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Another six Fibonacci-like sequences

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Abstract: We briefly describe six Fibonacci-like sequences or arbitrary order that give rise to a periodic or eventually periodic sequences. We provide some examples and demonstrate the explicit periods of these sequences.
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1 Motivation and Preliminaries

In the very recent paper Atanassov and Shannon introduced two Fibonacci-like sequences, which are given by the following recurrence relation [1]:

$$s_{n+m+1} = s_{n+m} - s_{n+m-1} + \cdots - s_n, \qquad m \text{ even}, \tag{1}$$

$$s_{n+m+1} = s_{n+m} - s_{n+m-1} + \cdots + s_n, \qquad m \text{ odd}. \tag{2}$$

In both cases, arbitrary initial conditions are imposed and set to be $s_0 = a_0, \ldots, s_m = a_m$. The main goal of [1] is to show that in this case both sequences are periodic and the authors provide an explicit description of their periods.

Inspired by these two examples, we present six more sequences of this type. For clarity, we will denote them as follows: $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$, $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$. These sequences are defined via recurrences:

$$\begin{aligned}
a_n &= -a_{n-2} - a_{n-4} - \cdots - a_{n-2m}, && a_0, \ldots, a_{2m-1} \in \mathbb{R}, && \\
b_n &= -b_{n-2} - b_{n-4} - \cdots - b_{n-2m}, && b_0, \ldots, b_{2m} \in \mathbb{R}, && \\
c_n &= c_{n-2} - c_{n-4} + \cdots + c_{n-2m}, && c_0, \ldots, c_{2m-1} \in \mathbb{R}, && m \text{ odd}, \\
d_n &= d_{n-2} - d_{n-4} + \cdots + d_{n-2m}, && d_0, \ldots, d_{2m} \in \mathbb{R}, && m \text{ odd}, \\
e_n &= e_{n-2} - e_{n-4} + \cdots - e_{n-2m}, && e_0, \ldots, e_{2m-1} \in \mathbb{R}, && m \text{ even}, \\
f_n &= d_{n-2} - f_{n-4} + \cdots - f_{n-2m}, && f_0, \ldots, f_{2m} \in \mathbb{R}, && m \text{ even}.
\end{aligned}$$

This will be our standard notation and a set of assumptions throughout the article for these sequences.

Note that the sequences obey the same recurrence relation, but with a different amount of initial conditions. We observe that the sequences $(c_n)_{n\in\mathbb{N}}$ to $(f_n)_{n\in\mathbb{N}}$ share a similar definition to that of (1) and (2), but our consideration takes every other term. In general, we show that all the sequences from (a_n) to (f_n) exhibit periodic pattern as (1) and (2) do.

We note that the study recurrence of arbitrary order is well-grounded, with many articles focusing on general properties, properties modulo an integer q, Pisano period and convolutions, among others (see for example [2–4]).

Recall that if a sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the property $u_{n+p} = u_n$ for some $p \geq 0$ and all $n \geq N \geq 0$, then this sequence is called eventually periodic provided $N > 0$ and periodic if $N = 0$, respectively. In any case, the period of $(u_n)_{n\in\mathbb{N}}$ is a finite subsequence of that sequence, denoted by

$$(u_N, \qquad u_{N+1}, \qquad \ldots \qquad u_{N+p-1}).$$

2 Examples

We begin our investigation with six examples, each related to one of the sequences and each exploring the periodic pattern for small m.

2.1 First example: The equence $(a_n)_{n\in\mathbb{N}}$

Let $m = 2$ and consider the sequence $(a_n)_{n\in\mathbb{N}}$, which means that $a_0, \ldots, a_3$ are fixed and the recurrence is

$$a_n = -a_{n-2} - a_{n-4}.$$

Iterating the equation reveals the following pattern:

$$\begin{aligned}
a_4 &= -a_2 - a_0, \\
a_5 &= -a_3 - a_1, \\
a_6 &= -(-a_2 - a_0) - a_2 = a_0, \\
a_7 &= -(-a_3 - a_1) - a_3 = a_1, \\
a_8 &= -a_0 - (-a_2 - a_0) = a_2, \\
a_9 &= -a_1 - (-a_3 - a_1) = a_3,
\end{aligned}$$

and therefore the sequence repeats and forms a periodic sequence with period

$$(a_0, \quad a_1, \quad a_2, \quad a_3, \quad -a_2 - a_0, \quad -a_3 - a_1).$$

2.2 Second example: The sequence $(b_n)_{n\in\mathbb{N}}$

Let $m = 2$ and consider the sequence $(b_n)_{n\in\mathbb{N}}$. Fix $b_0, \ldots, b_4$ and with the aid of the recurrence

$$b_n = -b_{n-2} - b_{n-4}$$

we reveal another pattern:

$$\begin{aligned}
b_5 &= -b_3 - b_1, \\
b_6 &= -b_4 - b_2, \\
b_7 &= -(-b_3 - b_1) - b_3 = b_1, \\
b_8 &= -(-b_4 - b_2) - b_4 = b_2, \\
b_9 &= -b_1 - (-b_3 - b_1) = b_3, \\
b_{10} &= -b_2 - (-b_4 - b_2) = b_4, \\
b_{11} &= -b_3 - b_1, \\
b_{12} &= -b_4 - b_2.
\end{aligned}$$

The sequence repeats again, but this time it forms an eventually periodic sequence with period

$$(b_1, \quad b_2, \quad b_3, \quad b_4, \quad -b_3 - b_1, \quad -b_4 - b_2).$$

2.3 Third example: The sequence $(c_n)_{n\in\mathbb{N}}$

Let $m = 3$ and consider the sequence $(c_n)_{n\in\mathbb{N}}$. Fix $c_0, \ldots, c_5$ and use the recurrence

$$c_n = c_{n-2} - c_{n-4} + c_{n-6}$$

to find out that:

$$\begin{aligned}
c_6 &= c_4 - c_2 + c_0, \\
c_7 &= c_5 - c_3 + c_1, \\
c_8 &= (c_4 - c_2 + c_0) - c_4 + c_2 = c_0, \\
c_9 &= (c_5 - c_3 + c_1) - c_5 + c_3 = c_1, \\
c_{10} &= c_0 - (c_4 - c_2 + c_0) + c_4 = c_2, \\
c_{11} &= c_1 - (c_5 - c_3 + c_1) + c_5 = c_3, \\
c_{12} &= c_2 - c_0 + (c_4 - c_2 + c_0) = c_4, \\
c_{13} &= c_3 - c_1 + (c_5 - c_3 + c_1) = c_5, \\
c_{14} &= c_4 - c_2 + c_0.
\end{aligned}$$

Thus we have found a periodic pattern with period

$$(c_0, \quad c_1, \quad c_2, \quad c_3, \quad c_4, \quad c_5, \quad c_4 - c_2 + c_0, \quad c_5 - c_3 + c_1).$$

2.4 Fourth example: The sequence $(d_n)_{n\in\mathbb{N}}$

Our next example is devoted to the sequence $(d_n)_{n\in\mathbb{N}}$. Set $m = 3$ again, fix $d_0, \dots, d_6$ and consider

$$d_n = d_{n-2} - d_{n-4} + d_{n-6}.$$

This time, we have the following pattern:

$$\begin{aligned}
d_7 &= d_5 - d_3 + d_1,\\
d_8 &= d_6 - d_4 + d_2,\\
d_9 &= (d_5 - d_3 + d_1) - d_5 + d_3 = d_1,\\
d_{10} &= (d_6 - d_4 + d_2) - d_6 + d_4 = d_2,\\
d_{11} &= d_1 - (d_5 - d_3 + d_1) + d_5 = d_3,\\
d_{12} &= d_2 - (d_6 - d_4 + d_2) + d_6 = d_4,\\
d_{13} &= d_3 - d_1 + (d_5 - d_3 + d_1) = d_5,\\
d_{14} &= d_4 - d_2 + (d_6 - d_4 + d_2) = d_6,\\
d_{15} &= d_5 - d_3 + d_1.
\end{aligned}$$

In this case we have an eventually periodic sequence with period

$$(d_1, \quad d_2, \quad d_3, \quad d_4, \quad d_5, \quad d_6, \quad d_5 - d_3 + d_1, \quad d_6 - d_4 + d_2).$$

2.5 Fifth and sixth examples: The sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$

Let $m = 2$ and consider $(e_n)_{n\in\mathbb{N}}$ with initial conidtions $e_0, \dots, e_3$. Here,

$$e_n = e_{n-2} - e_{n-4}$$

and thus:

$$\begin{aligned}
e_4 &= e_2 - e_0,\\
e_5 &= e_3 - e_1,\\
e_6 &= (e_2 - e_0) - e_2 = -e_0,\\
e_7 &= (e_3 - e_1) - e_3 = -e_1,\\
e_8 &= -e_0 - (e_2 - e_0) = -e_2,\\
e_9 &= -e_1 - (e_3 - e_1) = -e_3,\\
e_{10} &- e_2 + e_0,\\
e_{11} &= -e_3 + e_1,\\
e_{12} &= (-e_2 + e_0) + e_2 = e_0,\\
e_{13} &= (-e_3 + e_1) + e_3 = e_1,\\
e_{14} &= e_0 - (-e_2 + e_0) = e_2,\\
e_{15} &= e_1 - (-e_3 + e_1) = e_3.
\end{aligned}$$

The sequence loops and we have the following periodic pattern:

$$(e_0, \quad e_1, \quad e_2, \quad e_3, \quad e_2 - e_0, \quad e_3 - e_1, \quad -e_0, \quad -e_1, \quad -e_2, \quad -e_3, \quad -e_2 + e_0, \quad -e_3 + e_1).$$

We can also notice that the second half of the period is equal to the first half with signs changed.

Let $m = 2$ and in the final example we consider $(f_n)_{n\in\mathbb{N}}$ with initial conditions $f_0, \dots, f_4$. Similar consideration to the one in fifth example reveals that this sequence forms an eventually periodic sequence with period

$$(f_1, \quad f_2, \quad f_3, \quad f_4, \quad f_3-f_1, \quad f_4-f_2, \quad -f_1, \quad -f_2, \quad -f_3, \quad -f_4, \quad -f_3+f_1, \quad -f_4+f_2).$$

3 Main results

In this section we show the properties of all six sequences for general m.

Theorem 3.1. *Fix $m > 0$ according to the definition of one of the sequences under consideration. Then:*

1. *the sequence $(a_n)_{n\in\mathbb{N}}$ is periodic with period*

$$\Big(a_0, \quad a_1, \quad \dots \quad a_{2m-1}, \quad -\sum_{k=0}^{m-1} a_{2k}, \quad -\sum_{k=0}^{m-1} a_{2k+1}\Big), \tag{3}$$

2. *the sequence $(b_n)_{n\in\mathbb{N}}$ is eventually periodic (that is, the sequence $(b_n)_{n\in\mathbb{N}\setminus\{0\}}$ is periodic) with period*

$$\Big(b_1, \quad b_2, \quad \dots \quad b_{2m}, \quad -\sum_{k=0}^{m-1} b_{2k+1}, \quad -\sum_{k=1}^{m} b_{2k}\Big), \tag{4}$$

3. *the sequence $(c_n)_{n\in\mathbb{N}}$ is periodic with period*

$$\Big(c_0, \quad c_1, \quad \dots \quad c_{2m-1}, \quad \sum_{k=0}^{m-1}(-1)^k c_{2k}, \quad \sum_{k=0}^{m-1}(-1)^k c_{2k+1}\Big), \tag{5}$$

4. *the sequence $(d_n)_{n\in\mathbb{N}}$ is eventually periodic (that is, the sequence $(d_n)_{n\in\mathbb{N}\setminus\{0\}}$ is periodic) with period*

$$\Big(d_1, \quad d_3, \quad \dots \quad d_{2m}, \quad \sum_{k=0}^{m-1}(-1)^k d_{2k+1}, \quad \sum_{k=1}^{m}(-1)^k d_{2k}\Big), \tag{6}$$

5. *the sequence $(e_n)_{n\in\mathbb{N}}$ is periodic with period*

$$\Big(e_0, \quad e_1, \quad \dots \quad e_{2m-1}, \quad \sum_{k=0}^{m-1}(-1)^{k+1} e_{2k}, \quad \sum_{k=0}^{m-1}(-1)^{k+1} e_{2k+1},$$
$$-e_0, \quad -e_1, \quad \dots \quad -e_{2m-1}, \quad \sum_{k=0}^{m-1}(-1)^k e_{2k}, \quad \sum_{k=0}^{m-1}(-1)^k e_{2k+1}\Big), \tag{7}$$

6. *the sequence* $(f_n)_{n\in\mathbb{N}}$ *is eventually periodic (that is, the sequence* $(f_n)_{n\in\mathbb{N}\setminus\{0\}}$ *is periodic) with period*

$$\begin{aligned}\big(f_1, \quad f_2, \quad \dots \quad f_{2m}, \quad \sum_{k=0}^{m-1}(-1)^{k+1}f_{2k+1}, \quad \sum_{k=1}^{m}(-1)^{k}f_{2k},\\ -f_1, \quad -f_2, \quad \dots \quad -f_{2m}, \quad \sum_{k=0}^{m-1}(-1)^{k+1}f_{2k+1}, \quad \sum_{k=1}^{m}(-1)^{k+1}f_{2k}\big). \quad (8)\end{aligned}$$

Proof. We only prove item 4. and (6); the remaining items and patterns (3)–(5) and (7)–(8) are similar.

Let us perform a sequential computation of the next few terms:

$$\begin{aligned}d_{2m+1} &= d_{2m-1} - d_{2m-3} + \dots - d_1,\\ d_{2m+2} &= d_{2m} - d_{2m-2} + \dots - d_2,\\ d_{2m+3} &= (d_{2m-1} - d_{2m-3} + \dots - d_1) - d_{2m-1} + d_{2m-3} - \dots + d_3 = d_1,\\ d_{2m+4} &= (d_{2m} - d_{2m-2} + \dots - d_2) - d_{2m} + d_{2m-4} + \dots + d_4 = d_2.\end{aligned}$$

From now on, we can easily see that:

$$\begin{aligned}d_{2m+5} &= d_3,\\ d_{2m+6} &= d_4,\\ d_{2m+7} &= d_5,\\ d_{2m+7} &= d_6,\\ &\vdots\\ d_{2m+(m+1)} &= d_{2m-1},\\ d_{2m+(m+2)} &= d_{2m},\\ d_{2m+(m+3)} &= d_{2m-1} - d_{2m-3} + \dots - d_1,\\ d_{2m+(m+4)} &= d_{2m} - d_{2m-2} + \dots - d_2.\end{aligned}$$

This implies that $(d_n)_{n\in\mathbb{N}}$ forms an eventually periodic sequence with period described in (6).

We note that for items 5. and 6. we obtain a sign change in the second half of the period due to different parity of m. □

Remark 3.1. *Note that the number of elements in the period may be smaller than the period described in the main result. For example, if we consider the sequence* $(c_n)_{n\in\mathbb{N}}$ *with* $m = 2$, $c_0 = 1$, $c_1 = -1$, $c_2 = c_3 = 0$, *then the period is*

$$(1, \quad -1, \quad 0, \quad 0, \quad -1).$$

4 Conclusion

The patterns described in the six introduced sequences show that extending the classic Fibonacci sequence to an arbitrary order and then slightly modifying the coefficients can lead to periodic patterns, that are not present in the classic Fibonacci-like setting. Exploring more sequences of this kind seems interesting, as does finding a suitable periodic sequence with interesting patterns in the coefficients. The patterns considered originally in [1] alternate between 1 and -1, while those introduced in this article alternate between 0 and -1 or 1, 0, -1 and 0.

References

[1] Atanassov, K. T., & Shannon, A. G. (2025). Two Fibonacci-like sequences. *Notes on Number Theory and Discrete Mathematics*, 31(2), 335–339.

[2] Gryszka, K. (2022). A note on the Fibonacci m-step sequences modulo q. *Mathematical Communications*, 27(2), 215–223.

[3] Philippou, A. N. (1983). A note on the Fibonacci sequence of order k and the multinomial coefficients. *The Fibonacci Quarterly*, 21(2), 82–86.

[4] Shannon, A. G. (1976). Some number theoretic properties of arbitrary order recursive sequences. *Bulletin of the Australian Mathematical Society*, 14(1), 149–151.