Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 3, 583–587
DOI: 10.7546/nntdm.2025.31.3.583-587

On a logarithmic inequality by Shenton and Kemp

József Sándor

Department of Mathematics, Babeş-Bolyai University
Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro

Received: 10 May 2025 **Accepted:** 20 August 2025 **Online First:** 1 September 2025

Abstract: We offer new proof and refinement of a double inequality for $\ln^2(1+x)$, by L. R. Shenton and A. W. Kemp [11].

Keywords: Inequalities, Series expansions, Means.
2020 Mathematics Subject Classification: 26D05, 26D99.

1 Introduction

In 1989 L. R. Shenton and A. W. Kemp [11], by using a series expansion for $\ln^2(1+x)$, proved the following inequality:

$$\frac{x^2}{1+x+x^2/12} < \ln^2(1+x) < \frac{x^2}{1+x} \quad (x > -1). \tag{1}$$

Our aim in what follows is to offer a new proof and refinement of this relation based on the theory of means.

Let $a, b > 0$. The logarithmic mean $L = L(a,b)$ is defined by

$$\begin{aligned} L(a,b) &= \frac{b-a}{\ln b - \ln a} \quad \text{for } b \neq a, \\ L(a,a) &= a. \end{aligned} \tag{2}$$

The following classical inequalities for the logarithmic mean are well-known (see e.g. [5–8]):

$$G < L < A, \tag{3}$$

where $G = G(a,b) = \sqrt{ab}$ and $A = A(a,b) = \frac{a+b}{2}$ are the geometric, respectively, the arithmetic means of a and b.

There are many simple improvements of (3); we quote, e.g.,

$$\sqrt[3]{G^2A} < L < \frac{2G+A}{3}, \tag{4}$$

where the left side inequalities is due to E. B. Leach and M. C. Sholander [2], while the right side to G. Pólya and G. Szegő [4] and B. C. Carlson [1].

Another inequality is (see [3, 9, 10]):

$$L < A_{1/3}, \tag{5}$$

where $A_r = A_r(a,b) = \left(\frac{a^r+b^r}{2}\right)^{1/r}$, $r \neq 0$; $A_0 = G$ denotes the root power mean of a and b.

2 Proofs and refinements of (1)

1. First assume that $x > 0$.

Apply the left side of (3) for $a = 1$, $b = 1 + x$. Then we get the relation

$$\sqrt{1+x} < \frac{x}{\ln(1+x)}, \tag{6}$$

which is essentially the right side of (1).

Applying now the left side inequality of (4), we can deduce in the same manner that

$$\sqrt[3]{\frac{(1+x)(2+x)}{2}} < \frac{x}{\ln(1+x)}. \tag{7}$$

It is easy to see that

$$\sqrt[3]{\frac{(1+x)(2+x)}{2}} > \sqrt{1+x},$$

so (7) offers an improvement of (6).

Now, we shall apply inequality (5) in order to obtain a refinement of left side of (1).

With the application $a = 1$, $b = 1 + x$, inequality (5) becomes

$$\frac{x}{\ln(1+x)} < \left(\frac{\sqrt[3]{x+1}+1}{2}\right)^3. \tag{8}$$

In what follows, we shall prove that

$$\left(\frac{\sqrt[3]{x+1}+1}{2}\right)^3 < \sqrt{1+x+x^2/12}. \tag{9}$$

Let $x+1=y^3$ $(y>0)$, when (9) becomes:

$$\left(\frac{y+1}{2}\right)^6 < \frac{12y^3+(y^3-1)^2}{12}. \tag{10}$$

After some elementary computations, which we omit here, (10) can be written equivalently as

$$13y^6-18y^5-45y^4+100y^3-45y^2-18y+13>0. \tag{11}$$

By letting $y+\frac{1}{y}=t$, remark that equation (11) may be rewritten as

$$13t^3-18t^2-84t+136=0. \tag{12}$$

As this can be rewritten as

$$(t-2)^2\cdot(13t+34)>0,$$

inequality (12) follows, so the proof of (9) is completed.

Remark 1. *In a recent paper [10] we have shown the surprising fact*

$$A_{1/3}<\frac{2G+A}{3},$$

which shows that (5) *offers a refinement of the right side of* (4)*, too. We may ask, if eventually the right side of* (4) *(which is weaker than* (5)*) can provide an improvement of left side of* (1)*?*

As inequality $L<\frac{2G+A}{3}$ *yields that*

$$\frac{x}{\ln(1+x)}<\frac{2\sqrt{x+1}+\frac{x+2}{2}}{3}=\frac{4\sqrt{14}+x+2}{6}, \tag{13}$$

it will be sufficient to show that

$$\frac{4\sqrt{x+1}+x+2}{6}<\sqrt{1+x+\frac{x^2}{12}}. \tag{14}$$

By putting $x+1=y^2$*, after elementary computations this becomes*

$$y^4-4y^3+6y^2+1>0. \tag{15}$$

Remark that $y^4-4y^3+6y^2=y^2\cdot(y^2-4y+6)>0$ *as* $y^2>0$ *and* $y^2-4y+6>0$; *as the equation* $y^2-4y+6=0$ *has a negative discriminant* $\Delta=16-24=-8$.

By the above results, one has the refinement

$$\frac{x}{\ln(1+x)}<\frac{4\sqrt{x+1}+x+2}{6}<\sqrt{1+x+\frac{x^2}{12}}, \tag{16}$$

without knowing the fact that (5) is a refinement of right side of (4).

Remark 2. *Inequality* (5) *was discovered in fact by T. Rado (see [10]), and rediscovered by T.-P. Lin [3].*

2. Suppose now that $x < 0$. But $x = -X$, where $X > 0$. Then $0 < X < 1$ and the inequality (1) becomes

$$\frac{X}{1 - X + X^2/12} < \ln \frac{1}{1 - X} < \frac{X}{\sqrt{1 - X}}. \tag{17}$$

In this case, all can be repeated, as remark that

$$\frac{X}{\ln \frac{1}{1-X}} = \frac{1 - (1 - X)}{\ln 1 - \ln(1 - X)} = L(1 - X, 1),$$

so the application of the left side of (3) gives immediately the right side of (17). A similar refinement to (7) is

$$\sqrt[3]{\frac{(1 - X)(2 - X)}{2}} < \frac{X}{\ln 1/(1 - X)}. \tag{18}$$

The analogue of (8) and (9) will be:

$$\frac{X}{\ln 1/(1 - X)} < \left(\frac{\sqrt[3]{1 - X} + 1}{2}\right)^3 \tag{19}$$

and

$$\left(\frac{\sqrt[3]{1 - X} + 1}{2}\right)^3 < \sqrt{1 - X + X^2/12}. \tag{20}$$

The notation $1 - X = y^3$ leads again to inequality (10), which has been proved.

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