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New summation identities of hyperbolic k-Fibonacci and k-Lucas quaternions

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Abstract: In this paper, we introduce a set of identities involving hyperbolic k-Fibonacci quaternions and k-Lucas quaternions. Moreover, we derive summation identities for hyperbolic k-Fibonacci and k-Lucas quaternions by utilizing established properties of k-Fibonacci and k-Lucas numbers. These findings add valuable insight into the relationships between these quaternion sequences and offer valuable insights into their properties.
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1 Introduction

In 2007, Sergio Falcón and Ángel Plaza introduced generalized Fibonacci sequences, specifically the k-Fibonacci sequence $\phi_{k,n}$ and the k-Lucas sequence $\psi_{k,n}$, as a new way to extend the classic Fibonacci and Lucas numbers. These sequences depend on an integer parameter $k \geq 1$. We reproduce following Definitions 1.1 and 1.2 from [2].

Definition 1.1. *The k-Fibonacci sequence $(\phi_{k,n})$ is defined by the recurrence relation $\phi_{k,n+1} = k\phi_{k,n} + \phi_{k,n-1}$ with $\phi_{k,0} = 0$ and $\phi_{k,1} = 1$, for $n \geq 1$.*

Definition 1.2. *The k-Lucas sequence $(\psi_{k,n})$ is defined by the recurrence relation $\psi_{k,n+1} = k\psi_{k,n} + \psi_{k,n-1}$ with $\psi_{k,0} = 2$ and $\psi_{k,1} = k$, for $n \geq 1$.*

Hamilton is acknowledged with the invention of the quaternions in 1843. He demonstrated that they constitute a non-commutative division ring with four dimensions when multiplied together [7, 8].

Definition 1.3. (Horadam [9]) *A quaternion ω is an element of the form $\omega = \omega_0 + \omega_1 i + \omega_2 j + \omega_3 k$, where $\omega_0, \omega_1, \omega_2, \omega_3$ are real components and $1, i, j, k$ are basis elements satisfying the properties $i^2 = j^2 = k^2 = ijk = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.*

Horadam [9] introduced Fibonacci and Lucas quaternions in 1963, laying the foundation for further exploration of their properties. Ramírez's work [13] in 2015 expanded on this, introducing the concepts of k-Fibonacci quaternions and k-Lucas quaternions. For more detailed information, refer [11, 12] to works by Polatlı in 2015 and 2016. The study of various quaternion types has seen significant advancements in recent years, with multiple authors delving into their generalizations. The hyperbolic quaternions were first identified by Macfarlane [10] in the year 1900.

Definition 1.4. (Macfarlane [10]) *The hyperbolic quaternion ϑ is an element of the form $\vartheta = \vartheta_1 + \vartheta_2\epsilon_1 + \vartheta_3\epsilon_2 + \vartheta_4\epsilon_3 = \big(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\big)$, with real components ϑ_1, ϑ_2, ϑ_3, h_4 and $1, \epsilon_1, \epsilon_2, \epsilon_3$ are hyperbolic quaternion units that satisfy the non-commutative multiplication rules*

$$\epsilon_1{}^2 = \epsilon_2{}^2 = \epsilon_3{}^2 = \epsilon_1\epsilon_2\epsilon_3 = 1, \tag{1}$$

$$\epsilon_1\epsilon_2 = \epsilon_3 = -\epsilon_2\epsilon_1, \epsilon_2\epsilon_3 = \epsilon_1 = -\epsilon_3\epsilon_2, \epsilon_3\epsilon_1 = \epsilon_2 = -\epsilon_1\epsilon_3. \tag{2}$$

The hyperbolic quaternion is neither a commutative nor an associative algebraic structure. In the classical quaternion, every imaginary basis element has the property $e_n^2 = -1$, while in the hyperbolic quaternion all basis elements satisfy $\epsilon_m^2 = +1$. Read more about this in [10].

The hyperbolic k-Fibonacci and k-Lucas quaternions were defined in [3, 4, 6] by Godase and some of their identities were established.

Definition 1.5. (Godase [3, 4, 6]) *The hyperbolic k-Fibonacci quaternion $\xi_{k,n}$ is an element of the form $\xi_{k,n} = \phi_{k,n} + \phi_{k,n+1}\epsilon_1 + \phi_{k,n+2}\epsilon_2 + \phi_{k,n+3}\epsilon_3$, and the hyperbolic k-Lucas quaternion $\varrho_{k,n}$ is an element of the form $\varrho_{k,n} = \psi_{k,n} + \psi_{k,n+1}\epsilon_1 + \psi_{k,n+2}\epsilon_2 + \psi_{k,n+3}\epsilon_3$. The hyperbolic quaternion units 1, ϵ_1, ϵ_2 and ϵ_3 satisfy the multiplication rules defined in Definition 1.4 and $\phi_{k,n}$ and $\psi_{k,n}$ are k-Fibonacci and k-Lucas numbers.*

2 Preliminary results

The papers [3, 4, 6] explore the underlying characteristics of hyperbolic k-Fibonacci and k-Lucas quaternions. Below are a few of these properties that have been explored.

Theorem 2.1. (Godase [6]) *If $n \in \mathbb{Z}^+$, then*

(i)
$$\xi_{k,n+2} = k\xi_{k,n+1} + \xi_{k,n}, \tag{3}$$

(ii)
$$\varrho_{k,n+2} = k\varrho_{k,n+1} + \varrho_{k,n}, \tag{4}$$

(iii) $$\varrho_{k,n} = k\xi_{k,n} + 2\xi_{k,n-1}, \tag{5}$$

(iv) $$\varrho_{k,n} = \xi_{k,n+1} + \xi_{k,n-1}. \tag{6}$$

Theorem 2.2. (Godase [6]) **(Binet Formula).** *For every positive integer n, we have*

(1) $$\xi_{k,n} = \frac{\bar{\mu_1}{\mu_1}^n - \bar{\mu_2}{\mu_2}^n}{\mu_1 - \mu_2}, \tag{7}$$

(2) $$\varrho_{k,n} = \bar{\mu_1}{\mu_1}^n + \bar{\mu_2}{\mu_2}^n, \tag{8}$$

where $\bar{\mu_1} = 1+\mu_1\epsilon_1+{\mu_1}^2\epsilon_2+{\mu_1}^3\epsilon_3$, $\bar{\mu_2} = 1+\mu_2\epsilon_1+{\mu_2}^2\epsilon_2+{\mu_2}^3\epsilon_3$ and $1, \epsilon_1, \epsilon_2, \epsilon_3$ are hyperbolic quaternion units that satisfy the multiplication rule (1)–(2).

Theorem 2.3. (Godase [6]) **(Catalan's Identity).** *Let n, t be positive integers. Then show that*

(i). $$\xi_{k,n-t}\xi_{k,n+t} - {\xi_{k,n}}^2 = (-1)^{n-t}\phi_{k,t}\big(0, -2\phi_{k,t+1}, -2\phi_{k,t+2}, -\phi_{k,t+3} + \phi_{k,t-3} + \phi_{k,t+1} + \phi_{k,t-1}\big),$$

(ii). $$\varrho_{k,n-t}\varrho_{k,n+t} - {\varrho_{k,n}}^2 = \Delta(-1)^{n-t+1}\phi_{k,t}\big(0, -2\phi_{k,t+1}, -2\phi_{k,t+2}, -\phi_{k,t+3} + \phi_{k,t-3} + \phi_{k,t+1} + \phi_{k,t-1}\big).$$

Theorem 2.4. (Godase [6]) **(Cassini's Identity).** *For every $n \in \mathbb{Z}^+$. We have*

(i). $$\xi_{k,n-1}\xi_{k,n+1} - {\xi_{k,n}}^2 = (-1)^{n-1}\big(0, -2\phi_{k,2}, -2\phi_{k,3}, -\phi_{k,4}\big),$$

(ii). $$\varrho_{k,n-1}\varrho_{k,n+1} - {\varrho_{k,n}}^2 = \Delta(-1)^{n}\big(0, -2\phi_{k,2}, -2\phi_{k,3}, -\phi_{k,4}\big).$$

Theorem 2.5. (Godase [6]) **(d'Ocagne's Identity).** *Let n be any non-negative integer and t represents a natural number with $t \geq n+1$. Then prove that*

(i). $$\xi_{k,t}\xi_{k,n+1} - \xi_{k,t+1}\xi_{k,n} = (-1)^n\big(0,\ -2\phi_{k,t-n-1},\ 2\phi_{k,t-n-2},\ 2\phi_{k,t-n+3} - 2\phi_{k,t-n-3} + 2\phi_{k,t-n+1} + 2\phi_{k,t-n-1}\big),$$

(ii). $$\varrho_{k,t}\varrho_{k,n+1} - \varrho_{k,t+1}\varrho_{k,n} = (-1)^{n+1}\Delta\big(0,\ -2\phi_{k,t-n-1},\ 2\phi_{k,t-n-2},\ 2\phi_{k,t-n+3} - 2\phi_{k,t-n-3} + 2\phi_{k,t-n+1} + 2\phi_{k,t-n-1}\big).$$

3 Some new properties of hyperbolic k-Fibonacci and k-Lucas quaternions

In this section, we establish relationships between hyperbolic k-Fibonacci and k-Lucas quaternions. Our aim is to leverage the properties of the corresponding k-Fibonacci and k-Lucas numbers as demonstrated in the research by Godase in [5]. Our goal is to generate a novel set of identities for the k-Fibonacci and k-Lucas quaternions.

Theorem 3.1. *For all integers n, m, we have*

(i). $$2\xi_{k,n+m} = \phi_{k,n}\varrho_{k,m} + \psi_{k,n}\xi_{k,m},$$

(ii). $$2\varrho_{k,n+m} = \psi_{k,n}\varrho_{k,m} + (k^2+4)\phi_{k,n}\xi_{k,m}.$$

Proof. The Lemma 4.2.3 in [5] yields the following equations

$$2\psi_{k,n+m} = \psi_{k,n}\psi_{k,m} + (k^2+4)\phi_{k,n}\phi_{k,m}, \tag{9}$$
$$2\phi_{k,n+m} = \phi_{k,n}\psi_{k,m} + \psi_{k,n}\phi_{k,m}. \tag{10}$$

Equation (10) can be applied with Definition 1.5 to give

$$\begin{aligned}
2\xi_{k,n+m} &= 2\big(\phi_{k,n+m} + \phi_{k,n+m+1}\epsilon_1 + \phi_{k,n+m+2}\epsilon_2 + \phi_{k,n+m+3}\epsilon_3\big)\\
&= \phi_{k,n}\psi_{k,m} + \psi_{k,n}\phi_{k,m} + \big(\phi_{k,n}\psi_{k,m+1} + \psi_{k,n}\phi_{k,m+1}\big)\epsilon_1\\
&\quad + \big(\phi_{k,n}\psi_{k,m+2} + \psi_{k,n}\phi_{k,m+2}\big)\epsilon_2 + \big(\phi_{k,n}\psi_{k,m+3} + \psi_{k,n}\phi_{k,m+3}\big)\epsilon_3\\
&= \phi_{k,n}\big(\psi_{k,m} + \psi_{k,m+1}\epsilon_1 + \psi_{k,m+2}\epsilon_2 + \psi_{k,m+3}\epsilon_3\big)\\
&\quad + \psi_{k,n}\big(\phi_{k,m} + \phi_{k,m+1}\epsilon_1 + \phi_{k,m+2}\epsilon_2 + \phi_{k,m+3}\epsilon_3\big)\\
&= \phi_{k,n}\varrho_{k,m} + \psi_{k,n}\xi_{k,m}.
\end{aligned}$$

Again, using Definition 1.5 and Equation (9), we can write

$$\begin{aligned}
2\varrho_{k,n+m} &= 2\big(\psi_{k,n+m} + \psi_{k,n+m+1}\epsilon_1 + \psi_{k,n+m+2}\epsilon_2 + \psi_{k,n+m+3}\epsilon_3\big)\\
&= \psi_{k,n}\psi_{k,m} + (k^2+4)\phi_{k,n}\phi_{k,m} + \big(\psi_{k,n}\psi_{k,m+1} + (k^2+4)\phi_{k,n}\phi_{k,m+1}\big)\epsilon_1\\
&\quad + \big(\psi_{k,n}\psi_{k,m+2} + (k^2+4)\phi_{k,n}\phi_{k,m+2}\big)\epsilon_2 + \big(\psi_{k,n}\psi_{k,m+3}\\
&\quad + (k^2+4)\phi_{k,n}\phi_{k,m+3}\big)\epsilon_3\\
&= \psi_{k,n}\big(\psi_{k,m} + \psi_{k,m+1}\epsilon_1 + \psi_{k,m+2}\epsilon_2 + \psi_{k,m+3}\epsilon_3\big)\\
&\quad + (k^2+4)\phi_{k,n}\big(\phi_{k,m} + \phi_{k,m+1}\epsilon_1 + \phi_{k,m+2}\epsilon_2 + \phi_{k,m+3}\epsilon_3\big)\\
&= \psi_{k,n}\varrho_{k,m} + (k^2+4)\phi_{k,n}\xi_{k,m}.
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Theorem 3.2. *For every $n, m \in \mathbb{Z}$, we have*

(i). $$2(-1)^m\xi_{k,n-m} = \xi_{k,n}\psi_{k,m} - \varrho_{k,n}\phi_{k,m},$$

(ii). $$2(-1)^m\varrho_{k,n-m} = \varrho_{k,n}\psi_{k,m} - (k^2+4)\xi_{k,n}\phi_{k,m}.$$

Proof. As a result of Lemma 4.2.4 in [5], we can write

$$2(-1)^m\psi_{k,n-m} = \psi_{k,n}\psi_{k,m} - (k^2+4)\phi_{k,n}\phi_{k,m}, \tag{11}$$
$$2(-1)^m\phi_{k,n-m} = \phi_{k,n}\psi_{k,m} - \psi_{k,n}\phi_{k,m}. \tag{12}$$

Using Definition 1.5 and Equation (12), we have

$$\begin{aligned}
2(-1)^m\xi_{k,n-m} &= 2(-1)^m\big(\phi_{k,n-m} + \phi_{k,n-m+1}\epsilon_1 + \phi_{k,n-m+2}\epsilon_2 + \phi_{k,n-m+3}\epsilon_3\big)\\
&= \phi_{k,n}\psi_{k,m} - \psi_{k,n}\phi_{k,m} + \big(\phi_{k,n+1}\psi_{k,m} - \psi_{k,n+1}\phi_{k,m}\big)\epsilon_1\\
&\quad + \big(\phi_{k,n+2}\psi_{k,m} - \psi_{k,n+2}\phi_{k,m}\big)\epsilon_2 + \big(\phi_{k,n+3}\psi_{k,m} - \psi_{k,n+3}\phi_{k,m}\big)\epsilon_3\\
&= \psi_{k,m}\big(\phi_{k,n} + \phi_{k,n+1}\epsilon_1 + \phi_{k,n+2}\epsilon_2 + \phi_{k,n+3}\epsilon_3\big)\\
&\quad - \phi_{k,m}\big(\psi_{k,n} + \psi_{k,n+1}\epsilon_1 + \psi_{k,n+2}\epsilon_2 + \psi_{k,n+3}\epsilon_3\big)\\
&= \psi_{k,m}\xi_{k,n} - \phi_{k,m}\varrho_{k,n}.
\end{aligned}$$

The proof of the result (ii) is analogous to the proof of the result (i), so we omit the proof. □

Theorem 3.3. *If* $n, m \in \mathbb{Z}$*, then*

$(i).$ $$(-1)^m \xi_{k,n-m} + \xi_{k,n+m} = \xi_{k,n}\psi_{k,m},$$

$(ii).$ $$(-1)^m \varrho_{k,n-m} + \varrho_{k,n+m} = \varrho_{k,n}\psi_{k,m}.$$

Proof. From Lemma 4.2.5 of [5], it follows that

$$(-1)^m \psi_{k,n-m} + \psi_{k,n+m} = \psi_{k,n}\psi_{k,m}, \tag{13}$$

$$(-1)^m \phi_{k,n-m} + \phi_{k,n+m} = \phi_{k,n}\psi_{k,m}. \tag{14}$$

Using Definition 1.5 and Equation (14), we obtain

$$\begin{aligned}(-1)^m \xi_{k,n-m} + \xi_{k,n+m} &= (-1)^m\big(\phi_{k,n-m} + \phi_{k,n-m+1}\epsilon_1 + \phi_{k,n-m+2}\epsilon_2 + \phi_{k,n-m+3}\epsilon_3\big)\\ &\quad + \big(\phi_{k,n+m} + \phi_{k,n+m+1}\epsilon_1 + \phi_{k,n+m+2}\epsilon_2 + \phi_{k,n+m+3}\epsilon_3\big)\\ &= (-1)^m\phi_{k,n-m} + \phi_{k,n+m} + \big((-1)^m\big(\phi_{k,n-m+1} + \phi_{k,n+m+1}\big)\epsilon_1\\ &\quad + \big((-1)^m\big(\phi_{k,n-m+2} + \phi_{k,n+m+2}\big)\epsilon_2 + \big((-1)^m\big(\phi_{k,n-m+3} + \phi_{k,n+m+3}\big)\epsilon_3\\ &= \psi_{k,m}\phi_{k,n} + \psi_{k,m}\phi_{k,n+1}\epsilon_1 + \psi_{k,m}\phi_{k,n+2}\epsilon_2 + \psi_{k,m}\phi_{k,n+3}\epsilon_3\\ &= \psi_{k,m}\big(\phi_{k,n} + \phi_{k,n+1}\epsilon_1 + \phi_{k,n+2}\epsilon_2 + \phi_{k,n+3}\epsilon_3\big)\\ &= \psi_{k,m}\xi_{k,n}.\end{aligned}$$

Since the proof of the result (ii) is identical to the proof of the result (i), we omit it. □

Theorem 3.4. *For any integers* n, m *and the nonzero integer* t*, we have*

$(i).$ $$\xi_{k,n+t} = \phi_{k,n}\xi_{k,t+1} + \phi_{k,n-1}\xi_{k,t},$$

$(ii).$ $$\varrho_{k,n+t} = \phi_{k,n}\varrho_{k,t+1} + \phi_{k,n-1}\varrho_{k,t},$$

$(iii).$ $$\xi_{k,2n+t} = \psi_{k,n}\xi_{k,n+t} - (-1)^n\xi_{k,t},$$

$(iv).$ $$\varrho_{k,2n+t} = \psi_{k,n}\varrho_{k,n+t} - (-1)^n\varrho_{k,t}.$$

Proof. According to Theorem 3.2.1 in [5], we have

$$\phi_{k,n+t} = \phi_{k,n}\phi_{k,t+1} + \phi_{k,n-1}\phi_{k,t}, \tag{15}$$

$$\psi_{k,n+t} = \phi_{k,n}\psi_{k,t+1} + \phi_{k,n-1}\psi_{k,t}, \tag{16}$$

$$\phi_{k,2n+t} = \psi_{k,n}\phi_{k,n+t} - (-1)^n\phi_{k,t}, \tag{17}$$

$$\psi_{k,2n+t} = \psi_{k,n}\psi_{k,n+t} - (-1)^n\psi_{k,t}. \tag{18}$$

Applying Definition 1.5 and Equation (15), we get

$$\begin{aligned}\xi_{k,n+t} &= \phi_{k,n+t} + \phi_{k,n+t+1}\epsilon_1 + \phi_{k,n+t+2}\epsilon_2 + \phi_{k,n+t+3}\epsilon_3\\ &= \phi_{k,n}\phi_{k,t+1} + \phi_{k,n-1}\phi_{k,t} + \big(\phi_{k,n}\phi_{k,t+2} + \phi_{k,n-1}\phi_{k,t+1}\big)\epsilon_1\\ &\quad + \big(\phi_{k,n}\phi_{k,t+3} + \phi_{k,n-1}\phi_{k,t+2}\big)\epsilon_2 + \big(\phi_{k,n}\phi_{k,t+4} + \phi_{k,n-1}\phi_{k,t+3}\big)\epsilon_3\\ &= \phi_{k,n}\big(\phi_{k,t+1} + \phi_{k,t+2}\epsilon_1 + \phi_{k,t+3}\epsilon_2 + \phi_{k,t+4}\epsilon_3\big)\\ &\quad + \phi_{k,n-1}\big(\phi_{k,t} + \phi_{k,t+1}\epsilon_1 + \phi_{k,t+2}\epsilon_2 + \phi_{k,t+3}\epsilon_3\big)\\ &= \phi_{k,n}\xi_{k,t+1} - \phi_{k,n-1}\xi_{k,t}.\end{aligned}$$

Using Equation (16) and Definition 1.5, we can write

$$\begin{aligned}
\varrho_{k,n+t} &= \psi_{k,n+t} + \psi_{k,n+t+1}\epsilon_1 + \psi_{k,n+t+2}\epsilon_2 + \psi_{k,n+t+3}\epsilon_3 \\
&= \phi_{k,n}\psi_{k,t+1} + \phi_{k,n-1}\psi_{k,t} + \big(\phi_{k,n}\psi_{k,t+2} + \phi_{k,n-1}\psi_{k,t+1}\big)\epsilon_1 \\
&\quad + \big(\phi_{k,n}\psi_{k,t+3} + \phi_{k,n-1}\psi_{k,t+2}\big)\epsilon_2 + \big(\phi_{k,n}\psi_{k,t+4} + \phi_{k,n-1}\psi_{k,t+3}\big)\epsilon_3 \\
&= \phi_{k,n}\big(\psi_{k,t+1} + \psi_{k,t+2}\epsilon_1 + \psi_{k,t+3}\epsilon_2 + \psi_{k,t+4}\epsilon_3\big) \\
&\quad + \phi_{k,n-1}\big(\psi_{k,t} + \psi_{k,t+1}\epsilon_1 + \psi_{k,t+2}\epsilon_2 + \psi_{k,t+3}\epsilon_3\big) \\
&= \phi_{k,n}\varrho_{k,t+1} - \phi_{k,n-1}\varrho_{k,t}.
\end{aligned}$$

The proofs of the results (iii) and (iv) are analogous to the proofs for the results (i) and (ii). □

Theorem 3.5. *Given two integers s and t, we have*

(i). $$k\xi_{k,s+2t} = \phi_{k,2t}\varrho_{k,s+1} - \psi_{k,2t-1}\xi_{k,s},$$

(ii). $$k\varrho_{k,s+2t} = \phi_{k,2t}\Delta\xi_{k,s+1} - \psi_{k,2t-1}\varrho_{k,s}.$$

Proof. Theorem 3.2.7 of [5] allows us to write

$$k\phi_{k,s+2t} + \psi_{k,2t-1}\phi_{k,s} = \phi_{k,2t}\psi_{k,s+1}, \tag{19}$$

$$k\psi_{k,s+2t} + \psi_{k,2t-1}\psi_{k,s} = \phi_{k,2t}\Delta\phi_{k,s+1}. \tag{20}$$

Using Equation (19) and Definition 1.5, we have

$$\begin{aligned}
k\xi_{k,s+2t} &= k\big(\phi_{k,s+2t} + \phi_{k,s+2t+1}\epsilon_1 + \phi_{k,s+2t+2}\epsilon_2 + \phi_{k,s+2t+3}\epsilon_3\big) \\
&= \phi_{k,2t}\psi_{k,s+1} - \psi_{k,2t-1}\phi_{k,s} + \big(\phi_{k,2t}\psi_{k,s+2} - \psi_{k,2t-1}\phi_{k,s+1}\big)\epsilon_1 \\
&\quad + \big(\phi_{k,2t}\psi_{k,s+3} - \psi_{k,2t-1}\phi_{k,s+2}\big)\epsilon_2 + \big(\phi_{k,2t}\psi_{k,s+4} - \psi_{k,2t-1}\phi_{k,s+3}\big)\epsilon_3 \\
&= \phi_{k,2t}\big(\psi_{k,s+1} + \psi_{k,s+2}\epsilon_1 + \psi_{k,s+3}\epsilon_2 + \psi_{k,s+4}\epsilon_3\big) \\
&\quad - \psi_{k,2t-1}\big(\phi_{k,s} + \phi_{k,s+1}\epsilon_1 + \phi_{k,s+2}\epsilon_2 + \phi_{k,s+3}\epsilon_3\big) \\
&= \phi_{k,2t}\varrho_{k,s+1} - \psi_{k,2t-1}\xi_{k,s}.
\end{aligned}$$

The Definition 1.5 and Equation (20) are used to obtain

$$\begin{aligned}
k\varrho_{k,s+2t} &= k\big(\psi_{k,s+2t} + \psi_{k,s+2t+1}\epsilon_1 + \psi_{k,s+2t+2}\epsilon_2 + \psi_{k,s+2t+3}\epsilon_3\big) \\
&= \phi_{k,2t}\Delta\phi_{k,s+1} - \psi_{k,2t-1}\psi_{k,s} + \big(\phi_{k,2t}\Delta\phi_{k,s+2} - \psi_{k,2t-1}\psi_{k,s+1}\big)\epsilon_1 \\
&\quad + \big(\phi_{k,2t}\Delta\phi_{k,s+3} - \psi_{k,2t-1}\psi_{k,s+2}\big)\epsilon_2 + \big(\phi_{k,2t}\Delta\phi_{k,s+4} - \psi_{k,2t-1}\psi_{k,s+3}\big)\epsilon_3 \\
&= \phi_{k,2t}\Delta\big(\phi_{k,s+1} + \phi_{k,s+2}\epsilon_1 + \phi_{k,s+3}\epsilon_2 + \phi_{k,s+4}\epsilon_3\big) \\
&\quad - \psi_{k,2t-1}\big(\psi_{k,s} + \psi_{k,s+1}\epsilon_1 + \psi_{k,s+2}\epsilon_2 + \psi_{k,s+3}\epsilon_3\big) \\
&= \phi_{k,2t}\xi_{k,s+1}\Delta - \psi_{k,2t-1}\varrho_{k,s}.
\end{aligned}$$

The proof of Theorem 3.5 is now complete. □

Theorem 3.6. *For all integers s and t, we have*

(i). $$k\xi_{k,s+2t+1} = \psi_{k,2t+1}\xi_{k,s+1} - \phi_{k,2t}\varrho_{k,s}\sqrt{\Delta},$$

(ii). $$k\varrho_{k,s+2t+1} = \psi_{k,2t+1}\varrho_{k,s+1} - \Delta\phi_{k,2t}\xi_{k,s}.$$

Proof. As a result of Theorem 3.2.8 of [5], we can write

$$\phi_{k,s+2t+1} = \frac{\psi_{k,2t+1}}{k}\phi_{k,s+1} - \frac{\phi_{k,2t}}{k}\psi_{k,s}\sqrt{\Delta}, \tag{21}$$

$$\psi_{k,s+2t+1} = \frac{\psi_{k,2t+1}}{k}\psi_{k,s+1} - \Delta\frac{\phi_{k,2t}}{k}\phi_{k,s}. \tag{22}$$

Equation (21) and Definition 1.5 yield

$$\begin{aligned}
k\xi_{k,s+2t+1} &= k\big(\phi_{k,s+2t+1} + \phi_{k,s+2t+2}\epsilon_1 + \phi_{k,s+2t+3}\epsilon_2 + \phi_{k,s+2t+4}\epsilon_3\big) \\
&= \psi_{k,2t+1}\phi_{k,s+1} - \phi_{k,2t}\psi_{k,s}\sqrt{\Delta} + \big(\psi_{k,2t+1}\phi_{k,s+2} - \phi_{k,2t}\psi_{k,s+1}\sqrt{\Delta}\big)\epsilon_1 \\
&\quad + \big(\psi_{k,2t+1}\phi_{k,s+3} - \phi_{k,2t}\psi_{k,s+2}\sqrt{\Delta}\big)\epsilon_2 + \big(\psi_{k,2t+1}\phi_{k,s+4} - \phi_{k,2t}\psi_{k,s+3}\sqrt{\Delta}\big)\epsilon_3 \\
&= \psi_{k,2t+1}\big(\phi_{k,s+1} + \phi_{k,s+2}\epsilon_1 + \phi_{k,s+3}\epsilon_2 + \phi_{k,s+4}\epsilon_3\big) \\
&\quad - \sqrt{\Delta}\phi_{k,2t}\big(\psi_{k,s} + \psi_{k,s+1}\epsilon_1 + \psi_{k,s+2}\epsilon_2 + \psi_{k,s+3}\epsilon_3\big) \\
&= \psi_{k,2t+1}\xi_{k,s+1} - \sqrt{\Delta}\phi_{k,2t}\varrho_{k,s}.
\end{aligned}$$

Using Definition 1.5 and Equation (22), we obtain

$$\begin{aligned}
k\varrho_{k,s+2t+1} &= k\big(\psi_{k,s+2t+1} + \psi_{k,s+2t+2}\epsilon_1 + \psi_{k,s+2t+3}\epsilon_2 + \psi_{k,s+2t+4}\epsilon_3\big) \\
&= \psi_{k,2t+1}\psi_{k,s+1} - \Delta\phi_{k,2t}\phi_{k,s} + \big(\psi_{k,2t+1}\psi_{k,s+2} - \Delta\phi_{k,2t}\phi_{k,s+1}\big)\epsilon_1 \\
&\quad + \big(\psi_{k,2t+1}\psi_{k,s+3} - \Delta\phi_{k,2t}\phi_{k,s+2}\big)\epsilon_2 + \big(\psi_{k,2t+1}\psi_{k,s+4} - \Delta\phi_{k,2t}\phi_{k,s+3}\big)\epsilon_3 \\
&= \psi_{k,2t+1}\big(\psi_{k,s+1} + \psi_{k,s+2}\epsilon_1 + \psi_{k,s+3}\epsilon_2 + \psi_{k,s+4}\epsilon_3\big) \\
&\quad - \Delta\phi_{k,2t}\big(\phi_{k,s} + \phi_{k,s+1}\epsilon_1 + \phi_{k,s+2}\epsilon_2 + \phi_{k,s+3}\epsilon_3\big) \\
&= \psi_{k,2t+1}\varrho_{k,s+1} - \Delta\phi_{k,2t}\xi_{k,s}.
\end{aligned}$$

This concludes the proof of Theorem 3.6. □

Theorem 3.7. *For all integers s and t, we have*

(i). $\quad 4\xi_{k,a+b+c} = \psi_{k,a}\psi_{k,b}\xi_{k,c} + \phi_{k,a}\psi_{k,b}\varrho_{k,c} + \psi_{k,a}\phi_{k,b}\varrho_{k,c} + (k^2+4)\phi_{k,a}\phi_{k,b}\xi_{k,c},$

(ii). $\quad 4\varrho_{k,a+b+c} = \psi_{k,a}\psi_{k,b}\varrho_{k,c} + (k^2+4)\big(\psi_{k,a}\phi_{k,b}\xi_{k,c} + \phi_{k,a}\psi_{k,b}\xi_{k,c} + \phi_{k,a}\phi_{k,b}\varrho_{k,c}\big).$

Proof. As a result of Theorem 4.2.6 of [5], we have

$$4\phi_{k,a+b+c} = \psi_{k,a}\psi_{k,b}\phi_{k,c} + \phi_{k,a}\psi_{k,b}\psi_{k,c} + \psi_{k,a}\phi_{k,b}\psi_{k,c} + (k^2+4)\phi_{k,a}\phi_{k,b}\phi_{k,c}, \tag{23}$$

$$4\psi_{k,a+b+c} = \psi_{k,a}\psi_{k,b}\psi_{k,c} + (k^2+4)\big(\psi_{k,a}\phi_{k,b}\phi_{k,c} + \phi_{k,a}\psi_{k,b}\phi_{k,c} + \phi_{k,a}\phi_{k,b}\psi_{k,c}\big). \tag{24}$$

By applying Definition 1.5 and Equation (23), we get

$$\begin{aligned}
4\xi_{k,a+b+c} &= 4\big(\phi_{k,a+b+c} + \phi_{k,a+b+c+1}\epsilon_1 + \phi_{k,a+b+c+2}\epsilon_2 + \phi_{k,a+b+c+3}\epsilon_3\big) \\
&= \psi_{k,a}\psi_{k,b}\phi_{k,c} + \phi_{k,a}\psi_{k,b}\psi_{k,c} + \psi_{k,a}\phi_{k,b}\psi_{k,c} + (k^2+4)\phi_{k,a}\phi_{k,b}\phi_{k,c} \\
&\quad + \big(\psi_{k,a}\psi_{k,b}\phi_{k,c+1} + \phi_{k,a}\psi_{k,b}\psi_{k,c+1} + \psi_{k,a}\phi_{k,b}\psi_{k,c+1} + (k^2+4)\phi_{k,a}\phi_{k,b}\phi_{k,c+1}\big)\epsilon_1 \\
&\quad + \big(\psi_{k,a}\psi_{k,b}\phi_{k,c+2} + \phi_{k,a}\psi_{k,b}\psi_{k,c+2} + \psi_{k,a}\phi_{k,b}\psi_{k,c+2} + (k^2+4)\phi_{k,a}\phi_{k,b}\phi_{k,c+2}\big)\epsilon_2 \\
&\quad + \big(\psi_{k,a}\psi_{k,b}\phi_{k,c+3} + \phi_{k,a}\psi_{k,b}\psi_{k,c+3} + \psi_{k,a}\phi_{k,b}\psi_{k,c+3} + (k^2+4)\phi_{k,a}\phi_{k,b}\phi_{k,c+3}\big)\epsilon_3
\end{aligned}$$

$$
\begin{aligned}
&= \psi_{k,a}\psi_{k,b}\big(\phi_{k,c} + \phi_{k,c+1}\epsilon_1 + \phi_{k,c+2}\epsilon_2 + \phi_{k,c+3}\epsilon_3\big) \\
&\quad + \phi_{k,a}\psi_{k,b}\big(\psi_{k,c} + \psi_{k,c+1}\epsilon_1 + \psi_{k,c+2}\epsilon_2 + \psi_{k,c+3}\epsilon_3\big) \\
&\quad + \psi_{k,a}\phi_{k,b}\big(\psi_{k,c} + \psi_{k,c+1}\epsilon_1 + \psi_{k,c+2}\epsilon_2 + \psi_{k,c+3}\epsilon_3\big) \\
&\quad + (k^2+4)\phi_{k,a}\phi_{k,b}\big(\phi_{k,c} + \phi_{k,c+1}\epsilon_1 + \phi_{k,c+2}\epsilon_2 + \phi_{k,c+3}\epsilon_3\big) \\
&= \psi_{k,a}\psi_{k,b}\xi_{k,c} + \phi_{k,a}\psi_{k,b}\varrho_{k,c} + \psi_{k,a}\phi_{k,b}\varrho_{k,c} + (k^2+4)\phi_{k,a}\phi_{k,b}\xi_{k,c}.
\end{aligned}
$$

The proof of the result (ii) is analogous to the proof of the result (i). Therefore, we omit it. □

Theorem 3.8. *Given integers a, b and c, we have*

$(i).$ $\quad \xi_{k,a+b-1} = \xi_{k,a}\phi_{k,b} + \xi_{k,a-1}\phi_{k,b-1},$

$(ii).$ $\quad k\xi_{k,a+b-2} = \xi_{k,a}\phi_{k,b} - \xi_{k,a-2}\phi_{k,b-2},$

$(iii).$ $\quad k\xi_{k,a+b+c-3} = \xi_{k,a}\phi_{k,b}\phi_{k,c} + k\xi_{k,a-1}\phi_{k,b-1}\phi_{k,c-1} - \xi_{k,a-2}\phi_{k,b-2}\phi_{k,c-2}.$

Proof. From Proposition 5 of [1], it follows that

$$
\phi_{k,a+b-1} = \phi_{k,a}\phi_{k,b} + \phi_{k,a-1}\phi_{k,b-1}, \tag{25}
$$

$$
\phi_{k,a+b-2} = \frac{1}{k}\big(\phi_{k,a}\phi_{k,b} - \phi_{k,a-2}\phi_{k,b-2}\big), \tag{26}
$$

$$
\phi_{k,a+b+c-3} = \frac{1}{k}\big(\phi_{k,a}\phi_{k,b}\phi_{k,c} + k\phi_{k,a-1}\phi_{k,b-1}\phi_{k,c-1} - \phi_{k,a-2}\phi_{k,b-2}\phi_{k,c-2}\big) \tag{27}
$$

By using Equation (26) and Definition 1.5, we can write

$$
\begin{aligned}
k\xi_{k,a+b-2} &= k\big(\phi_{k,a+b-2} + \phi_{k,a+b-1}\epsilon_1 + \phi_{k,a+b}\epsilon_2 + \phi_{k,a+b+1}\epsilon_3\big) \\
&= \phi_{k,a}\phi_{k,b} - \phi_{k,a-2}\phi_{k,b-2} + \big(\phi_{k,a+1}\phi_{k,b} - \phi_{k,a-1}\phi_{k,b-1}\big)\epsilon_1 \\
&\quad + \big(\phi_{k,a+2}\phi_{k,b} - \phi_{k,a}\phi_{k,b-1}\big)\epsilon_2 + \big(\phi_{k,a+3}\phi_{k,b} - \phi_{k,a+1}\phi_{k,b-1}\big)\epsilon_3 \\
&= \phi_{k,b}\big(\phi_{k,a} + \phi_{k,a+1}\epsilon_1 + \phi_{k,a+2}\epsilon_2 + \phi_{k,a+3}\epsilon_3\big) \\
&\quad - \phi_{k,b-2}\big(\phi_{k,a-2} + \phi_{k,a-1}\epsilon_1 + \phi_{k,a}\epsilon_2 + \phi_{k,a+1}\epsilon_3\big) \\
&= \xi_{k,a}\phi_{k,b} - \xi_{k,a-2}\phi_{k,b-2}.
\end{aligned}
$$

Finally, using Definition 1.5 and Equation (27), we obtain

$$
\begin{aligned}
k\xi_{k,a+b+c-3} &= k\big(\phi_{k,a+b+c-3} + \phi_{k,a+b+c-2}\epsilon_1 + \phi_{k,a+b+c-1}\epsilon_2 + \phi_{k,a+b+c}\epsilon_3\big) \\
&= \phi_{k,a}\phi_{k,b}\phi_{k,c} + k\phi_{k,a-1}\phi_{k,b-1}\phi_{k,c-1} - \phi_{k,a-2}\phi_{k,b-2}\phi_{k,c-2} \\
&\quad + \big(\phi_{k,a+1}\phi_{k,b}\phi_{k,c} + k\phi_{k,a}\phi_{k,b-1}\phi_{k,c-1} - \phi_{k,a-1}\phi_{k,b-2}\phi_{k,c-2}\big)\epsilon_1 \\
&\quad + \big(\phi_{k,a+2}\phi_{k,b}\phi_{k,c} + k\phi_{k,a+1}\phi_{k,b-1}\phi_{k,c-1} - \phi_{k,a}\phi_{k,b-2}\phi_{k,c-2}\big)\epsilon_2 \\
&\quad + \big(\phi_{k,a+3}\phi_{k,b}\phi_{k,c} + k\phi_{k,a+2}\phi_{k,b-1}\phi_{k,c-1} - \phi_{k,a+1}\phi_{k,b-2}\phi_{k,c-2}\big)\epsilon_3 \\
&= \phi_{k,b}\phi_{k,c}\big(\phi_{k,a} + \phi_{k,a+1}\epsilon_1 + \phi_{k,a+2}\epsilon_2 + \phi_{k,a+3}\epsilon_3\big) \\
&\quad + k\phi_{k,b-1}\phi_{k,c-1}\big(\phi_{k,a-1} + \phi_{k,a}\epsilon_1 + \phi_{k,a+1}\epsilon_2 + \phi_{k,a+2}\epsilon_3\big) \\
&\quad - \phi_{k,b-2}\phi_{k,c-2}\big(\phi_{k,a-2} + \phi_{k,a-1}\epsilon_1 + \phi_{k,a}\epsilon_2 + \phi_{k,a+1}\epsilon_3\big) \\
&= \xi_{k,a}\phi_{k,b}\phi_{k,c} + k\xi_{k,a-1}\phi_{k,b-1}\phi_{k,c-1} - \xi_{k,a-2}\phi_{k,b-2}\phi_{k,c-2}.
\end{aligned}
$$

The proof of the result (i) is identical to that result (ii). So, we omit it. □

4 Sums of hyperbolic k-Fibonacci and k-Lucas quaternions

In this section, distinct summation identities are derived for hyperbolic k-Fibonacci and k-Lucas quaternions.

Theorem 4.1. *Let $n \in \mathbb{N}$. Then prove that*

(i).
$$\sum_{i=0}^{n} \binom{n}{i} k^i \xi_{k,i} = \xi_{k,2n},$$

(ii).
$$\sum_{i=0}^{n} \binom{n}{i} k^i \varrho_{k,i} = \varrho_{k,2n}.$$

Theorem 4.2. *Let $n \in \mathbb{N}$. Then prove that*

(i).
$$\sum_{i=1}^{n} \xi_{k,i} = \frac{\xi_{k,n} + \xi_{k,n+1} - \xi_{k,0} - \xi_{k,1}}{k},$$

(ii).
$$\sum_{i=1}^{n} \varrho_{k,i} = \frac{\varrho_{k,n} + \varrho_{k,n+1} - \varrho_{k,0} - \varrho_{k,1}}{k}.$$

Proof. (i) From Theorem 2.1(i), we have

$$\begin{aligned} k\xi_{k,1} + \xi_{k,0} &= \xi_{k,2}, \\ k\xi_{k,2} + \xi_{k,1} &= \xi_{k,3}, \\ k\xi_{k,3} + \xi_{k,2} &= \xi_{k,4}, \\ &\vdots \\ k\xi_{k,n} + \xi_{k,n-1} &= \xi_{k,n+1}. \end{aligned}$$

Consequently, adding all these relations, we get

$$\begin{aligned} k\big(\xi_{k,1} + \xi_{k,2} + \xi_{k,3} + \cdots + \xi_{k,n}\big) + \big(\xi_{k,0} + \xi_{k,1} + \xi_{k,2} + \cdots + \xi_{k,n-1}\big) \\ = \xi_{k,2} + \xi_{k,3} + \xi_{k,4} + \cdots + \xi_{k,n+1}, \\ k\big(\xi_{k,1} + \xi_{k,2} + \xi_{k,3} + \cdots + \xi_{k,n}\big) = \xi_{k,n} + \xi_{k,n+1} - \xi_{k,0} - \xi_{k,1}. \end{aligned}$$

It gives that

$$\sum_{i=1}^{n} \xi_{k,i} = \frac{\xi_{k,n} + \xi_{k,n+1} - \xi_{k,0} - \xi_{k,1}}{k}.$$

The proof of (ii) is analogous to (i) using Theorem 2.1(ii). Hence, we omit the proof. □

Theorem 4.3. *Let $n \in \mathbb{N}$. Then prove that*

(i).
$$\sum_{i=1}^{n} \xi_{k,2i} = \frac{\xi_{k,2n+1} - \xi_{k,1}}{k},$$

(ii).
$$\sum_{i=1}^{n} \varrho_{k,2i} = \frac{\varrho_{k,2n+1} - \varrho_{k,1}}{k}.$$

Proof. (i) From Theorem 2.1(i), we can write

$$\begin{aligned} k\xi_{k,2} &= \xi_{k,3} - \xi_{k,1}, \\ k\xi_{k,4} &= \xi_{k,5} - \xi_{k,3}, \\ k\xi_{k,6} &= \xi_{k,7} - \xi_{k,5}, \\ &\vdots \\ k\xi_{k,2n} &= \xi_{k,2n+1} - \xi_{k,2n-1}. \end{aligned}$$

Adding all these relations, we obtain

$$\begin{aligned} k\big(\xi_{k,2} + \xi_{k,4} + \xi_{k,6} + \cdots + \xi_{k,2n}\big) &= \big(\xi_{k,3} + \xi_{k,5} + \xi_{k,7} + \cdots + \xi_{k,2n+1}\big) \\ &\quad - \big(\xi_{k,1} + \xi_{k,3} + \xi_{k,5} + \cdots + \xi_{k,2n-1}\big), \\ k\big(\xi_{k,2} + \xi_{k,4} + \xi_{k,6} + \cdots + \xi_{k,2n}\big) &= \xi_{k,2n+1} - \xi_{k,1}. \end{aligned}$$

It can be written in the form

$$\sum_{i=1}^{n} \xi_{k,2i} = \frac{\xi_{k,2n+1} - \xi_{k,1}}{k}.$$

The proof of (ii) is similar to (i) using Theorem 2.1(ii). Hence, we omit the proof. □

Theorem 4.4. *Let $n \in \mathbb{N}$. Then prove that*

(i).
$$\sum_{i=1}^{n} \xi_{k,2i-1} = \frac{\xi_{k,2n} - \xi_{k,0}}{k},$$

(ii).
$$\sum_{i=1}^{n} \varrho_{k,2i-1} = \frac{\varrho_{k,2n} - \varrho_{k,0}}{k}.$$

Proof. (i) From Theorem 2.1(i), we can write

$$\begin{aligned} k\xi_{k,1} &= \xi_{k,2} - \xi_{k,0}, \\ k\xi_{k,3} &= \xi_{k,4} - \xi_{k,2}, \\ k\xi_{k,5} &= \xi_{k,6} - \xi_{k,4}, \\ &\vdots \\ k\xi_{k,2n-1} &= \xi_{k,2n} - \xi_{k,2n-2}. \end{aligned}$$

By adding all these relations, we get

$$\begin{aligned} k\big(\xi_{k,1} + \xi_{k,3} + \xi_{k,5} + \cdots + \xi_{k,2n-1}\big) &= \big(\xi_{k,2} + \xi_{k,4} + \xi_{k,6} + \cdots + \xi_{k,2n}\big) \\ &\quad - \big(\xi_{k,0} + \xi_{k,2} + \xi_{k,4} + \cdots + \xi_{k,2n-2}\big), \\ k\big(\xi_{k,1} + \xi_{k,3} + \xi_{k,5} + \cdots + \xi_{k,2n+1}\big) &= \xi_{k,2n} - \xi_{k,0}. \end{aligned}$$

Finally, we can write

$$\sum_{i=1}^{n} \xi_{k,2i-1} = \frac{\xi_{k,2n} - \xi_{k,0}}{k}.$$

The proof of (ii) is similar to (i), using Theorem 2.1 (ii). Hence, we omit the proof. □

Theorem 4.5. *Let* $n \in \mathbb{N}$. *Then prove that*

$$\sum_{i=1}^{n}(\xi_{k,i}+\varrho_{k,i}) = \frac{1}{k}\Big((k+3)\xi_{k,n+1}-(k-3)\xi_{k,n}-\hat{a}\Big),$$

where

$$\hat{a} = (k+3)+(k^2+2k+3)\epsilon_1+(k^3+2k^2+4k+3)\epsilon_2+(k^4+2k^3+5k^2+5k+3)\epsilon_3.$$

Proof. By using Theorem 4.2, we can write

$$\sum_{i=1}^{n}\big(\xi_{k,i}+\varrho_{k,i}\big) = \frac{\xi_{k,n}+\xi_{k,n+1}-\xi_{k,0}-\xi_{k,1}+\varrho_{k,n}+\varrho_{k,n+1}-\varrho_{k,0}-\varrho_{k,1}}{k}.$$

Now, using Equation (4), we get

$$\begin{aligned}\sum_{i=1}^{n}\big(\xi_{k,i}+\varrho_{k,i}\big) &= \frac{1}{k}\big(\xi_{k,n}+\xi_{k,n+1}+k\xi_{k,n}+2\xi_{k,n-1}+k\xi_{k,n+1}+2\xi_{k,n}\\ &\quad -\varrho_{k,0}-\varrho_{k,1}-\xi_{k,0}-\xi_{k,1}\big)\\ &= \frac{1}{k}\big((k+3)\xi_{k,n}+(k+1)\xi_{k,n+1}+2\xi_{k,n-1}-\varrho_{k,0}-\varrho_{k,1}-\xi_{k,0}-\xi_{k,1}\big).\end{aligned}$$

Finally, using Equation (3), we can write

$$\begin{aligned}\sum_{i=1}^{n}\big(\xi_{k,i}+\varrho_{k,i}\big) &= \frac{1}{k}\big((k+3)\xi_{k,n}+(k+1)\xi_{k,n+1}+2(\xi_{k,n+1}-\xi_{k,n})\\ &\quad -\varrho_{k,0}-\varrho_{k,1}-\xi_{k,0}-\xi_{k,1}\big)\\ &= \frac{1}{k}\Big((k+3)\xi_{k,n+1}-(k-3)\xi_{k,n}-(k+3)+(k^2+2k+3)\epsilon_1\\ &\quad +(k^3+2k^2+4k+3)\epsilon_2+(k^4+2k^3+5k^2+5k+3)\epsilon_3\Big)\\ &= \frac{1}{k}\Big((k+3)\xi_{k,n+1}-(k-3)\xi_{k,n}-\hat{a}\Big).\end{aligned}$$

This concludes the proof of Theorem 4.5. □

Theorem 4.6. *Let* $n \in \mathbb{N}$. *Then prove that*

$$(i). \qquad \sum_{i=1}^{n}\xi_{k,i}^2 = \begin{cases}\frac{1}{k}(\xi_{k,n}\xi_{k,n+1}-\xi_{k,0}\xi_{k,1}), & \textit{if } n \textit{ is an even positive integer;}\\ \frac{1}{k}(\xi_{k,n+1}\xi_{k,n}-\xi_{k,0}\xi_{k,1}), & \textit{if } n \textit{ is an even positive integer,}\end{cases} \tag{28}$$

$$(ii). \qquad \sum_{i=1}^{n}\varrho_{k,i}^2 = \begin{cases}\frac{1}{k}(\varrho_{k,n}\varrho_{k,n+1}-\varrho_{k,0}\varrho_{k,1}), & \textit{if } n \textit{ is an even positive integer;}\\ \frac{1}{k}(\varrho_{k,n+1}\varrho_{k,n}-\varrho_{k,0}\varrho_{k,1}), & \textit{if } n \textit{ is an even positive integer.}\end{cases} \tag{29}$$

Proof. (i) Using Equation (3), we can write

$$\begin{aligned}
k\xi_{k,1}^2 &= k\xi_{k,1}\xi_{k,1} = (\xi_{k,2} - \xi_{k,0})\xi_{k,1} = \xi_{k,2}\xi_{k,1} - \xi_{k,0}\xi_{k,1},\\
k\xi_{k,2}^2 &= k\xi_{k,2}\xi_{k,2} = \xi_{k,2}(\xi_{k,3} - \xi_{k,1}) = \xi_{k,2}\xi_{k,3} - \xi_{k,2}\xi_{k,1},\\
k\xi_{k,3}^2 &= k\xi_{k,3}\xi_{k,3} = (\xi_{k,4} - \xi_{k,2})\xi_{k,3} = \xi_{k,4}\xi_{k,3} - \xi_{k,2}\xi_{k,3},\\
k\xi_{k,4}^2 &= k\xi_{k,4}\xi_{k,4} = \xi_{k,4}(\xi_{k,5} - \xi_{k,3}) = \xi_{k,4}\xi_{k,5} - \xi_{k,4}\xi_{k,3},\\
k\xi_{k,5}^2 &= k\xi_{k,5}\xi_{k,5} = (\xi_{k,6} - \xi_{k,4})\xi_{k,5} = \xi_{k,6}\xi_{k,5} - \xi_{k,4}\xi_{k,5},\\
&\vdots
\end{aligned}$$

$$k\sum_{i=1}^{n}\xi_{k,n}^2 = \begin{cases} \xi_{k,n}\xi_{k,n+1} - \xi_{k,n}\xi_{k,n-1}, & \text{if } n \text{ is an even positive integer;}\\ \xi_{k,n+1}\xi_{k,n} - \xi_{k,n-1}\xi_{k,n}, & \text{if } n \text{ is an odd positive integer.} \end{cases}$$

Consequently, by adding all these relations, we get

$$\sum_{i=1}^{n}\xi_{k,i}^2 = \begin{cases} \frac{1}{k}(\xi_{k,n}\xi_{k,n+1} - \xi_{k,0}\xi_{k,1}), & \text{if } n \text{ is an even positive integer;}\\ \frac{1}{k}(\xi_{k,n+1}\xi_{k,n} - \xi_{k,0}\xi_{k,1}), & \text{if } n \text{ is an even positive integer.} \end{cases}$$

(ii) By using Equation 4, we write

$$\begin{aligned}
k\varrho_{k,1}^2 &= k\varrho_{k,1}\varrho_{k,1} = (\varrho_{k,2} - \varrho_{k,0})\varrho_{k,1} = \varrho_{k,2}\varrho_{k,1} - \varrho_{k,0}\varrho_{k,1},\\
k\varrho_{k,2}^2 &= k\varrho_{k,2}\varrho_{k,2} = \varrho_{k,2}(\varrho_{k,3} - \varrho_{k,1}) = \varrho_{k,2}\varrho_{k,3} - \varrho_{k,2}\varrho_{k,1},\\
k\varrho_{k,3}^2 &= k\varrho_{k,3}\varrho_{k,3} = (\varrho_{k,4} - \varrho_{k,2})\varrho_{k,3} = \varrho_{k,4}\varrho_{k,3} - \varrho_{k,2}\varrho_{k,3},\\
k\varrho_{k,4}^2 &= k\varrho_{k,4}\varrho_{k,4} = \varrho_{k,4}(\varrho_{k,5} - \varrho_{k,3}) = \varrho_{k,4}\varrho_{k,5} - \varrho_{k,4}\varrho_{k,3},\\
k\varrho_{k,5}^2 &= k\varrho_{k,5}\varrho_{k,5} = (\varrho_{k,6} - \varrho_{k,4})\varrho_{k,5} = \varrho_{k,6}\varrho_{k,5} - \varrho_{k,4}\varrho_{k,5},\\
&\vdots
\end{aligned}$$

$$k\sum_{i=1}^{n}\varrho_{k,n}^2 = \begin{cases} \varrho_{k,n}\varrho_{k,n+1} - \varrho_{k,n}\varrho_{k,n-1}, & \text{if } n \text{ is an even positive integer;}\\ \varrho_{k,n+1}\varrho_{k,n} - \varrho_{k,n-1}\varrho_{k,n}, & \text{if } n \text{ is an odd positive integer.} \end{cases}$$

Again, by adding all these relations, we obtain

$$\sum_{i=1}^{n}\varrho_{k,i}^2 = \begin{cases} \frac{1}{k}(\varrho_{k,n}\varrho_{k,n+1} - \varrho_{k,0}\varrho_{k,1}), & \text{if } n \text{ is an even positive integer;}\\ \frac{1}{k}(\varrho_{k,n+1}\varrho_{k,n} - \varrho_{k,0}\varrho_{k,1}), & \text{if } n \text{ is an even positive integer.} \end{cases}$$

This completes the proof of Theorem 4.6. □

Theorem 4.7. *Let $n \in \mathbb{N}$. Then prove that*

$$\sum_{i=1}^{n}(\xi_{k,i}^2 + \varrho_{k,i}^2) = \begin{cases} \frac{1}{k}(\xi_{k,n}\xi_{k,n+1} + \varrho_{k,n}\varrho_{k,n+1} - \xi_{k,0}\xi_{k,1} - \varrho_{k,0}\varrho_{k,1}),\\ \qquad \textit{if } n \textit{ is an even positive integer;}\\ \frac{1}{k}(\xi_{k,n+1}\xi_{k,n} + \varrho_{k,n+1}\varrho_{k,n} - \xi_{k,0}\xi_{k,1} - \varrho_{k,0}\varrho_{k,1}),\\ \qquad \textit{if } n \textit{ is an odd positive integer.} \end{cases}$$

5 Conclusion

The identities and summation formulas contained within not only improve our understanding of quaternion sequences but also offer potential applications in diverse fields, including computer graphics and quantum mechanics. By delving into these connections, scholars can uncover fresh opportunities for employing quaternions in advanced mathematical and scientific settings.

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