

A note on solitary numbers

Sagar Mandal 

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur
Kalyanpur, Kanpur, Uttar Pradesh 208016, India
e-mail: sagarmandal31415@gmail.com

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Abstract: Does 14 have a friend? Until now, this has been an open question. In this note, we prove that a potential friend F of 14 is an odd, non-square positive integer. 7 appears in the prime factorization of F with an even exponent while at most two prime divisors of F can have odd exponents in the prime factorization of F . If $p \mid F$ such that p is congruent to 7 modulo 8, then $p^{2a} \parallel F$, for some positive integer a . Further, no prime divisor of F has an exponent congruent to 7 modulo 8 and no prime divisor can exceed $1.4\sqrt{F}$. The primes 3, 5 cannot appear simultaneously in the prime factorization of F . If $(3, F) > 1$ or $(5, F) > 1$, then $\omega(F) \geq 4$, otherwise $\omega(F) \geq 8$.

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1 Introduction

In number theory, the sum of divisors function $\sigma(n)$ plays a central role in studying the properties of integers. For a positive integer n , the abundancy index is defined as $I(n) = \frac{\sigma(n)}{n}$. More generally, abundancy index can be considered as a measure of perfection of an integer, the abundancy index can be used to classify numbers as perfect, abundant, or deficient. A number is perfect if $I(n) = 2$, abundant if $I(n) > 2$, and deficient if $I(n) < 2$. Two distinct positive integers m and n are called friends if they share the same abundancy index, that is, $I(m) = I(n)$.



For example, all perfect numbers (OEIS A000396) are friends of each other, since they all have abundancy index 2. If a number has no friend, it is called solitary. It is easy to prove [1] that if a positive integer n is co-prime to $\sigma(n)$, then n is a solitary number (for example, see OEIS A014567). Anderson and Hickerson [1] stated that the density of such solitary numbers is zero. Although the concept of friendly numbers is simple, many interesting and difficult problems remain unsolved. It is not known whether 10 has a friend, though many necessary conditions have been proposed [4, 5, 7, 8]. Moreover, among numbers less than 100, those known to have friends are

$$6, 12, 24, 28, 30, 40, 42, 56, 60, 66, 78, 80, 84, 96.$$

For further results on the subject, see [1–3, 9].

Recent works have focused on specific integers and their possible friends. For example, Ward [8] investigated whether 10 has a friend, while Terry [6] studied the case of 15. The status of several positive integers, including 14, 15, and 20, is still unresolved. If any of the suspected solitary numbers up to 372 is actually a friendly number, then its smallest friend must be strictly greater than 10^{30} (see OEIS A074902). In this paper, we investigate friends of 14 and we give certain properties of a potential friend of 14.

2 Properties of the abundancy index

Some elementary properties of the abundancy index are given below, and the proofs of the following lemmas may be found in [3, 9].

Lemma 2.1. *$I(n)$ is weakly multiplicative, that is, for any two co-prime positive integers n and m we have $I(nm) = I(n)I(m)$.*

Lemma 2.2. *If γ, n are two positive integers and $\gamma > 1$. Then $I(\gamma n) > I(n)$.*

Lemma 2.3. *If $p_1, p_2, p_3, \dots, p_k$ are k distinct prime numbers and $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_k$ are positive integers, then*

$$I\left(\prod_{i=1}^k p_i^{\gamma_i}\right) = \prod_{i=1}^k \left(\sum_{j=0}^{\gamma_i} p_i^{-j}\right) = \prod_{i=1}^k \frac{p_i^{\gamma_i+1} - 1}{p_i^{\gamma_i}(p_i - 1)}.$$

Lemma 2.4. *If p_1, \dots, p_k are distinct prime numbers, q_1, \dots, q_k are distinct prime numbers such that $p_i \leq q_i$ for all $1 \leq i \leq k$. If $\gamma_1, \gamma_2, \dots, \gamma_k$ are positive integers, then*

$$I\left(\prod_{i=1}^k p_i^{\gamma_i}\right) \geq I\left(\prod_{i=1}^k q_i^{\gamma_i}\right).$$

Lemma 2.5. *If $n = \prod_{i=1}^k p_i^{\gamma_i}$, then $I(n) < \prod_{i=1}^k \frac{p_i}{p_i - 1}$.*

Throughout this article, we use $p, p_1, \dots, p_{\omega(F)}, p_k$ for denoting prime numbers. Further, we assume that the numbers $a_1, a_2, \dots, a_{\omega(F)}, a_k$ are positive integers.

3 Main results

Note that, $I(14) = \frac{12}{7}$, therefore, a positive integer F is said to be a friend of 14 if $I(F) = I(14) = \frac{12}{7}$. The following results describe certain characteristics of a friend of 14.

Theorem 3.1. *Let F be a friend of 14, then F is an odd positive non-square integer.*

Proof. Let a positive integer F be a friend of 14. Then F must be greater than 14 as for any positive integer less than 14 we have $I(F) \neq \frac{12}{7}$. Since $\frac{\sigma(F)}{F} = I(F) = \frac{12}{7}$ we have $7\sigma(F) = 12F$, from which it follows that $7 \mid F$ and $12 \mid \sigma(F)$ as $(7, 12) = 1$. Therefore, we can write $F = 7F'$ for some positive integer $F' > 2$.

Let us assume that F' is an even positive integer. Then we can rewrite F as $F = 14F''$ for some positive integer $F'' > 1$, but then $I(F) > I(14)$ by Lemma 2.2. Therefore, F' is an odd positive integer and thus F is an odd positive integer.

Now if $F = 7^a$ for some positive integer $a > 2$, then it cannot be a friend of 14 as by Lemma 2.5 we have $I(7^a) < \frac{7}{6} < \frac{12}{7}$. Therefore, F must be written as $F = 7^{a_1} \cdot \prod_{i=2}^{\omega(F)} p_i^{a_i} (p_1 = 7)$.

Let us suppose that all a_i are even. Then

$$I(F) = \frac{\sigma(F)}{F} = \frac{12}{7},$$

using Lemma 2.1 we get

$$I(7^{a_1}) \cdot \prod_{i=2}^{\omega(F)} I(p_i^{a_i}) = \frac{12}{7}.$$

This implies

$$\sigma(7^{a_1}) \cdot \prod_{i=2}^{\omega(F)} \sigma(p_i^{a_i}) = 12 \cdot 7^{a_1-1} \cdot \prod_{i=2}^{\omega(F)} p_i^{a_i},$$

that is

$$(1 + 7 + \cdots + 7^{a_1}) \cdot \prod_{i=2}^{\omega(F)} (1 + p_i + \cdots + p_i^{a_i}) = 12 \cdot 7^{a_1-1} \cdot \prod_{i=2}^{\omega(F)} p_i^{a_i}.$$

Since $p_i > 2$, for all $1 \leq i \leq \omega(F)$ the right-hand side of the above expression is an even integer but the left-hand side is odd since a_i are even, which immediately implies that $(1 + p_i + \cdots + p_i^{a_i})$ is odd, which is absurd. Hence, all a_i cannot be even integers. Therefore, F is a non-square positive integer. This proves that F is an odd positive non-square integer. \square

Remark 3.1. *If F is a friend of 14, then from the proof of Theorem 3.1, we can note that $4 \parallel \sigma(F)$, as F is an odd positive integer.*

Remark 3.1 is very crucial as we will be using it enormously in the upcoming proofs.

Theorem 3.2. *Let F be a friend of 14. If $p \mid F$ such that $p \equiv 7 \pmod{8}$, then p appears in the prime factorization of F to an even exponent.*

Proof. Suppose, for contradiction, that p congruent to 7 modulo 8 is a prime divisor of F with an odd exponent, that is, $p^{2a+1} \parallel F$, for some non-negative integer a . Then

$$\sigma(p^{2a+1}) = 1 + p + p^2 + \cdots + p^{2a+1} \equiv 1 - 1 + 1 - \cdots - 1 \pmod{8} = 0.$$

This implies that $8 \mid \sigma(p^{2a+1})$ and so $8 \mid \sigma(F)$, but this is a contradiction by Remark 3.1. Therefore, the exponent of p in the prime factorization of F must be an even positive integer. \square

An immediate consequence of the preceding theorem, the following corollary describes that the exponent of the prime divisor 7 of F cannot be odd.

Corollary 3.1. *If F is a friend of 14, then $7^{2a} \parallel F$, for some positive integer a .*

Proof. Since F is a friend of 14, 7 is a prime divisor of F . As 7 is a prime that satisfies $7 \equiv 7 \pmod{8}$, it follows from Theorem 3.2 that $7^{2a} \parallel F$, for some positive integer a . \square

Theorem 3.3. *If F is a friend of 14, then no prime divisor of F has an exponent congruent to 7 modulo 8.*

Proof. Let us assume that p is a prime divisor of F with an exponent congruent to 7 modulo 8, that is, $p^a \parallel F$ where $a \equiv 7 \pmod{8}$. Then

$$\sigma(p^a) = 1 + p + \cdots + p^a \equiv \begin{cases} 1 + (\pm 1) + \cdots + (\pm 1)^a \pmod{8} \\ 1 + (\pm 3) + \cdots + (\pm 3)^a \pmod{8} \end{cases}$$

since $a \equiv 7 \pmod{8}$ we have

$$1 + (\pm 1) + \cdots + (\pm 1)^a = \begin{cases} a + 1 \equiv 0 \pmod{8} \\ 0 \equiv 0 \pmod{8} \end{cases}$$

and

$$1 + (\pm 3) + \cdots + (\pm 3)^a = \begin{cases} \frac{3^{a+1} - 1}{2} \equiv 0 \pmod{8} \\ \frac{1 - 3^{a+1}}{4} \equiv 0 \pmod{8} \end{cases}.$$

This shows that $8 \mid \sigma(p^a)$, which implies $8 \mid \sigma(F)$ but $4 \nmid \sigma(F)$ from Remark 3.1. Hence, no prime divisor of F has an exponent congruent to 7 modulo 8. \square

We may ask how many distinct prime divisors of F can have odd exponents in the prime factorization of F ? The following theorem answers the question.

Theorem 3.4. *If F is a friend of 14, then at most two distinct prime divisors of F have odd exponents in the prime factorization of F .*

Proof. Suppose, for contradiction, that F has three distinct prime divisors p_1, p_2, p_3 with odd exponents a_1, a_2, a_3 , respectively, in the prime factorization of F . Then

$$\sigma(p_i^{a_i}) = 1 + p_i + \cdots + p_i^{a_i} \equiv 1 + 1 + \cdots + 1 \pmod{2} = 0.$$

This implies that $2 \mid \sigma(p_i^{a_i})$, for $i = 1, 2, 3$. Thus, we get $8 \mid \sigma(p_1^{a_1})\sigma(p_2^{a_2})\sigma(p_3^{a_3})$, that is, $8 \mid \sigma(F)$ but this contradicts Remark 3.1. Therefore, we conclude that at most two distinct prime divisors of F can have odd exponents in the prime factorization of F . \square

Theorem 3.5. *If 3 is a divisor of a friend F of 14, then $3 \parallel F$.*

Proof. Let $F = 3^a \cdot 7^{2b} \cdot m$ be a friend of 14, where a, b, m are positive integers. If $a \geq 3$, then using Lemma 2.2 we get

$$I(3^a \cdot 7^{2b} \cdot m) \geq I(3^3 \cdot 7^2) = \frac{760}{441} > I(14).$$

Therefore $a \leq 2$. Let us suppose that $a = 2$. Then

$$\frac{\sigma(3^2 \cdot 7^{2b} \cdot m)}{3^2 \cdot 7^{2b} \cdot m} = I(3^2 \cdot 7^{2b} \cdot m) = \frac{12}{7},$$

which is equivalent to

$$\sigma(3^2) \cdot \sigma(7^{2b}) \cdot \sigma(m) = 12 \cdot 3^2 \cdot 7^{2b-1} \cdot m$$

since $\sigma(3^2) = 13$, $13 \mid 12 \cdot 3^2 \cdot 7^{2b-1} \cdot m$, that is, $13 \mid m$. Therefore, let $m = 13m'$, for some positive integer m' . Then we have $F = 3^2 \cdot 7^{2b} \cdot 13m'$. Using Lemma 2.2, we get that

$$I(F) \geq I(3^2 \cdot 7^2 \cdot 13) = \frac{38}{21} > I(14).$$

Hence a cannot be 2. This completes the proof. \square

Lemma 3.1. *If F is a friend of 14, then 3 and 5 cannot appear simultaneously in the prime factorization of F .*

Proof. Let F be a friend of 14 and if possible, assume that 3, 5 appears simultaneously in the prime factorization of F . Then, using Lemma 2.2, we have

$$I(F) \geq I(3 \cdot 5 \cdot 7^2) = \frac{456}{245} > I(14).$$

Therefore, it follows that either 3 or 5 can appear in the prime factorization of F , but not together. \square

We now give the lower bounds for $\omega(F)$ according to the prime divisors of F .

Theorem 3.6. *If F is a friend of 14, then $\omega(F) \geq 4$ whenever $3 \mid F$ or $5 \mid F$. Further, if $(3, F) = (5, F) = 1$, then $\omega(F) \geq 8$.*

Proof. Let F be a friend of 14. If $3 \mid F$, then $5 \nmid F$ by Lemma 3.1, therefore all prime divisors of F are greater than 5. Let us suppose that F has exactly three distinct prime divisors, that is, $F = 3 \cdot 7^{2a} \cdot p^b$, where $p > 7$ is a prime and a, b are positive integers. Then, using Lemma 2.1, Lemma 2.4 and Lemma 2.5, we get

$$I(F) \leq I(3 \cdot 7^{2a} \cdot 11^b) = I(3) \cdot I(7^{2a} \cdot 11^b) < \frac{4}{3} \cdot \frac{7}{6} \cdot \frac{11}{10} = \frac{77}{45} < I(14).$$

Therefore, F cannot have exactly three distinct prime divisors, hence $\omega(F) \geq 4$.

If $5 \mid F$, then $3 \nmid F$ by Lemma 3.1, therefore all prime divisors of F are greater than 3. Let us suppose that F has exactly three distinct prime divisors, that is, $F = 5^a \cdot 7^{2b} \cdot p^c$, where $p > 7$ is a prime and $a, b, c \in \mathbb{Z}^+$. Then, using Lemma 2.4 and Lemma 2.5, we get

$$I(F) \leq I(5^a \cdot 7^{2b} \cdot 11^c) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} = \frac{77}{48} < I(14).$$

Therefore, F cannot have exactly three distinct prime divisors, hence $\omega(F) \geq 4$.

Let $(3, F) = (5, F) = 1$. Then every prime divisors of F are strictly greater than 5. Let us suppose that F has at most seven distinct prime divisors, that is, $F = 7^{2a} \cdot \prod_{i=1}^k p_i^{a_i}$, where $p_{i+1} > p_i > 7$ and $k \leq 6$. Then, by Lemma 2.2, Lemma 2.4, and Lemma 2.5, we get

$$\begin{aligned} I(F) &\leq I(7^{2a} \cdot \prod_{i=5}^{k+4} q_i^{a_{i-4}}) \quad (q_i \text{ is the } i\text{-th prime number}) \\ &\leq I(7^{2a} \cdot 11^{a_1} \cdot 13^{a_2} \cdot 17^{a_3} \cdot 19^{a_4} \cdot 23^{a_5} \cdot 29^{a_6}) \\ &< \frac{7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29}{6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdot 28} \\ &= \frac{2800733}{1658880} \\ &< I(14). \end{aligned}$$

Therefore, F cannot have at most seven distinct prime divisors, hence $\omega(F) \geq 8$. This completes the proof. \square

Theorem 3.7. *No prime divisor of a friend F of 14 can exceed $1.4\sqrt{F}$.*

Proof. Let p be a prime divisor of F . Then we can write $F = p^a \cdot 7^{2b} \cdot m$ where a, b, m are positive integers such that $(7p, m) = 1$. Since

$$I(F) = \frac{\sigma(F)}{F} = \frac{\sigma(p^a) \cdot \sigma(7^{2b}) \cdot \sigma(m)}{p^a \cdot 7^{2b} \cdot m} = \frac{12}{7},$$

we have

$$\sigma(F) = \sigma(p^a) \cdot \sigma(7^{2b}) \cdot \sigma(m) = 12 \cdot p^a \cdot 7^{2b-1} \cdot m = \frac{12F}{7}.$$

Note that, $p \mid \sigma(F)$ and $\sigma(p^a) \mid \sigma(F)$, since $(p^a, \sigma(p^a)) = 1$ we have $p^a \cdot \sigma(p^a) \mid \sigma(F)$. Therefore,

$$p^2 \leq p^a \cdot \sigma(p^a) \leq \sigma(F) = \frac{12F}{7},$$

that is

$$p \leq \sqrt{\frac{12F}{7}} < 1.4\sqrt{F}.$$

For the prime divisor 7 of F , the proof proceeds in the same way. \square

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