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Three versions of an inequality

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Abstract: Three versions of the inequality

$$\sum_{i=1}^{n-1} \frac{1}{a_i(a_{i+1}+1)} \geq \frac{n}{1+\prod_{i=1}^{n} a_i} - \frac{1}{a_n(a_1+1)}$$

are formulated and proved, where $a_1, \ldots, a_n > 0$ are real numbers. An open problem is formulated.
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1 Introduction

The areas of number theory and combinatorial analysis contain a lot of different inequalities (see, e.g., [1–4]). In the present papers, three versions of an inequality are discussed with respect to the values of the real positive numbers that participate in them. This is a generalization of the

Inequality 21 in [1, p. 185], that is:

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc},$$

where $a, b, c > 0$ are real numbers.

2 Main results

Here, we will formulate three assertions, related to one inequality, but with different values of its components.

Theorem 1. *Let us have* $n \geq 3$ *real numbers* $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. *Then*

$$\sum_{i=1}^{n-1} \frac{1}{a_i(a_{i+1}+1)} \geq \frac{n}{1+\prod_{i=1}^{n} a_i} - \frac{1}{a_n(a_1+1)}. \tag{1}$$

Proof. We see directly that for $a_1, a_2 \geq 1$

$$\begin{aligned} \frac{1}{a_1(a_2+1)} + \frac{1}{a_2(a_1+1)} - \frac{2}{1+a_1a_2} &= \frac{(1+a_1a_2)(2a_1a_2+a_1+a_2) - 2a_1a_2(a_1a_2+a_1+a_2+1)}{a_1a_2(a_1+1)(a_2+1)(1+a_1a_2)} \\ &= \frac{a_1+a_2-a_1^2a_2-a_1a_2^2}{a_1a_2(a_1+1)(a_2+1)(1+a_1a_2)} \\ &= \frac{(1-a_1a_2)(a_1+a_2)}{a_1a_2(a_1+1)(a_2+1)(1+a_1a_2)} \leq 0 \end{aligned}$$

and the equality exists only when $a_1 = a_2 = 1$, i.e., (1) is not valid for arbitrary a_1, a_2, i.e., the condition $n \geq 3$ is important.

In [1] there is a proof of (1) for $n = 3$, but here we will give another proof (the author read the book [1] after proving the theorem).

For $n = 3$ we check for the real numbers $a_1, a_2, a_3 \geq 1$ that

$$\frac{1}{a_1(a_2+1)} + \frac{1}{a_2(a_3+1)} + \frac{1}{a_3(a_1+1)} - \frac{3}{1+a_1a_2a_3} \geq 0,$$

because each of the denominators is a positive number and when we translate under a common denominator all terms, we obtain that:

$$\begin{aligned} &(1+a_1a_2a_3)(a_1a_2(a_2+1)(a_3+1) + a_2a_3(a_1+1)(a_3+1) + a_1a_3(a_1+1)(a_2+1)) \\ &\quad - 3a_1a_2a_3(a_1+1)(a_2+1)(a_3+1) \\ &= a_1a_2(a_1a_3-1)^2(a_2-1) + a_2a_3(a_1a_2-1)^2(a_3-1) + a_1a_3(a_2a_3-1)^2(a_1-1) \\ &\geq 0 \end{aligned}$$

It is important to mention that the inequality for $n = 3$ is valid for arbitrary real numbers $a_1, a_2, a_3 > 0$. This fact is valid for the next two theorems, too.

Let us assume that (1) is valid and let $a_n \geq a_{n+1} \geq 1$. Then, we obtain that:

$$\begin{aligned}
X &\equiv \sum_{i=1}^{n} \frac{1}{a_i(a_{i+1}+1)} - \frac{n+1}{1+\prod_{i=1}^{n+1} a_i} + \frac{1}{a_{n+1}(a_1+1)} \\
&= \sum_{i=1}^{n-1} \frac{1}{a_i(a_{i+1}+1)} + \frac{1}{a_n(a_{n+1}+1)} + \frac{1}{a_{n+1}(a_1+1)} - \frac{n}{1+\prod_{i=1}^{n} a_i} \\
&\quad + \frac{(na_{n+1}-n-1)(\prod_{i=1}^{n} a_i) - 1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} \\
&\geq \frac{1}{a_n(a_{n+1}+1)} + \frac{1}{a_{n+1}(a_1+1)} + \frac{(na_{n+1}-n-1)(\prod_{i=1}^{n} a_i) - 1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} - \frac{1}{a_n(a_1+1)} \\
&= \frac{1}{a_n(a_{n+1}+1)} + \frac{a_n - a_{n+1}}{a_n a_{n+1}(a_1+1)} + \frac{(na_{n+1}-n-1)(\prod_{i=1}^{n} a_i) - 1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} \\
&\geq \frac{1}{a_n(a_{n+1}+1)} + \frac{(na_{n+1}-n-1)(\prod_{i=1}^{n} a_i) - 1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)}.
\end{aligned}$$

Now, there are three cases to consider.

Case 1: If $(na_{n+1}-n-1)\prod_{i=1}^{n} a_i \geq 1$, then, obviously, $X > 0$.

Case 2: If $0 \leq (na_{n+1}-n-1)\prod_{i=1}^{n} a_i < 1$, then from $a_{n+1} \geq 1$ it follows that:

$$\begin{aligned}
X &> \frac{1}{a_n(a_{n+1}+1)} + \frac{(na_{n+1}-n-1)(\prod_{i=1}^{n} a_i) - 1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} \\
&\geq \frac{1}{a_n(a_{n+1}+1)} - \frac{1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} \\
&\geq \frac{1}{a_n(a_{n+1}+1)} - \frac{1}{(1+a_n)(1+a_n a_{n+1})} \\
&= \frac{1+a_n^2 a_{n+1}}{(1+a_n)(1+a_n a_{n+1})} \\
&\geq 0.
\end{aligned}$$

Case 3: If $(na_{n+1}-n-1)\prod_{i=1}^{n} a_i < 0$, i.e., if $na_{n+1}-n-1<0$, then

$$1 \le a_{n+1} < \frac{n+1}{n}.$$

From $a_{n+1} \ge 1$ it follows that $na_{n+1}-n-1 > -1$ and

$$\begin{aligned} X &\ge \frac{1}{a_n(a_{n+1}+1)} - \frac{1+\prod_{i=1}^{n} a_i}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} \\ &= \frac{1}{a_n(a_{n+1}+1)} - \frac{1}{1+\prod_{i=1}^{n+1} a_i} \\ &\ge \frac{1}{a_n(a_{n+1}+1)} - \frac{1}{1+a_1a_na_{n+1}} \ge 0, \end{aligned}$$

because the denominators are positive numbers and

$$\begin{aligned} 1+a_1a_na_{n+1}-a_n-a_na_{n+1} &\ge 1+a_1a_na_{n+1}-a_1-a_na_{n+1} \\ &= (1-a_na_{n+1})(1-a_1) \\ &\ge 0. \end{aligned}$$

This proves Theorem 1. □

Theorem 2. *Let us have $n \ge 2$ real numbers $1 \ge a_1 \ge a_2 \ge \cdots \ge a_n > 0$ so that for each i $(1 \le i \le n)$*

$$a_i \le \frac{1}{i}. \tag{2}$$

Then (1) *is valid.*

Proof. First, for $n=2$, we can see as above that for every two real numbers $a_1, a_2 \ge 1$:

$$\frac{1}{a_1(a_2+1)} + \frac{1}{a_2(a_1+1)} - \frac{2}{1+a_1a_2} = \frac{(1-a_1a_2)(a_1+a_2)}{a_1a_2(a_1+1)(a_2+1)(1+a_1a_2)} \ge 0.$$

Therefore, here (1) is valid for $n=2$.

Let us assume that (1) is valid and $a_n \ge a_{n+1} > 0$. Now, we use (1) and as above obtain:

$$\begin{aligned} X &\ge \frac{1}{a_n(a_{n+1}+1)} + \frac{(na_{n+1}-n-1)(\prod_{i=1}^{n} a_i)-1}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)} \\ &= \frac{1}{a_n(a_{n+1}+1)} - \frac{1+(n+1-na_{n+1})\prod_{i=1}^{n} a_i}{(1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i)}. \end{aligned}$$

X is greater than or equal to 0 because $n \geq 3, a_i \leq \frac{1}{i}$, each of the denominators is a positive number and the numerator is

$$
\begin{aligned}
Y &\equiv (1+\prod_{i=1}^{n} a_i)(1+\prod_{i=1}^{n+1} a_i) - a_n(a_{n+1}+1)(1+(n+1-na_{n+1})\prod_{i=1}^{n} a_i) \\
&= 1 + \left(\prod_{i=1}^{n} a_i\right)^2 a_{n+1} + \left(\prod_{i=1}^{n} a_i\right) + \left(\prod_{i=1}^{n+1} a_i\right) + \left(\prod_{i=1}^{n} a_i\right)^2 a_{n+1} \\
&\quad - a_n - a_n a_{n+1} - a_n(n+1-na_{n+1})\prod_{i=1}^{n} a_i - a_n - a_n a_{n+1} - a_n(n+1-na_{n+1})\prod_{i=1}^{n+1} a_i \\
&= 1 - a_n - a_n a_{n+1} + \prod_{i=1}^{n} a_i \left(1 + a_{n+1} + \prod_{i=1}^{n+1} a_i + na_n a_{n+1}^2 - a_n a_{n+1} - na_n - a_n\right) \\
&\geq 1 - a_n - a_n a_{n+1} + \prod_{i=1}^{n} a_i \left(1 + a_{n+1} - a_n - a_n a_{n+1} - na_i(1-a_{n+1}^2)\right) \\
&\geq 1 - a_n - a_n a_{n+1} + \prod_{i=1}^{n} a_i \left(1 + a_{n+1} - a_n - a_n a_{n+1} - 1 + a_{n+1}^2\right) \\
&= 1 - a_n - a_n a_{n+1} - \prod_{i=1}^{n} a_i \left(a_n - a_{n+1} + a_n a_{n+1} - a_{n+1}^2\right) \\
&\geq 1 - a_n - a_n a_{n+1} - (a_n - a_{n+1})(1+a_{n+1}) \\
&\geq 1 - 2a_n(1+a_{n+1}) \\
&\geq 1 - \frac{2}{3}\cdot\frac{5}{4} \\
&= 1 - \frac{5}{6} = \frac{1}{6} > 0,
\end{aligned}
$$

i.e., the numerator $Y \geq 0$. This proves Theorem 2. □

Theorem 3. *Let us have $n \geq 8$ real numbers $a_1, a_2, \ldots, a_n$ and let*

$$\frac{n+2}{2} \leq k \leq n-3. \tag{3}$$

Let $0 < a_k \leq a_{k-1} \leq \cdots \leq a_1 \leq 1 \leq a_n \leq a_{n-1} \leq \cdots \leq a_{k+1}$ and let $a_1, a_2, \ldots, a_k$ satisfy inequality (2), and

$$\left(\prod_{i=1}^{k-1} a_i\right)\left(\prod_{i=k+1}^{n-1} a_i\right) \geq 1.$$

Then (1) is valid.

Proof. Let the real numbers $a_1, a_2, \ldots, a_n$ satisfy the conditions from the Theorem 3. Then

$$
\begin{aligned}
Z &\equiv \sum_{i=1}^{n-1} \frac{1}{a_i(a_{i+1}+1)} - \frac{n}{1+\prod_{i=1}^{n} a_i} + \frac{1}{a_n(a_1+1)} \\
&= \sum_{i=1}^{k-1} \frac{1}{a_i(a_{i+1}+1)} + \frac{1}{a_k(a_{k+1}+1)} + \sum_{i=k+1}^{n-1} \frac{1}{a_i(a_{i+1}+1)} - \frac{n}{1+\prod_{i=1}^{n} a_i} + \frac{1}{a_n(a_1+1)}
\end{aligned}
$$

$$\geq \frac{k-1}{1+\prod_{i=1}^{k-1} a_i} - \frac{1}{a_k(a_1+1)} + \frac{1}{a_k(a_{k+1}+1)} + \frac{n-k-1}{1+\prod_{i=k+1}^{n-1} a_i} - \frac{1}{a_n(a_{k+1}+1)} - \frac{n}{1+\prod_{i=1}^{n} a_i} + \frac{1}{a_n(a_1+1)}$$

$$= \frac{k-1}{1+\prod_{i=1}^{k-1} a_i} + \frac{n-k-1}{1+\prod_{i=k+1}^{n-1} a_i} - \frac{n}{1+\prod_{i=1}^{n} a_i} - \frac{a_n - a_k}{a_k a_n (a_1+1)} + \frac{a_n - a_k}{a_k a_n (a_{k+1}+1)}$$

$$= \frac{k-1}{1+\prod_{i=1}^{k-1} a_i} + \frac{n-k-1}{1+\prod_{i=k+1}^{n-1} a_i} - \frac{n}{1+\prod_{i=1}^{n} a_i} + \frac{(a_n - a_k)(a_{k+1} - a_1)}{a_k a_n (a_1+1)(a_{k+1}+1)}$$

$$\geq \frac{(k-1)(1+\prod_{i=k+1}^{n-1} a_i)(1+\prod_{i=1}^{n} a_i) + (n-k-1)(1+\prod_{i=1}^{k-1} a_i)(1+\prod_{i=1}^{n} a_i) - n(1+\prod_{i=1}^{k-1} a_i)(1+\prod_{i=k+1}^{n-1} a_i)}{(1+\prod_{i=1}^{k-1} a_i)(1+\prod_{i=k+1}^{n-1} a_i)(1+\prod_{i=1}^{n} a_i)}.$$

Z is greater than or equal to 0 because $n \geq 8, k \geq 5$, (3) are valid by condition, the denominator is a positive number and the numerator is greater than

$$(k-1)(1+\prod_{i=k+1}^{n-1} a_i)(1+\prod_{i=1}^{n} a_i) + (n-k-1)(1+\prod_{i=1}^{k-1} a_i)(1+\prod_{i=1}^{n} a_i) - n(1+\prod_{i=1}^{k-1} a_i)(1+\prod_{i=k+1}^{n-1} a_i)$$

$$\geq 2(k-1)(1+\prod_{i=k+1}^{n-1} a_i) + 2(n-k-1)(1+\prod_{i=1}^{k-1} a_i) - n(1+\prod_{i=1}^{k-1} a_i)(1+\prod_{i=k+1}^{n-1} a_i)$$

$$\geq (2k-2-n\prod_{i=1}^{k-1} a_i)\prod_{i=k+1}^{n-1} a_i + n - 4 + (1+\prod_{i=1}^{k-1} a_i) + (3n-2-2k)\prod_{i=1}^{k-1} a_i > 0,$$

because from (2):

$$\begin{aligned} 2k-2-n\prod_{i=1}^{k-1} a_i \geq 2k-2-n &\geq 0, \\ n-4 &> 0, \\ 3n-2-2k &> 0. \end{aligned}$$

This proves Theorem 3. □

3 An Open problem instead of Conclusion

We will finish with the following

Open problem: *Is it possible to improve or relax the conditions in the three theorems?*

References

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