved using the method of mathematical induction over each k_i, where the base of induction corresponds to $k_i = 0$ (cf. [2]). A better understanding of the nature of rational cycles may be helpful in the future for attempting to solve $3x+1$ problem.

6 Conclusion

In the paper some generalizations of Collatz conjecture or $3x + 1$ problem are studied. Some results are obtained proving that special linear combinations of the terms of rational cycles are integers. Demonstrations of these results on some concrete examples are given. These results are then used to explain some patterns of digits in p-adic representations of the rational cycles.

Acknowledgements

This work was completed with the support of ADA University Faculty Research and Development Fund. I also thank the anonymous reviewers for their suggestions and comments which resulted in much clearer presentation of the results.

References

[1] Aliyev, Y. N. (2020). The $3x + 1$ problem for rational numbers: Invariance of periodic sequences in the $3x + 1$ problem. *Proceedings of IEEE 14th International Conference on Application of Information and Communication Technologies (AICT)*, 7–20 October 2020, Tashkent, Uzbekistan, pp. 1–4. DOI: 10.1109/AICT50176.2020.9368585.

[2] Aliyev, Y. N. (2023). Existence of invariants in periods of generalized Collatz sequences. *Proceedings of 5th International Conference on Problems of Cybernetics and Informatics (PCI)*, 28–30 August 2023, Baku, Azerbaijan, pp. 1–3. DOI: 10.1109/PCI60110.2023.10326001.

[3] Böhm, C., & Sontacchi, G. (1978). On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n+1$. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, 64(3), 260–264.

[4] Conway, J. H. (2013). On unsettleable arithmetical problems. *The American Mathematical Monthly*, 120(3), 192–198.

[5] Everett, C. J. (1977). Iteration of the number-theoretic function $f(2n) = n,\ f(2n + 1) = 3n + 2$. *Advances in Mathematics*, 25(1), 42–45.

[6] Gelfond, A. O. (1954). On the partition of the natural series into classes by the group of linear substitutions. *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya*, 18(4), 297–306.

[7] Kaufman, R., & Castañeda, P. (2025). Any positive number is a factor of a number that descends to 1 in the $\omega x + 1$ problem, for odd ω. *The College Mathematics Journal*, 56(3), 200–202.

[8] Kaufman, R., & Lagarias, J. (2023). Problem 12426. In: Problems and Solutions. (Ullman, D. H., Velleman, D. J., Wagon, S., & West, D. B. (eds.)). *The American Mathematical Monthly*, 130(10), 952–962.

[9] Korec, I. (1994). A density estimate for the $3x + 1$ problem. *Mathematica Slovaca*, 44(1), 85–89.

[10] Laarhoven, T., & de Weger, B. (2013). The Collatz conjecture and De Bruijn graphs. *Indagationes Mathematicae*, 24(4), 971–983.

[11] Lagarias, J. C. (1990). The set of rational cycles for the $3x + 1$ problem. *Acta Arithmetica*, 56(1), 33–53.

[12] Lagarias, J. C. (Ed.). (2010). *The Ultimate Challenge: The* $3x + 1$ *Problem*. American Mathematical Society, New York.

[13] Lehtonen, E. (2008). Two undecidable variants of Collatz's problems. *Theoretical Computer Science*, 407(1–3), 596–600.

[14] Levy, D. (2004). Injectivity and surjectivity of Collatz functions. *Discrete Mathematics*, 285(1–3), 191–199.

[15] Lutsko, C. (2023). Problem 270. In: Solved and unsolved problems. *European Mathematical Society Magazine*, 127, 53–61.

[16] Lutsko, C. (2023). Problem 270 (Solution by the proposer). In: Solved and unsolved problems. *European Mathematical Society Magazine*, 129, 47–55.

[17] Matthews, K., & Watts, A. (1984). A generalization of Hasse's generalization of the Syracuse algorithm. *Acta Arithmetica*, 43(2), 167–175.

[18] Matthews, K., & Watts, A. (1985). A Markov approach to the generalized Syracuse algorithm. *Acta Arithmetica*, 45(1), 29–42.

[19] Pranesachar C. R. (2025). Problem 12426 (Solution). In: Problems and Solutions. (Ullman, D. H., Velleman, D. J., Wagon, S., & West, D. B. (eds.)). *The American Mathematical Monthly*, 132(7), 706–715.

[20] Rouet, J. L., & Feix, M. R. (2002). A generalization of the Collatz problem. Building cycles and a stochastic approach. *Journal of Statistical Physics*, 107, 1283–1298.

[21] Tao, T. (2022). Almost all orbits of the Collatz map attain almost bounded value. *Forum of Mathematics, Pi*, 10, Article e12.

[22] Terras, R. (1976). A stopping time problem on the positive integers. *Acta Arithmetica*, 30(3), 241–252.