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On circulant matrices with Fibonacci quaternions

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Abstract: In literature, there exist many papers that compute determinants and some kinds of norms of circulant matrices involving some well-known number sequences. In this paper, we obtain an explicit formula for the determinant of a circulant matrix involving the well-known Fibonacci quaternions. Then, we obtain the Euclidean and spectral norms of these matrices.

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1 Introduction

A real quaternion is defined as

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{1}$$

where q_0, q_1, q_2 and q_3 are real numbers, and the set $\{1, i, j, k\}$ is the imaginary units that satisfy the non-commutative multiplication conditions:

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1, \\ ij &= k = -ji, \;\; jk = i = -kj, \;\; ki = j = -ik. \end{aligned} \tag{2}$$

The set of quaternions is represented by

$$H = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{R}\}. \tag{3}$$

In literature, many authors have studied the matrix applications of quaternions. In [10], the authors describe some properties of the real quaternion matrices. The authors, in [3], give several new amazing linear representations of matrix quaternions by using the general linear representation form of matrix quaternions. In [6] and [8], the authors express the matrix representations of dual quaternions by using some algebraic properties. The authors represent the algebraic properties of real quaternions in detail; see [12]. In [1], Akbıyık *et al.* examine the 4×4 matrix representation of Pauli quaternions and give the Euler's and De Moivre's formulas for these quaternions. Also, they provide De Moivre's formula for the light-like Pauli quaternions. Moreover, some researchers focus on the norms of special matrices such as maximum column and row norms, Euclidean (Frobenius) norm, [2, 11, 13]. A. F. Horadam, in [9], defines the n-th quaternions as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

where F_n is the n-th Fibonacci number, and i, j, k are as defined in (2).

The mostly studied Toeplitz matrices arise in different areas of science, such as solutions to differential and integral equations, spline functions, problems in physics, mathematics, statistics, and etc. A special case of Toeplitz matrices, called circulant matrices, is an n-square matrix given by the following form:

$$\mathcal{C}_n = \begin{pmatrix} \mathcal{C}_0 & \mathcal{C}_1 & \cdots & \mathcal{C}_{n-2} & \mathcal{C}_{n-1} \\ \mathcal{C}_{n-1} & \mathcal{C}_0 & \cdots & \mathcal{C}_{n-3} & \mathcal{C}_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{C}_2 & \mathcal{C}_3 & \cdots & \mathcal{C}_0 & \mathcal{C}_1 \\ \mathcal{C}_1 & \mathcal{C}_2 & \cdots & \mathcal{C}_{n-1} & \mathcal{C}_0 \end{pmatrix}, \tag{4}$$

where each row is a cyclic shift of the row above it [5]. The circulant matrices have many applications in many fields of science, such as statistics, algebraic coding theory, acoustics, numerical analysis, number theory, graph theory, and so on. The properties of circulant matrices are well-known and mostly presented in [5]. The eigenvalues of $\mathcal{C}_n$ are

$$\lambda_m = \sum_{l=0}^{n-1} c_l \omega^{ml}, \quad m = 0, 1, 2, 3, \ldots, n-1,$$

where $\omega = e^{\left(\frac{2\pi i}{n}\right)}$ and $i = \sqrt{-1}$. Therefore, we can write the d eterminant of a non-singular circulant matrix as:

$$\det \mathcal{C}_n = \prod_{m=0}^{n-1} \left(\sum_{l=0}^{n-1} c_l \omega^{ml} \right),$$

where $l = 0, 1, \ldots, n-1$.

The norm of a matrix is a non-negative real number, which is a measure of the magnitude of the matrix. There are several different ways of defining a matrix norm, but they all share the same certain properties. Let $A = (a_{tm})$ be a n by n matrix, then the maximum column length norm is

$$c_1(A) = \max_m \sqrt{\sum_t |a_{tm}|^2} \tag{5}$$

and the maximum row length norm is

$$r_1(A) = \max_t \sqrt{\sum_m |a_{tm}|^2}. \tag{6}$$

The ℓ_p norm of A is

$$\|A\|_p = \left(\sum_{t=1}^{n} \sum_{m=1}^{n} |a_{tm}|^p\right)^{\frac{1}{p}}. \tag{7}$$

For $p = 2$, the ℓ_p norm is called the Euclidean norm and is denoted by $\|A\|_{\mathbb{E}}$. Let A^H be the conjugate transpose of matrix A and λ_t be the eigenvalue of matrix AA^H, then the spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \le t \le n} \lambda_t}. \tag{8}$$

If the matrix A equals to the Hadamard product of B and C (i.e., $A = B \circ C$), then the following relation is satisfied

$$\|A\|_2 \le r_1(B) c_1(C). \tag{9}$$

The Euclidean and spectral norm of the matrix A satisfy the following inequality:

$$\frac{1}{\sqrt{n}} \|A\|_{\mathbb{E}} \le \|A\|_2 \le \|A\|_{\mathbb{E}}. \tag{10}$$

We remind that the Chebyshev polynomials of second kind, denoted by $\mathcal{U}_n(x)$, satisfy the three-term recurrence relations

$$\mathcal{U}_{n+1}(x) = 2x\mathcal{U}_n(x) - \mathcal{U}_{n-1}(x)$$

with initial contidions $\mathcal{U}_0(x) = 1$ and $\mathcal{U}_1(x) = 2x$, or, equivalently,

$$\mathcal{U}_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0, 1, 2 \ldots$. It is also known (see, e.g., [7]) that

$$\det \begin{pmatrix} a & b & & \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix}_{n \times n} = \left(\sqrt{bc}\right)^n \mathcal{U}_n\left(\frac{a}{2\sqrt{bc}}\right).$$

In [4], the authors give that if

$$\mathcal{D}_n = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ a & b & & & & \\ c & a & b & & & \\ & c & a & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \end{pmatrix}, \tag{11}$$

then

$$\det \mathcal{D}_n = \sum_{l=1}^{n} z_l b^{n-l} \left(-\sqrt{bc}\right)^{l-1} \mathcal{U}_{l-1}\left(\frac{a}{2\sqrt{bc}}\right), \tag{12}$$

where $\mathcal{U}_l(x)$ is the l-th Chebyshev polynomial of second kind.

In this paper, we consider the n-square circulant matrix,

$$\mathcal{T}_n := \text{circ}\left(Q_1, Q_2, \ldots, Q_n\right) \tag{13}$$

where Q_n is the n-th Fibonacci quaternion. Then, we get the determinant of the matrix $\mathcal{T}_n$ by using the second-kind Chebyshev polynomials. Furthermore, we get some norms for the circulant matrices with Fibonacci quaternion coefficients.

2 Main results

This section is devoted to the determinant formula of circulant matrices with Fibonacci quaternion entries.

Let us consider $n-$square matrices P_n and H_n, as below:

$$P_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

and

$$H_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

Then, we have the following property.

Lemma 2.1. *The determinants of the matrices in (14) and (15) are*

$$\det(P_n)=\det(H_n)=\begin{cases} -1, & \begin{array}{l} n\equiv 3 \pmod 4 \\ n\equiv 0 \pmod 4 \end{array} \\ 1, & \begin{array}{l} n\equiv 1 \pmod 4 \\ n\equiv 2 \pmod 4 \end{array} \end{cases}.$$

Proof. It is known that equality holds for H_n (see [14]). By using Laplace expansion on the first row, the proof can be easily seen for P_n. □

Theorem 2.1. *For $n>3$, we have*

$$\begin{aligned} \det(\mathcal{T}_n) &= -Q_2\sum_{l=1}^{n-1} Q_{l+2}b^{l-1}\left(-\sqrt{bc}\right)^{n-l-1}\mathcal{U}_{n-l-1}\left(\frac{a}{2\sqrt{bc}}\right) \\ &\quad -Q_1\sum_{l=2}^{n-1} Q_{l+2}b^{l-2}\left(-\sqrt{bc}\right)^{n-l}\mathcal{U}_{n-l}\left(\frac{a}{2\sqrt{bc}}\right) \\ &\quad -\left(Q_1^2+Q_1Q_n\right)b^{n-2}\mathcal{U}_0\left(\frac{a}{2\sqrt{bc}}\right), \end{aligned}$$

where

$$a=Q_2-Q_{n+2},\ b=Q_1-Q_{n+1},\ c=Q_0-Q_n.$$

Proof. Let us multiply the matrices as below:

$$S_n=P_n\mathcal{T}_nH_n.$$

Then, we obtain the following matrix:

$$S_n=\begin{bmatrix} -Q_1 & -Q_{n+1} & -Q_n & \cdots & -Q_4 & -Q_2 \\ Q_2 & Q_1+Q_n & Q_{n+1} & \cdots & Q_5 & Q_3 \\ 0 & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c & a & b \end{bmatrix}, \tag{16}$$

where

$$a=Q_2-Q_{n+2},\ b=Q_1-Q_{n+1},\ c=Q_0-Q_n.$$

By adding the first column to the n-th column, we have

$$S_n=\begin{bmatrix} -Q_1 & -Q_{n+1} & -Q_n & \cdots & -Q_4 & -Q_3 \\ Q_2 & Q_1+Q_n & Q_{n+1} & \cdots & Q_5 & Q_4 \\ 0 & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c & a & b \end{bmatrix}. \tag{17}$$

Then

$$\det(S_n) = \det(P_n \mathcal{T}_n H_n) = \det(P_n)\det(\mathcal{T}_n)\det(H_n).$$

By Lemma 2, it is seen that

$$\det(S_n) = \det(\mathcal{T}_n).$$

So,

$$\det(\mathcal{T}_n) = \begin{vmatrix} -Q_1 & -Q_{n+1} & -Q_n & \cdots & -Q_4 & -Q_3 \\ Q_2 & Q_1 + Q_n & Q_{n+1} & \cdots & Q_5 & Q_4 \\ 0 & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c & a & b \end{vmatrix}. \tag{18}$$

By using the Laplace expansion on the first column and (12), the following equation is satisfied:

$$\begin{aligned} \det(\mathcal{T}_n) &= -Q_2 \sum_{l=1}^{n-1} Q_{l+2} b^{l-1} \left(-\sqrt{bc}\right)^{n-l-1} \mathcal{U}_{n-l-1}\left(\frac{a}{2\sqrt{bc}}\right) \\ &\quad -Q_1 \sum_{l=2}^{n-1} Q_{l+2} b^{l-2} \left(-\sqrt{bc}\right)^{n-l} \mathcal{U}_{n-l}\left(\frac{a}{2\sqrt{bc}}\right) \\ &\quad -\left(Q_1^2 + Q_1 Q_n\right) b^{n-2} \mathcal{U}_0\left(\frac{a}{2\sqrt{bc}}\right) \end{aligned}$$

where $\mathcal{U}_l(x)$ is the l-th Chebyshev polynomial of second kind. □

Example 1. For $n = 4$,

$$\mathcal{T}_4 = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ Q_4 & Q_1 & Q_2 & Q_3 \\ Q_3 & Q_4 & Q_1 & Q_2 \\ Q_2 & Q_3 & Q_4 & Q_1 \end{bmatrix},$$

i.e.,

$$\mathcal{T}_4 = \begin{bmatrix} 1+i+2j+3k & 1+2i+3j+5k & 2+3i+5j+8k & 3+5i+8j+13k \\ 3+5i+8j+13k & 1+i+2j+3k & 1+2i+3j+5k & 2+3i+5j+8k \\ 2+3i+5j+8k & 3+5i+8j+13k & 1+i+2j+3k & 1+2i+3j+5k \\ 1+2i+3j+5k & 2+3i+5j+8k & 3+5i+8j+13k & 1+i+2j+3k \end{bmatrix}.$$

Since

$$S_n = P_n \mathcal{T}_n H_n,$$

where

$$P_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

and

$$H_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

then the following matrix is obtained:

$$\mathcal{S}_4 = \begin{bmatrix} -Q_1 & -Q_5 & -Q_4 & -Q_3 \\ Q_2 & Q_1 + Q_4 & Q_5 & Q_4 \\ 0 & Q_2 - Q_6 & Q_2 - Q_5 & 0 \\ 0 & Q_0 - Q_4 & Q_2 - Q_6 & Q_1 - Q_5 \end{bmatrix},$$

i.e.,

$$\mathcal{S}_4 = \begin{bmatrix} -1 - i - 2j - 3k & -5 - 8i - 13j - 21k & -3 - 5i - 8j - 13k & -2 - 3i - 5j - 8k \\ 1 + 2i + 3j + 5k & 4 + 6i + 10j + 16k & 5 + 8i + 13j + 21k & 3 + 5i + 8j + 13k \\ 0 & -7 - 11i - 18j - 29k & -4 - 6i - 10j - 16k & 0 \\ 0 & -3 - 4i - 7j - 11k & -7 - 11i - 18j - 29k & -4 - 6i - 10j - 16k \end{bmatrix}.$$

Since

$$\det(S_4) = \det(P_4 \mathcal{T}_4 H_4) = \det(\mathcal{T}_4),$$

we have

$$\begin{aligned} \det(\mathcal{T}_4) &= -Q_2 \sum_{l=1}^{3} Q_{l+2} b^{l-1} \left(-\sqrt{bc}\right)^{3-l} \mathcal{U}_{3-l}\left(\frac{a}{2\sqrt{bc}}\right) \\ &\quad -Q_1 \sum_{l=2}^{3} Q_{l+2} b^{l-2} \left(-\sqrt{bc}\right)^{4-l} \mathcal{U}_{4-l}\left(\frac{a}{2\sqrt{bc}}\right) \\ &\quad -\left(Q_1^2 + Q_1 Q_4\right) b^2 \mathcal{U}_0\left(\frac{a}{2\sqrt{bc}}\right) \\ &= -Q_2 \left[Q_3 b^0 \left(-\sqrt{bc}\right)^2 \mathcal{U}_2\left(\frac{a}{2\sqrt{bc}}\right) + Q_4 b^1 \left(-\sqrt{bc}\right)^1 \mathcal{U}_1\left(\frac{a}{2\sqrt{bc}}\right) \right. \\ &\quad \left. + Q_5 b^2 \left(-\sqrt{bc}\right)^0 \mathcal{U}_0\left(\frac{a}{2\sqrt{bc}}\right) \right] \\ &\quad -Q_1 \left[Q_4 b^0 \left(-\sqrt{bc}\right)^2 \mathcal{U}_2\left(\frac{a}{2\sqrt{bc}}\right) + Q_5 b^1 \left(-\sqrt{bc}\right)^1 \mathcal{U}_1\left(\frac{a}{2\sqrt{bc}}\right) \right] \\ &\quad -\left(Q_1^2 + Q_1 Q_4\right) b^2 \mathcal{U}_0\left(\frac{a}{2\sqrt{bc}}\right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\det(\mathcal{T}_4) &= -Q_2Q_3\left(-\sqrt{bc}\right)^2\left(4\left(\frac{a}{2\sqrt{bc}}\right)^2-1\right)-Q_2Q_4b\left(-\sqrt{bc}\right)2\left(\frac{a}{2\sqrt{bc}}\right)\\
&\quad -Q_2Q_5b^2-Q_1Q_4\left(-\sqrt{bc}\right)^2\left(4\left(\frac{a}{2\sqrt{bc}}\right)^2-1\right)\\
&\quad -Q_1Q_5b\left(-\sqrt{bc}\right)2\left(\frac{a}{2\sqrt{bc}}\right)-\left(Q_1^2+Q_1Q_4\right)b^2\\
&= Q_2Q_3bc-a^2Q_2Q_3+Q_2Q_4ab-Q_2Q_5b^2-a^2Q_1Q_4+Q_1Q_4bc\\
&\quad +Q_1Q_5ab-(Q_1^2+Q_1Q_4)b^2\\
&= -61287+18370i+26782j+47671k
\end{aligned}$$

where $a=Q_2-Q_6,\ b=Q_1-Q_5,\ c=Q_0-Q_4,\ \mathcal{U}_0(x)=1,\ \mathcal{U}_1(x)=2x,\ \mathcal{U}_2(x)=4x^2-1.$

3 On norms of $\mathcal{T}_n$ matrices

In this section, we initially obtain summation formulas for the Fibonacci numbers. Then, we obtain some norms for them.

Lemma 3.1. *Let F_n be the n-th Fibonacci number. Then, the following summation formulas are satisfied:*

- $F_1F_2+F_2F_3+\cdots+F_nF_{n+1}=\begin{cases}F_{n+1}^2, & \text{if } n \text{ odd}\\ F_{n+1}^2-1, & \text{if } n \text{ even}\end{cases}$
- $F_1F_3+F_2F_4+\cdots+F_nF_{n+2}=\begin{cases}F_{n+1}F_{n+2}, & \text{if } n \text{ odd}\\ F_{n+1}F_{n+2}-1, & \text{if } n \text{ even}\end{cases}$
- $F_1F_4+F_2F_5+\cdots+F_nF_{n+3}=\begin{cases}F_{n+1}F_{n+3}, & \text{if } n \text{ odd}\\ F_{n+1}F_{n+3}-2, & \text{if } n \text{ even}\end{cases}$

Proof. We apply the mathematical induction on n. For $n=1$,

$$F_1F_2=1=F_2^2$$

and for $n=2$,

$$F_1F_2+F_2F_3=1.1+1.2=F_3^2-1$$

are satisfied. Assume that for $n=2k+1$ an arbitrary odd number where k is an integer,

$$F_1F_2+F_2F_3+\cdots+F_nF_{n+1}=F_1F_2+F_2F_3+\cdots+F_{2k+1}F_{2k+2}=F_{2k+2}^2$$

and for $n = 2k$ an arbitrary even number,

$$F_1F_2 + F_2F_3 + \cdots + F_nF_{n+1} = F_1F_2 + F_2F_3 + \cdots + F_{2k}F_{2k+1} = F_{2k}^2 - 1$$

hold. For $n \to n+1 = 2k+2$,

$$\begin{aligned} F_1F_2 + F_2F_3 + \cdots + F_{n+1}F_{n+2} &= F_1F_2 + F_2F_3 + \cdots + F_{2k+2}F_{2k+3} \\ &= F_{2k+2}(F_{2k+2} + F_{2k+3}) \\ &= F_{2k+2}F_{2k+4} \\ &= (-1)^{2k+3} + F_{2k+3}^2 \\ &= F_{n+2}^2 - 1 \end{aligned}$$

when n is odd. For $n \to n+1 = 2k+1$,

$$F_1F_2 + F_2F_3 + \cdots + F_{n+1}F_{n+2} = F_1F_2 + F_2F_3 + \cdots + F_{2k+1}F_{2k+2} = F_{2k+1}^2$$

when n is even.
As a result, we can say that

$$F_1F_2 + F_2F_3 + \cdots + F_nF_{n+1} = F_{n+1}^2$$

when n is an odd number, and

$$F_1F_2 + F_2F_3 + \cdots + F_nF_{n+1} = F_{n+1}^2 - 1$$

when n is even. The other identities can be obtained by following similar steps of the proof. □

Theorem 3.1. *The maximum column length norm and the maximum row length norm of $\mathcal{T}_n$ is given as follows:*

- *If n is odd, then*

$$c_1(\mathcal{T}_n) = r_1(\mathcal{T}_n) = \sqrt{-F_{n+1}^2 + F_{2n+6} + 9 + 2iF_{n+1}^2 + 2jF_{n+1}F_{n+2} + 2kF_{n+1}F_{n+3}}.$$

- *If n is even, then*
$c_1(\mathcal{T}_n) = r_1(\mathcal{T}_n) = \sqrt{-F_{n+1}^2 + F_{2n+6} + 9 + 2i(F_{n+1}^2 - 1) + 2j(F_{n+1}F_{n+2} - 1) + 2k(F_{n+1}F_{n+3} - 2)},$
where $Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$ is the n-th Fibonacci quaternion.

Proof. From the definition of the maximum column length norm and the maximum row length norm, we get

$$\begin{aligned} c_1(\mathcal{T}_n) &= \max_m \sqrt{\sum_t |\mathcal{T}_{tm}|^2} \\ &= r_1(\mathcal{T}_n) \\ &= \max_t \sqrt{\sum_m |\mathcal{T}_{tm}|^2} \\ &= \sqrt{Q_1^2 + Q_2^2 + \cdots + Q_n^2} \\ &= \sqrt{(F_1 + iF_2 + jF_3 + kF_4)^2 + \cdots + (F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})^2}, \end{aligned}$$

where $\mathcal{T}_{tm}$ is t-th column and m-th row of $\mathcal{T}_n$.

The square of any quaternion $q = a + ib + jc + kd$ can be calculated as

$$q^2 = (a + ib + jc + kd)^2 = a^2 - b^2 - c^2 - d^2 + 2iab + 2jac + 2kad,$$

where $a, b, c, d \in \mathbb{R}$. So, we obtain the following squares:

$$\begin{aligned}
(F_1 + iF_2 + jF_3 + kF_4)^2 &= F_1^2 - F_2^2 - F_3^2 - F_4^2 \\
&\quad +2iF_1F_2 + 2jF_1F_3 + 2kF_1F_4 \\
(F_2 + iF_3 + jF_4 + kF_5)^2 &= F_2^2 - F_3^2 - F_4^2 - F_5^2 \\
&\quad +2iF_2F_3 + 2jF_2F_4 + 2kF_2F_5 \\
&\ \vdots \\
(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})^2 &= F_n^2 - F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2 \\
&\quad +2iF_nF_{n+1} + 2jF_nF_{n+2} + 2kF_nF_{n+3}.
\end{aligned}$$

The summation of the right-hand sides of all equations is

$$\begin{aligned}
&(F_1^2 + F_2^2 + \cdots + F_n^2) - (F_2^2 + F_3^2 + \cdots + F_{n+1}^2) - (F_3^2 + F_4^2 + \cdots + F_{n+2}^2) - (F_4^2 + F_5^2 + \cdots + F_{n+3}^2) \\
&+ 2i(F_1F_2 + F_2F_3 + \cdots + F_nF_{n+1}) + 2j(F_1F_3 + F_2F_4 + \cdots + F_nF_{n+2}) \\
&+ 2k(F_1F_4 + F_2F_5 + \cdots + F_nF_{n+3}) \\
&= F_nF_{n+1} - F_{n+1}F_{n+2} - F_{n+2}F_{n+3} - F_{n+3}F_{n+4} + 9 \\
&\quad + \begin{cases} 2iF_{n+1}^2 + 2jF_{n+1}F_{n+2} + 2kF_{n+1}F_{n+3}, & \text{if } n \text{ odd} \\ 2i(F_{n+1}^2 - 1) + 2j(F_{n+1}F_{n+2} - 1) + 2k(F_{n+1}F_{n+3} - 2), & \text{if } n \text{ even} \end{cases} \\
&= -F_{n+1}^2 + F_{2n+6} + 9 + \begin{cases} 2iF_{n+1}^2 + 2jF_{n+1}F_{n+2} + 2kF_{n+1}F_{n+3}, & \text{if } n \text{ odd} \\ 2i(F_{n+1}^2 - 1) + 2j(F_{n+1}F_{n+2} - 1) + 2k(F_{n+1}F_{n+3} - 2), & \text{if } n \text{ even} \end{cases}.
\end{aligned}$$

□

Theorem 3.2. *If $\mathcal{T}_n$ is a circulant matrix of Fibonacci quaternions in (13), then the Euclidean norm is*

$$\|\mathcal{T}_n\|_E = \sqrt{n}c_1(\mathcal{T}_n) = \sqrt{n}r_1(\mathcal{T}_n).$$

Proof. From the definition of Euclidean norm of a matrix, we obtain

$$\begin{aligned}
\|\mathcal{T}_n\|_E &= \Big(\sum_{t=1}^{n}\sum_{m=1}^{n} |\mathcal{T}_{tm}|^2\Big)^{\frac{1}{2}} \\
&= \sqrt{n(Q_1^2 + Q_2^2 + \cdots + Q_n^2)} \\
&= \sqrt{n}\sqrt{Q_1^2 + Q_2^2 + \cdots + Q_n^2} \\
&= \sqrt{n}c_1(\mathcal{T}_n) = \sqrt{n}r_1(\mathcal{T}_n).
\end{aligned}$$

□

4 Conclusion

In literature, there are a huge amount of papers on determinants of circulant matrices with some famous number sequence entries. In this paper, we consider the circulant matrix family $\mathcal{T}_n$ whose entries are Fibonacci quaternions. Then, we compute the determinant of $\mathcal{T}_n$ by exploiting the well-known Chebyshev polynomials of the second kind. Moreover, we obtain some norms for $\mathcal{T}_n$ matrices.

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