Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 3, 646–657
DOI: 10.7546/nntdm.2025.31.3.646-657

Congruences for the Apéry numbers modulo p^3

Zhi-Hong Sun

School of Mathematics and Statistics, Huaiyin Normal University
Huaian, Jiangsu 223300, P. R. China
e-mail: zhsun@hytc.edu.cn

Received: 3 May 2025 **Revised:** 3 September 2025
Accepted: 14 September 2025 **Online First:** 25 September 2025

Abstract: Let $\{A'_n\}$ be the Apéry numbers given by $A'_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$. For any prime $p \equiv 3 \pmod 4$ we show that

$$A'_{\frac{p-1}{2}} \equiv \frac{p^2}{3\binom{(p-3)/2}{(p-3)/4}^2} \pmod{p^3}.$$

Let $\{t_n\}$ be given by $t_0 = 1$, $t_1 = 5$ and $t_{n+1} = (8n^2+12n+5)t_n - 4n^2(2n+1)^2 t_{n-1}$ $(n \ge 1)$. We also establish the congruences for $t_p \pmod{p^3}$, $t_{p-1} \pmod{p^2}$ and $t_{\frac{p-1}{2}} \pmod{p^2}$, where p is an odd prime.

Keywords: Apéry number, Congruence, Combinatorial identity, Binary quadratic form, Euler number.
2020 Mathematics Subject Classification: 11A07, 05A10, 05A19, 11B68, 11E25.

1 Introduction

For $s > 1$ let $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$. In 1979, in order to prove that $\zeta(3)$ and $\zeta(2)$ are irrational, Apéry [2] introduced the Apéry numbers $\{A_n\}$ and $\{A'_n\}$ given by

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \quad \text{and} \quad A'_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

The first few values of A_n and A'_n are shown below:

$$A_1 = 5,\ A_2 = 73,\ A_3 = 1445,\ A_4 = 33001,\ A_5 = 819005,\ A_6 = 21460825,$$
$$A'_1 = 3,\ A'_2 = 19,\ A'_3 = 147,\quad A'_4 = 1251,\quad A'_5 = 11253,\quad A'_6 = 104959.$$

It is well known (see [4]) that

$$(n+1)^3A_{n+1} = (2n+1)(17n(n+1)+5)A_n - n^3A_{n-1} \quad (n \ge 1),$$
$$(n+1)^2A'_{n+1} = (11n(n+1)+3)A'_n + n^2A'_{n-1} \quad (n \ge 1).$$

Let $\mathbb{Z}^+$ denote the set of positive integers. In [3] Beukers showed that for any prime $p > 3$ and $m, r \in \mathbb{Z}^+$,

$$A_{mp^r-1} \equiv A_{mp^{r-1}-1} \pmod{p^{3r}},\quad A'_{mp^r-1} \equiv A'_{mp^{r-1}-1} \pmod{p^{3r}}.$$

In [4] Beukers conjectured that for any odd prime p,

$$A'_{\frac{p-1}{2}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2}, & \text{if } 4 \mid p-1 \text{ and so } p = x^2+4y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

This was proved by several authors including Ishikawa [7] ($p \equiv 1 \pmod 4$), Van Hamme [19] ($p \equiv 3 \pmod 4$) and Ahlgren [1].

In Section 2, we establish the congruence for $A'_{\frac{p-1}{2}}$ modulo p^3, where p is an odd prime. In particular, we prove that

$$A'_{\frac{p-1}{2}} \equiv \frac{p^2}{3\binom{(p-3)/2}{(p-3)/4}^2} \pmod{p^3} \quad \text{for any prime } p \equiv 3 \pmod 4, \tag{1.2}$$

which was conjectured by the author in [13].

In Section 3, we investigate the identities and congruences for $\{t_n\}$, where the sequence $\{t_n\}$ is given by

$$t_0 = 1,\ t_1 = 5 \quad \text{and} \quad t_{n+1} = (8n^2+12n+5)t_n - 4n^2(2n+1)^2t_{n-1}\ (n \ge 1). \tag{1.3}$$

The initial values of t_n are shown below:

$$t_1 = 5,\ t_2 = 89,\ t_3 = 3429,\ t_4 = 230481,\ t_5 = 23941125,\ t_6 = 3555578025.$$

We show that

$$t_n^2 = -(2n+1)!^2 \sum_{k=0}^{2n+1} \binom{2n+1+k}{2k}\binom{2k}{k}^2 \frac{1}{(-4)^k} \sum_{i=1}^{k} \frac{1}{(2i-1)^2} \quad (n = 0, 1, 2, \ldots),$$

and obtain the congruences for $t_p \pmod{p^3}$, $t_{p-1} \pmod{p^2}$ and $t_{\frac{p\pm1}{2}} \pmod{p^2}$, where p is an odd prime. For example,

$$t_p \equiv \big(1 + 4(-1)^{\frac{p-1}{2}}\big)p^2 \pmod{p^3} \quad \text{and} \quad t_{\frac{p-1}{2}} \equiv pB_{p-1} - p + 2^{p-1} - 1 \pmod{p^2},$$

where $\{B_n\}$ are the Bernoulli numbers given by $B_0 = 1$ and $\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0$ $(n \geq 2)$. We also prove that for any prime $p > 3$ of the form $4k+3$,

$$A'_{\frac{p-1}{2}} \equiv \frac{4}{3}p^2 t^2_{\frac{p-3}{4}} \pmod{p^3},$$

which can be viewed as the connection between A'_n and t_n.

Throughout this paper, the harmonic numbers $\{H_n\}$ are given by

$$H_0 = 0 \quad \text{and} \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\ (n \geq 1),$$

the Fermat quotient $q_p(a) = (a^{p-1} - 1)/p,$ and the Euler numbers $\{E_n\}$ are defined by

$$E_{2n-1} = 0, \quad E_0 = 1 \quad \text{and} \quad E_{2n} = -\sum_{k=0}^{n-1}\binom{2n}{2k}E_{2k} \quad (n \geq 1).$$

2 Congruences for $A'_{\frac{p-1}{2}}$ modulo p^3

Let $\{D_n\}$ be defined by $D_0 = 0$ and

$$D_n = 2\sum_{1\leq i<j\leq n}\frac{1}{(2i-1)(2j-1)} = \Big(\sum_{i=1}^{n}\frac{1}{2i-1}\Big)^2 - \sum_{i=1}^{n}\frac{1}{(2i-1)^2} \quad (n \geq 1).$$

Lemma 2.1. *For* $n = 0, 1, 2, \dots$ *we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\binom{2k}{k}}{4^k}D_k = \frac{\binom{2n}{n}}{4^n}D_n.$$

Proof. Let

$$S_1(n) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\binom{2k}{k}}{4^k}D_k \quad \text{and} \quad S_2(n) = \frac{\binom{2n}{n}}{4^n}D_n.$$

Then $S_0(0) = 0 = S_2(0),\ S_1(1) = 0 = S_2(1),\ S_1(2) = \frac{1}{4} = S_2(2)$. Using the software *Sigma* we find that for $i = 1, 2$,

$$\begin{aligned}&8(n+1)(n+2)(n+3)S_i(n+3) - 12(n+1)(n+2)(2n+3)S_i(n+2)\\&+ 2(n+1)(12n^2+24n+13)S_i(n+1) - (2n+1)^3S_i(n) = 0 \quad (n = 0, 1, 2, \dots).\end{aligned}$$

Hence, $S_1(n) = S_2(n)$ for $n = 0, 1, 2, \dots$. This proves the lemma. □

Lemma 2.2 ([11, Theorem 2.2]). *Let* $\{a_n\}$ *be a sequence satisfying*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k = a_n \quad (n = 0, 1, 2, \dots).$$

Then

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k a_k = 0 \quad (n = 1, 3, 5, \dots).$$

Lemma 2.3. *Let p be an odd prime. Then*

$$A'_{\frac{p-1}{2}} \equiv 1 + \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k}\Big(1 - p\sum_{i=1}^{k}\frac{1}{2i-1} + \frac{p^2}{2}\Big(\Big(\sum_{i=1}^{k}\frac{1}{2i-1}\Big)^2 - 3\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big)\Big) \pmod{p^3}.$$

Proof. Clearly

$$\begin{aligned}
&A'_{\frac{p-1}{2}} - 1 \\
&= \sum_{k=1}^{\frac{p-1}{2}} \frac{(\frac{p-1}{2})^2(\frac{p-1}{2}-1)^2\cdots(\frac{p-1}{2}-k+1)^2}{k!^2} \cdot \frac{(\frac{p-1}{2}+1)(\frac{p-1}{2}+2)\cdots(\frac{p-1}{2}+k)}{k!} \\
&= \sum_{k=1}^{\frac{p-1}{2}} \frac{(p-1)(p-3)\cdots(p-(2k-1))\cdot(p^2-1^2)(p^2-3^2)\cdots(p^2-(2k-1)^2)}{2^{3k}\cdot k!^3} \\
&\equiv \sum_{k=1}^{(p-1)/2} \frac{(1\cdot 3\cdot\cdots\cdot(2k-1))^3}{2^{3k}\cdot k!^3}\Big(1 - p\sum_{i=1}^{k}\frac{1}{2i-1} + \frac{p^2}{2}D_k\Big)\Big(1 - p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big) \\
&\equiv \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k}\Big(1 - p\sum_{i=1}^{k}\frac{1}{2i-1} + \frac{p^2}{2}\Big(\Big(\sum_{i=1}^{k}\frac{1}{2i-1}\Big)^2 - 3\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big)\Big) \pmod{p^3}. \quad \square
\end{aligned}$$

Lemma 2.4 ([14, Theorem 4.1]). *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}, & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod 4, \\ -\frac{p^2}{4}\binom{(p-3)/2}{(p-3)/4}^{-2} \pmod{p^3}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

For an odd prime p and rational p-integer x, the p-adic Gamma function $\Gamma_p(x)$ is defined by

$$\Gamma_p(0) = 1, \quad \Gamma_p(n) = (-1)^n \prod_{\substack{k\in\{1,2,\ldots,n-1\} \\ p\nmid k}} k \qquad \text{for } n = 1,2,3,\ldots$$

and

$$\Gamma_p(x) = \lim_{\substack{n\in\{0,1,\ldots\} \\ |x-n|_p\to 0}} \Gamma_p(n).$$

Lemma 2.5 ([18, (9)]). *Let p be an odd prime. Then*

$$\Gamma_p\Big(\frac{1}{4}\Big)^4 \equiv \begin{cases} -\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1 - \frac{p^2}{2}E_{p-3}\Big) \pmod{p^3}, & \text{if } 4 \mid p-1, \\ 2^{p-3}(16 + 32p + (48 - 8E_{p-3})p^2)\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2} \pmod{p^3}, & \text{if } 4 \mid p-3. \end{cases}$$

Lemma 2.6 ([12, Theorem 2.8]). *Let p be a prime of the form $4k+1$ and so $p = x^2 + 4y^2$ with $x, y \in \mathbb{Z}$. Then*

$$\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1 - \frac{p^2}{2}E_{p-3}\Big) \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}.$$

Lemma 2.7 ([17]). *For any prime $p>3$,*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\sum_{i=1}^{k}\frac{1}{2i-1}\equiv\begin{cases}0 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ -\frac{p}{12}\Gamma_p\left(\frac14\right)^4 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4)\end{cases}$$

and

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\equiv\begin{cases}\frac12\Gamma_p(\frac14)^4E_{p-3} \pmod{p}, & \text{if } p\equiv 1 \pmod 4,\\ -\frac{1}{16}\Gamma_p\left(\frac14\right)^4 \pmod{p}, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Theorem 2.1. *Let p be an odd prime.*

(i) *If $p\equiv 3 \pmod 4$, then*

$$A'_{\frac{p-1}{2}}\equiv\frac{p^2}{3\binom{(p-3)/2}{(p-3)/4}^2} \pmod{p^3}.$$

(ii) *If $p\equiv 1 \pmod 4$ and so $p=x^2+4y^2$ with $x,y\in\mathbb{Z}$, then*

$$A'_{\frac{p-1}{2}}\equiv 4x^2-2p-\frac{p^2}{4x^2}+3p^2x^2E_{p-3}+\frac{p^2}{2}\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(\sum_{i=1}^{k}\frac{1}{2i-1}\Big)^2 \pmod{p^3}.$$

Proof. Since $A'_1=3$, the result is true for $p=3$. Now assume that $p>3$. For $\frac{p}{2}<k<p$ we see that $p\mid\binom{2k}{k}$. Thus, from Lemmas 2.5–2.7 we have

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\sum_{i=1}^{k}\frac{1}{2i-1}\equiv\begin{cases}0 \pmod{p^2}, & \text{if } 4\mid p-1,\\ -\frac{p}{12}\cdot 16\cdot 2^{p-3}\binom{(p-3)/2}{(p-3)/4}^{-2}\equiv-\frac{p}{3}\binom{(p-3)/2}{(p-3)/4}^{-2} \pmod{p^2}, & \text{if } 4\mid p-3\end{cases}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\sum_{i=1}^{k}\frac{1}{(2i-1)^2}$$
$$\equiv\begin{cases}\frac12(-4x^2)E_{p-3}=-2x^2E_{p-3} \pmod{p}, & \text{if } p\equiv 1 \pmod 4,\\ -2^{p-3}\binom{(p-3)/2}{(p-3)/4}^{-2}\equiv-\frac14\binom{(p-3)/2}{(p-3)/4}^{-2} \pmod{p}, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

By Lemmas 2.1 and 2.2,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{\binom{2k}{k}}{(-4)^k}D_k=0 \quad \text{for } n=1,3,5,\ldots. \tag{2.1}$$

Note that $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$. By [10, Lemma 2.2],

$$\binom{\frac{p-1}{2}+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2} \qquad \text{for } k=1,2,\ldots,\frac{p-1}{2}. \tag{2.2}$$

Hence, for $p\equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}D_k\equiv\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}\frac{\binom{2k}{k}}{(-4)^k}D_k=0 \pmod{p^2}.$$

Now, from the above and Lemmas 2.3 and 2.4 we deduce that for $p\equiv 3 \pmod 4$,

$$\begin{aligned}A'_{\frac{p-1}{2}}&\equiv 1+\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(1-p\sum_{i=1}^k\frac{1}{2i-1}+\frac{p^2}{2}\Big(D_k-2\sum_{i=1}^k\frac{1}{(2i-1)^2}\Big)\Big)\\&\equiv -\frac{p^2}{4}\binom{(p-3)/2}{(p-3)/4}^{-2}+\frac{p^2}{3}\binom{(p-3)/2}{(p-3)/4}^{-2}+\frac{p^2}{4}\binom{(p-3)/2}{(p-3)/4}^{-2}\\&=\frac{p^2}{3}\binom{(p-3)/2}{(p-3)/4}^{-2} \pmod{p^3},\end{aligned}$$

and for $p=x^2+4y^2\equiv 1 \pmod 4$,

$$\begin{aligned}A'_{\frac{p-1}{2}}&\equiv 1+\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(1-p\sum_{i=1}^k\frac{1}{2i-1}+\frac{p^2}{2}\Big(\Big(\sum_{i=1}^k\frac{1}{2i-1}\Big)^2-3\sum_{i=1}^k\frac{1}{(2i-1)^2}\Big)\Big)\\&\equiv 4x^2-2p-\frac{p^2}{4x^2}+\frac{p^2}{2}\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(\sum_{i=1}^k\frac{1}{2i-1}\Big)^2-\frac{3}{2}p^2(-2x^2E_{p-3}) \pmod{p^3}.\end{aligned}$$

This completes the proof. □

Conjecture 2.1. *Let p be a prime of the form $4k+1$ and so $p=x^2+4y^2$ with $x,y\in\mathbb{Z}$. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\Big(\sum_{i=1}^k\frac{1}{2i-1}\Big)^2\equiv\frac{2}{3}x^2E_{p-3} \pmod p.$$

Remark 2.1. *In [15, Conjecture 22.36], the author conjectured that for any prime $p=4k+1$ $=x^2+4y^2$ $(x,y\in\mathbb{Z})$,*

$$\begin{aligned}A'_{\frac{p-1}{2}}&\equiv\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1+\frac{p^2}{3}E_{p-3}\Big)\equiv 4x^2-2p+p^2\Big(\frac{10}{3}x^2E_{p-3}-\frac{1}{4x^2}\Big)\\&\equiv\frac{5}{3}\cdot\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2-\frac{2}{3}\Big(4x^2-2p-\frac{p^2}{4x^2}\Big) \pmod{p^3}.\end{aligned}$$

3 Identities and congruences for t_n

Let $\{t_n\}$ be defined by (1.3). In this section, we investigate the properties of t_n.

Theorem 3.1. *For $n=0,1,2,\dots$ we have*

$$t_n=(2n+1)!\sum_{k=0}^n\frac{\binom{2k}{k}}{4^k(2(n-k)+1)}.$$

Proof. For $|x|<1$ it is well known that

$$\begin{aligned}\operatorname{arctanh}(x)&=\frac12\ln\frac{1+x}{1-x}=\sum_{m=0}^\infty\frac{x^{2m+1}}{2m+1},\\\frac{1}{\sqrt{1-x^2}}&=\sum_{k=0}^\infty\binom{-\frac12}{k}(-x^2)^k=\sum_{k=0}^\infty\frac{\binom{2k}{k}}{4^k}x^{2k}.\end{aligned}$$

From [8, A028353],

$$\frac{\operatorname{arctanh}(x)}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} t_n \frac{x^{2n+1}}{(2n+1)!} \quad (|x|<1).$$

Thus,

$$\sum_{n=0}^{\infty} t_n \frac{x^{2n+1}}{(2n+1)!} = \Big(\sum_{m=0}^{\infty} \frac{x^{2m+1}}{2m+1}\Big)\Big(\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k} x^{2k}\Big).$$

Comparing the coefficients of x^{2n+1} on both sides yields the result. □

Theorem 3.2. *For* $n = 0, 1, 2, \ldots$ *we have*

$$\begin{aligned} t_n^2 &= -(2n+1)!^2 \sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k}^2 \frac{1}{(-4)^k} \sum_{i=1}^{k} \frac{1}{(2i-1)^2} \\ &= -(2n+1)!^2 \sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k}^2 \frac{1}{(-4)^k} \Big(\sum_{i=1}^{k} \frac{1}{2i-1}\Big)^2. \end{aligned}$$

Proof. Set

$$\begin{aligned} S_1(n) &= \sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k}^2 \frac{1}{(-4)^k} \sum_{i=1}^{k} \frac{1}{(2i-1)^2}, \\ S_2(n) &= -\Big(\sum_{k=0}^{n} \frac{\binom{2k}{k}}{4^k(2(n-k)+1)}\Big)^2. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &S_1(0) = -1 = S_2(0), \quad S_1(1) = -\frac{25}{36} = S_2(1), \\ &S_1(2) = -\Big(\frac{89}{120}\Big)^2 = S_2(2), \quad S_1(3) = -\Big(\frac{381}{560}\Big)^2 = S_2(3). \end{aligned}$$

Using the *Maple* software *doublesum.mpl* and the method in [5], we find that for $i = 1, 2$ and $n = 0, 1, 2, \ldots$,

$$\begin{aligned} &4(n+4)^2(2n+7)^2(2n+9)^2(4n+9)(75+72n+16n^2)S_i(n+4) \\ &- (2n+7)^2(6913575 + 17355348n + 18370228n^2 + 10658464n^3 + 3670400n^4 \\ &\quad + 751872n^5 + 84992n^6 + 4096n^7)S_i(n+3) \\ &+ (4n+11)(18889425 + 56173260n + 72583012n^2 + 53324832n^3 + 24399376n^4 \\ &\quad + 7128000n^5 + 1299328n^6 + 135168n^7 + 6144n^8)S_i(n+2) \\ &- 8(n+2)^2(1254375 + 3543600n + 4277038n^2 + 2861712n^3 + 1146240n^4 \\ &\quad + 274560n^5 + 36352n^6 + 2048n^7)S_i(n+1) \\ &+ 16(n+1)^2(n+2)^2(2n+3)^2(4n+13)(163+104n+16n^2)S_i(n) = 0. \end{aligned}$$

Hence, for $n = 0, 1, 2, \ldots$ we have $S_1(n) = S_2(n)$. That is,

$$\Big(\sum_{k=0}^{n} \frac{\binom{2k}{k}}{4^k(2(n-k)+1)}\Big)^2 = -\sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k}^2 \frac{1}{(-4)^k} \sum_{i=1}^{k} \frac{1}{(2i-1)^2}.$$

This together with Theorem 3.1 and (2.1) yields the result. □

Theorem 3.3. *Let p be an odd prime. Then*

$$\begin{aligned}t_p &\equiv \big(1+4(-1)^{\frac{p-1}{2}}\big)p^2 \pmod{p^3},\\ t_{p-1} &\equiv (-1)^{\frac{p-1}{2}}\big(2p+2^p-2+(pB_{p-1})^2\big) \pmod{p^2},\\ t_{\frac{p-1}{2}} &\equiv pB_{p-1}-p+2^{p-1}-1 \pmod{p^2},\\ t_{\frac{p+1}{2}} &\equiv pB_{p-1}-3p+2^{p-1}-1 \pmod{p^2}\end{aligned}$$

and

$$t_{\frac{p-3}{4}} \equiv \pm\frac{1}{\binom{(p-1)/2}{(p-3)/4}} \pmod{p} \quad \textit{for } p\equiv 3 \pmod 4.$$

Proof. Since $p \mid \binom{2k}{k}$ for $\frac{p}{2}<k<p$ and

$$\frac{(2p+1)!}{p^2} = (p-1)!(p+1)\cdots(2p-1)\cdot 2(2p+1) \equiv 2(p-1)!^2 \equiv 2 \pmod{p},$$

we derive that

$$\begin{aligned}t_p &= (2p+1)!\sum_{k=0}^{p}\frac{\binom{2k}{k}}{4^k(2(p-k)+1)}\\ &\equiv (2p+1)!\Big(\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k(2(p-k)+1)}+\frac{\binom{p+1}{(p+1)/2}}{4^{\frac{p+1}{2}}\cdot p}+\frac{\binom{2p}{p}}{4^p}\Big)\\ &= (2p+1)!\Big(\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k(2p+1-2k)}+\frac{\binom{p-1}{(p-1)/2}}{4^{\frac{p-1}{2}}(p+1)}+\frac{\binom{2p-1}{p-1}}{2\cdot 4^{p-1}}\Big)\\ &\equiv 2p^2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k(p+1-2k)}+2p^2\cdot(-1)^{\frac{p-1}{2}}+p^2\\ &= p^2\sum_{k=0}^{(p-1)/2}\binom{-1/2}{k}(-1)^k\frac{1}{\frac{p+1}{2}-k}+2p^2(-1)^{\frac{p-1}{2}}+p^2\\ &\equiv p^2\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(-1)^k\frac{1}{\frac{p+1}{2}-k}+2(-1)^{\frac{p-1}{2}}p^2+p^2 \pmod{p^3}.\end{aligned}$$

By [6, (1.43)],

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{x-k} = \frac{(-1)^n}{(x-n)\binom{x}{n}}. \tag{3.1}$$

Hence

$$\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(-1)^k\frac{1}{\frac{p+1}{2}-k} = \frac{(-1)^{\frac{p-1}{2}}}{\big(\frac{p+1}{2}-\frac{p-1}{2}\big)\binom{(p+1)/2}{(p-1)/2}} \equiv 2(-1)^{\frac{p-1}{2}} \pmod{p}.$$

Therefore,

$$t_p \equiv 2(-1)^{\frac{p-1}{2}}p^2+2(-1)^{\frac{p-1}{2}}p^2+p^2 = \big(1+4(-1)^{\frac{p-1}{2}}\big)p^2 \pmod{p^3}.$$

Since $p \mid (2p-1)!$ and $p \mid \binom{2k}{k}$ for $\frac{p}{2} < k < p$, we see that

$$
\begin{aligned}
t_{p-1} &= (2p-1)! \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{4^k(2p-1-2k)} \\
&\equiv (2p-1)! \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{4^k(2p-1-2k)} + (2p-1)! \frac{\binom{p-1}{(p-1)/2}}{4^{\frac{p-1}{2}} \cdot p} \\
&\equiv \frac{(2p-1)!}{2} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(-4)^k}(-1)^k \frac{1}{\frac{2p-1}{2} - k} + (p-1)!^2 \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} \pmod{p^2}.
\end{aligned}
$$

It is well known (see, for example, [9]) that $H_{\frac{p-1}{2}} \equiv -2q_p(2) \pmod{p}$ and $(p-1)! \equiv pB_{p-1} - p \pmod{p^2}$. Thus,

$$
\begin{aligned}
\frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} &= \frac{(p-1)(p-2)\cdots(p-\frac{p-1}{2})}{(\frac{p-1}{2})! \cdot 2^{p-1}} \equiv (-1)^{\frac{p-1}{2}} \frac{1 - pH_{\frac{p-1}{2}}}{2^{p-1}} \\
&\equiv (-1)^{\frac{p-1}{2}} \frac{(1+pq_p(2))^2}{2^{p-1}} = (-1)^{\frac{p-1}{2}} 2^{p-1} \pmod{p^2}
\end{aligned}
$$

and so

$$
\begin{aligned}
(p-1)!^2 \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} &\equiv (-1)^{\frac{p-1}{2}} (p-1)!^2 (2^{p-1} - 1 + 1) \\
&\equiv (-1)^{\frac{p-1}{2}} (2^{p-1} - 1) + (-1)^{\frac{p-1}{2}} (p-1)!^2 \\
&\equiv (-1)^{\frac{p-1}{2}} \big(2^{p-1} - 1 + (pB_{p-1} - p)^2\big) \\
&\equiv (-1)^{\frac{p-1}{2}} \big(2^{p-1} - 1 + (pB_{p-1})^2 + 2p\big) \pmod{p^2}.
\end{aligned}
$$

Since $\frac{\binom{2k}{k}}{(-4)^k} = \binom{-\frac{1}{2}}{k} \equiv \binom{\frac{p-1}{2}}{k} \pmod{p}$ for $k \le \frac{p-1}{2}$, using (3.1) we deduce that

$$
\begin{aligned}
&\frac{(2p-1)!}{2} \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(-4)^k}(-1)^k \frac{1}{\frac{2p-1}{2} - k} \\
&\equiv \frac{(2p-1)!}{2} \sum_{k=0}^{(p-3)/2} \binom{\frac{p-1}{2}}{k}(-1)^k \frac{1}{\frac{2p-1}{2} - k} \\
&= \frac{(2p-1)!}{2} \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k}(-1)^k \frac{1}{\frac{2p-1}{2} - k} - (-1)^{\frac{p-1}{2}} (p^2 - 1^2) \cdots (p^2 - (p-1)^2) \\
&= \frac{p \cdot (p^2 - 1^2) \cdots (p^2 - (p-1)^2)}{2} \cdot \frac{(-1)^{\frac{p-1}{2}}}{(\frac{2p-1}{2} - \frac{p-1}{2})\binom{(2p-1)/2}{(p-1)/2}} \\
&\quad - (-1)^{\frac{p-1}{2}} (p^2 - 1^2) \cdots (p^2 - (p-1)^2) \\
&\equiv (-1)^{\frac{p-1}{2}} (p-1)!^2 \frac{1}{\frac{(\frac{p}{2}+1)(\frac{p}{2}+2)\cdots(\frac{p}{2}+\frac{p-1}{2})}{(\frac{p-1}{2})!}} - (-1)^{\frac{p-1}{2}} (p-1)!^2 \\
&\equiv (-1)^{\frac{p-1}{2}} (p-1)!^2 \frac{1}{1 + \frac{p}{2} H_{\frac{p-1}{2}}} - (-1)^{\frac{p-1}{2}} (p-1)!^2
\end{aligned}
$$

$$\equiv (-1)^{\frac{p-1}{2}}(p-1)!^2\Big(\frac{1}{1-pq_p(2)}-1\Big)$$
$$\equiv (-1)^{\frac{p-1}{2}}(p-1)!^2 pq_p(2) \equiv (-1)^{\frac{p-1}{2}}(2^{p-1}-1) \pmod{p^2}.$$

Therefore,

$$t_{p-1} \equiv (-1)^{\frac{p-1}{2}}\big(2(2^{p-1}-1)+(pB_{p-1})^2+2p\big) \pmod{p^2}.$$

From [16],

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k k} \equiv \sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{4^k k} \equiv 2q_p(2) \pmod p.$$

Thus,

$$t_{\frac{p-1}{2}} = p!\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k(p-2k)} \equiv (p-1)!\Big(1-\frac{p}{2}\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k k}\Big)$$
$$\equiv (p-1)!(1-pq_p(2)) \equiv (pB_{p-1}-p)(1-pq_p(2))$$
$$\equiv pB_{p-1}-p+2^{p-1}-1 \pmod{p^2}.$$

From (1.3) and the above congruence for $t_{\frac{p-1}{2}}$ modulo p^2 we deduce that

$$t_{\frac{p+1}{2}} \equiv \Big(8\Big(\frac{p-1}{2}\Big)^2+12\Big(\frac{p-1}{2}\Big)+5\Big)t_{\frac{p-1}{2}} \equiv (2p+1)t_{\frac{p-1}{2}}$$
$$\equiv (2p+1)\big(pB_{p-1}-p+2^{p-1}-1\big) \equiv pB_{p-1}-3p+2^{p-1}-1 \pmod{p^2}.$$

For $p\equiv 3 \pmod 4$, from Theorem 3.2 and (2.2) we see that

$$t^2_{\frac{p-3}{4}} = -\Big(\frac{p-1}{2}\Big)!^2\sum_{k=0}^{(p-1)/2}\binom{\frac{p-1}{2}+k}{2k}\binom{2k}{k}^2\frac{1}{(-4)^k}\sum_{i=1}^{k}\frac{1}{(2i-1)^2}$$
$$\equiv -\Big(\frac{p-1}{2}\Big)!^2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\sum_{i=1}^{k}\frac{1}{(2i-1)^2} \pmod{p^2}.$$

Since $p\equiv 3 \pmod 4$ we have $\big(\frac{p-1}{2}\big)!^2 \equiv -(p-1)! \equiv 1 \pmod p$. By the proof of Theorem 2.1,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k}\sum_{i=1}^{k}\frac{1}{(2i-1)^2} \equiv -\frac{1}{4\binom{(p-3)/2}{(p-3)/4}^2} \pmod p.$$

Hence,

$$t^2_{\frac{p-3}{4}} \equiv \frac{1}{4\binom{(p-3)/2}{(p-3)/4}^2} \pmod p \tag{3.2}$$

and so

$$t_{\frac{p-3}{4}} \equiv \mp\frac{1}{2\binom{(p-3)/2}{(p-3)/4}} \equiv \pm\frac{1}{\binom{(p-1)/2}{(p-3)/4}} \pmod p.$$

This completes the proof. □

Corollary 3.1. *Let $p>3$ be a prime of the form $4k+3$. Then*

$$A'_{\frac{p-1}{2}} \equiv \frac{4}{3}p^2 t^2_{\frac{p-3}{4}} \pmod{p^3}.$$

Proof. This is immediate from (3.2) and Theorem 2.1(i). □

Acknowledgements

The author was supported by the National Natural Science Foundation of China (Grant No. 12271200).

References

[1] Ahlgren, S. (1999). Gaussian hypergeometric series and combinatorial congruences. In: *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics (Garvan, F. G., & Ismail, M. E. H. (eds).)*. Developments in Mathematics, vol 4. Springer, Boston, MA, 1–12.

[2] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61, 11–13.

[3] Beukers, F. (1985). Some congruences for the Apery numbers. *Journal of Number Theory*, 21(2), 141–155.

[4] Beukers, F. (1987). Another congruence for the Apéry numbers. *Journal of Number Theory*, 25(2), 201–210.

[5] Chen, W. Y. C., Hou, Q.-H., & Mu, Y.-P. (2006). A telescoping method for double summations. *Journal of Computational and Applied Mathematics*, 196(2), 553–566.

[6] Gould, H. W. (1972). *Combinatorial Identities. A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*. West Virginia University, Morgantown, WV.

[7] Ishikawa, T. (1990). Super congruence for the Apéry numbers. *Nagoya Mathematical Journal*, 118, 195–202.

[8] Sloane, N. J. A. *The On-Line Encyclopedia of Integer Sequences*. Available online at: `https://oeis.org/`.

[9] Sun, Z.-H. (2000). Congruences concerning Bernoulli numbers and Bernoulli polynomials. *Discrete Applied Mathematics*, 105(1–3), 193–223.

[10] Sun, Z.-H. (2011). Congruences concerning Legendre polynomials. *Proceedings of the American Mathematical Society*, 139(6), 1915–1929.

[11] Sun, Z.-H. (2017). Some further properties of even and odd sequences. *International Journal of Number Theory*, 13(6), 1419–1442.

[12] Sun, Z.-H. (2018). Super congruences for two Apéry-like sequences. *Journal of Difference Equations and Applications*, 24(10), 1685–1713.

[13] Sun, Z.-H. (2020). Congruences involving binomial coefficients and Apéry-like numbers. *Publicationes Mathematicae Debrecen*, 96(3–4), 315–346.

[14] Sun, Z.-H. (2022). Supercongruences involving Apéry-like numbers and binomial coefficients. *AIMS Mathematics*, 7(2), 2729–2781.

[15] Sun, Z.-H. (2025). *Binomial Coefficients, Recurrence Sequences and Congruences*. Science Press, Beijing.

[16] Tauraso, R. (2010). Congruences involving alternating multiple harmonic sums. *Electronic Journal of Combinatorics*, 17, Article ID R16.

[17] Tauraso, R. (2018). Supercongruences related to ${}_3F_2(1)$ involving harmonic numbers. *International Journal of Number Theory*, 14(4), 1093–1109.

[18] Tauraso, R. (2020). A supercongruence involving cubes of Catalan numbers. *Integers*, 20, Article ID A44.

[19] Van Hamme, L. (1987). Proof of a conjecture of Beukers on Apéry numbers. *Proceedings of the Conference on p-adic Analysis (De Grande-De Kimpe, N., & van Hamme, L. (eds.))*. Houthalen, 189–195, Vrije Univ. Brussel, Brussels, 1986.