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Binomial transform of the bivariate Fibonacci quaternion polynomials and its properties

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Abstract: The primary aim of this work is to deal with binomial transforms of bivariate Fibonacci quaternion polynomial sequence. The binomial sequence of the bivariate Fibonacci quaternion polynomial is found, and then results are obtained for the recurrence relation, generating function and Binet formula. Furthermore, different types of sums are being sought for the polynomials. While working with bivariate Fibonacci quaternion polynomial sequence, we found the general formula for all sequences with quadratic recurrence relation, which includes the formulas in general terms for binomial transform. In the last part, matrix representations are derived for the corresponding binomial sequence.
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1 Introduction

Integer sequences are a cornerstone of mathematical research and application. They appear in theoretical work, practical applications, educational resources, and numerous databases, reflecting their fundamental role across various fields of study. Many mathematical papers delve into the properties and behaviors of specific integer sequences. There is no doubt that the Fibonacci number sequence has attracted the most attention in the literature. This is a sequence in which each number is the sum of the two preceding ones, starting from 0 and 1. The sequence progresses as $0, 1, 1, 2, 3, 5, 8, 13,$ and so forth [15].

The bivariate Fibonacci polynomial is an extension of the Fibonacci sequence into two variables, in [10]. It is defined to generalize the properties of the Fibonacci sequence using polynomial expressions. Below is a closer look at the bivariate Fibonacci polynomial and its properties:

$$F_n(x,y) = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ xF_{n-1}(x,y) + yF_{n-2}(x,y), & \text{if } n \geq 2. \end{cases} \tag{1}$$

The characteristic equation of bivariate Fibonacci polynomial is

$$h^2 - xh - y = 0.$$

Also its roots are $h_1(x,y) = \frac{x+\sqrt{x^2+4y}}{2}$ and $h_2(x,y) = \frac{x-\sqrt{x^2+4y}}{2}$. Hence, the Binet formula and the generating function are as follows, respectively,

$$F_n(x,y) = \frac{h_1^n - h_2^n}{h_1 - h_2} \text{ for } n \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} F_n(x,y)p^n = \frac{p}{1 - xp - yp}.$$

It should be noted that the information given above about bivariate Fibonacci polynomials was originally presented by Catalani in [2] and [3].

Table 1 below demonstrates that, by assigning appropriate values to the two-variable Fibonacci polynomial, other Fibonacci-type polynomials of second order are obtained.

Table 1. Fibonacci-type polynomials of the second order

		Fibonacci-type polynomials
x	y	bivariate Fibonacci polynomial
x	1	Fibonacci polynomial
$2x$	1	Pell polynomial
1	$2x$	Jacobsthal polynomial
$2x$	-1	Chebyshev polynomial of the second kind
$3x$	-2	Fermat polynomial

The interest in this polynomial has continued to this day, as seen in [8] and [13]. Sequences of numbers with coefficients that are quaternions or quaternion polynomials have been studied extensively, and it is observed from the literature that interest in this topic remains high even today.

Consider a sequence $\{q_n\}$ where each q_n is a quaternion. That is, each term in the sequence can be written as $q = a_n + b_n i + c_n j + d_n k$ where a_n, b_n, c_n, and d_n are real sequences and i,

j and k are the basic quaternion units. For example, in [12], the authors introduced a bivariate Fibonacci quaternion polynomial sequence as follows. In that work, the authors used QBF instead of bivariate Fibonacci quaternion polynomial sequence for abbreviation purposes.

Definition 1.1. [12] *A quaternion polynomial, which is called QBF, is defined by*

$$QBF_n(x,y) \;=\; \sum_{r=0}^{3} F_{n+r}(x,y)e_r \; (e_0=1, e_1=i, e_2=j, e_3=k),$$

where $F_{n+r}(x,y)$ is the $(n+r)$-th bivariate Fibonacci polynomial with the initial conditions

$$QBF_0(x,y) = e_1 + xe_2 + (x^2+y)e_3$$

and

$$QBF_1(x,y) = e_0 + xe_1 + (x^2+y)e_2 + (x^3+2xy)e_3.$$

Moreover, let us consider (1), then we have

$$\begin{aligned} QBF_{n+1}(x,y) &= \sum_{r=0}^{3} F_{n+1+r}(x,y)e_r \\ &= \sum_{r=0}^{3} \left(xF_{n+r}(x,y) + yF_{n-1+r}(x,y)\right) e_r \\ &= x\sum_{r=0}^{3} F_{n+r}(x,y)e_r + y\sum_{r=0}^{3} F_{n-1+r}(x,y)e_r. \end{aligned}$$

This leads to the following recurrence relation (see [12]):

$$QBF_{n+1}(x,y) = xQBF_n(x,y) + yQBF_{n-1}(x,y). \tag{2}$$

In the same study, the iteration relation for QBF was established, then Binet formula, generating function, and some identities including Cassini, Catalan, and d 'Ocagne were obtained. Some authors (e.g., [9, 11]) have studied the binomial transform of quaternion. The binomial transform is a versatile mathematical tool that finds applications across various fields, [4,5,7,14]. It can simplify problems, enable transforms between different forms.

2 Binomial transform of bivariate Fibonacci quaternion polynomial

In this section, it is aimed to find the binomial transform of the two-variable Fibonacci quaternion polynomial. In [14], for a sequence $\{a_1, a_2, a_3, \ldots\}$ the binomial transform is defined by:

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.$$

Definition 2.1. *f_n is called a binomial transform of bivariate Fibonacci quaternion polynomial if it is defined by*

$$f_n = \sum_{r=0}^{n} \binom{n}{r} QBF_r(x,y), \tag{3}$$

where $QBF_n(x,y)$ is the n-th bivariate Fibonacci quaternion polynomial sequence.

Lemma 2.1. *The following equation is valid*

$$f_{n+1} = \sum_{r=0}^{n} \binom{n}{r} (QBF_r(x,y) + QBF_{r+1}(x,y)),$$

where f_n is the binomial transform of bivariate Fibonacci quaternion polynomial.

Proof. In view of (3), we have

$$f_n = \sum_{r=0}^{n} \binom{n}{r} QBF_r(x,y).$$

Then we replace n with $n+1$

$$\begin{aligned} f_{n+1} &= \sum_{r=0}^{n+1} \binom{n+1}{r} QBF_r(x,y) \\ &= \sum_{r=1}^{n+1} \binom{n+1}{r} QBF_r(x,y) + QBF_0(x,y). \end{aligned}$$

From known binomial equations $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ and $\binom{n}{n+1} = 0$, we find

$$f_{n+1} = \sum_{r=1}^{n+1} \left[\binom{n}{r-1} + \binom{n}{r} \right] QBF_r(x,y) + QBF_0(x,y),$$

so we get

$$f_{n+1} = \sum_{r=0}^{n} \binom{n}{r} (QBF_r(x,y) + QBF_{r+1}(x,y)).$$

This finishes the proof. □

Corollary 2.1. *Let f_n represent the binomial transform of the bivariate Fibonacci quaternion polynomial sequence. Then*

$$f_{n+1} = f_n + \sum_{r=0}^{n} \binom{n}{r} QBF_{r+1}(x,y).$$

Theorem 2.1. *For $n \geq 1$, the binomial transform for the bivariate Fibonacci quaternion polynomial f_n is constructed by the recursive relation:*

$$f_{n+1} = (x+2)f_n + (y-x-1)f_{n-1},$$

where $f_0 = QBF_0(x,y)$ and $f_1 = QBF_0(x,y) + QBF_1(x,y)$.

Proof. According to Lemma 2.1, we have

$$\begin{aligned}
f_{n+1} &= \sum_{r=0}^{n}\binom{n}{r}(QBF_r(x,y)+QBF_{r+1}(x,y))\\
&= \sum_{r=1}^{n}\binom{n}{r}(QBF_r(x,y)+QBF_{r+1}(x,y))+QBF_0(x,y)+QBF_1(x,y)\\
&= (x+1)f_n+y\sum_{i=1}^{n}\binom{n}{i}QBF_{i-1}(x,y)+QBF_1(x,y)-yQBF_0(x,y).
\end{aligned}$$

By substituting $n-1$ for n, we obtain

$$\begin{aligned}
f_n &= (x+1)f_{n-1}+y\sum_{r=1}^{n-1}\binom{n-1}{r}QBF_{r-1}(x,y)+QBF_1(x,y)-yQBF_0(x,y)\\
&= xf_{n-1}+\sum_{r=1}^{n}\left[\binom{n-1}{r-1}+y\binom{n-1}{r}\right]QBF_{r-1}(x,y)+QBF_1(x,y)-yQBF_0(x,y)\\
&= xf_{n-1}+\sum_{r=1}^{n}\left[\binom{n-1}{r-1}+y\binom{n-1}{r}\pm y\binom{n-1}{r-1}\right]QBF_{r-1}(x,y)+QBF_1(x,y)\\
&\quad -yQBF_0(x,y)\\
&= xf_{n-1}+(1-y)f_{n-1}+y\sum_{r=1}^{n}QBF_{r-1}(x,y)+QBF_1(x,y)-yQBF_0(x,y),
\end{aligned}$$

so we get

$$y\sum_{r=1}^{n}\binom{n}{r}QBF_{r-1}(x,y)+QBF_1(x,y)-yQBF_0(x,y)=f_n+(y-x-1)f_{n-1}.$$

As a result of necessary arrangements we have

$$f_{n+1}=(x+2)f_n+(y-x-1)f_{n-1}. \qquad \square$$

Moreover, the characteristic equation is

$$h^2-(x+2)h-(y-x-1)=0.$$

The roots of this equation are

$$\alpha=\frac{x+2+\sqrt{x^2+4y}}{2} \quad \text{and} \quad \beta=\frac{x+2-\sqrt{x^2+4y}}{2}.$$

Based on the roots, the following equations are valid;

$$\begin{aligned}
\alpha+\beta &= x+2,\\
\alpha-\beta &= \sqrt{x^2+4y}=\Delta,\\
\alpha\beta &= x-y+1,\\
\alpha^2 &= (x+2)\alpha+(y-x-1),\\
\beta^2 &= (x+2)\beta+(y-x-1),\\
\alpha^2+(y-x-1) &= \alpha\Delta,\\
\beta^2+(y-x-1) &= -\beta\Delta.
\end{aligned} \tag{4}$$

Theorem 2.2. *Binet-like formula for the binomial transform of bivariate Fibonacci quaternion polynomial sequence is given by*

$$f_n = \frac{f_1 - f_0\beta}{\alpha - \beta}\alpha^n + \frac{f_0\alpha - f_1}{\alpha - \beta}\beta^n.$$

Proof. To put it in the most general terms,

$$f_n = c_1\alpha^n + c_2\beta^n.$$

So we get

- for $n = 0,\ f_0 = c_1 + c_2$,
- for $n = 1,\ f_1 = c_1\alpha + c_2\beta$.

Then

$$c_1 = \frac{f_1 - f_0\beta}{\alpha - \beta} \quad \text{and} \quad c_2 = \frac{f_0\alpha - f_1}{\alpha - \beta}.$$

If $c_1 = \frac{f_1 - f_0\beta}{\alpha - \beta}$ and $c_2 = \frac{f_0\alpha - f_1}{\alpha - \beta}$ are substituted into f_n, we find

$$f_n = \frac{f_1 - f_0\beta}{\alpha - \beta}\alpha^n + \frac{f_0\alpha - f_1}{\alpha - \beta}\beta^n.$$

The proof is completed. □

For convenience, we can write the Binet-like formula for f_n as follows:

$$f_n = \kappa\alpha^n + \tau\beta^n, \tag{5}$$

where $\kappa = \frac{f_1 - f_0\beta}{\alpha - \beta}$ and $\tau = \frac{f_0\alpha - f_1}{\alpha - \beta}$.

Theorem 2.3. *The generating function of the related binomial transform is*

$$f(t) = \frac{(1 - xt - 2t)f_0 + f_1 t}{1 - xt - 2t - yt^2 + xt^2 + t^2}$$

where

$$f_0 = QBF_0(x,y) = i + xj + (x^2 + y)k \quad \textit{and}$$
$$f_1 = QBF_0(x,y) + QBF_1(x,y) = 1 + (1+x)i + (x + x^2 + y)j + (x^3 + x^2 + 2xy + y)k.$$

Proof. First, using the relevant Binet formula, and then using the equation (4) and geometric series, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}(\kappa\alpha^n + \tau\beta^n)\,t^n &= \sum_{n=0}^{\infty}\kappa(\alpha t)^n + \sum_{n=0}^{\infty}\tau(\beta t)^n \\
&= \left(\frac{\kappa}{1 - \alpha t} + \frac{\tau}{1 - \beta t}\right) \\
&= \frac{1}{\alpha - \beta}\left(\frac{-f_0\beta - f_1\beta t + f_0\beta^2 t + f_0\alpha - f_0\alpha^2 t + f_1\alpha t}{(y - x - 1)t^2 - t(x+2) + 1}\right) \\
&= \frac{1}{\alpha - \beta}\left(\frac{f_0(\alpha - \beta) + t(-f_0(\beta + \alpha)(\alpha - \beta) + f_1(\alpha - \beta))}{(y - x - 1)t^2 - t(x+2) + 1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f_0 + t(-f_0(\alpha+\beta) + f_1)}{(y-x-1)t^2 - (x+2)t + 1} \\
&= \frac{(1 - xt - 2t)f_0 + f_1 t}{1 - xt - 2t - yt^2 + xt^2 + t^2},
\end{aligned}
$$

as we declared. □

The theorem to be given now is about exponential generating function.

Theorem 2.4. *The exponential generating function for f_n is given by:*

$$\sum_{n=0}^{\infty} f_n \frac{t^n}{n!} = \kappa e^{\alpha t} + \tau e^{\beta t}.$$

Proof. If we apply the Binet formula, we obtain

$$\sum_{n=0}^{\infty} f_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (\kappa\alpha^n + \tau\beta^n) \frac{t^n}{n!} = \kappa \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} + \tau \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!}.$$

The desired result is achieved with the necessary basic operations and arrangements:

$$\sum_{n=0}^{\infty} f_n \frac{t^n}{n!} = \kappa e^{\alpha t} + \tau e^{\beta t}.$$

□

The following result is valid for f_n.

Theorem 2.5. *For f_n, for any $n \in \mathbb{N}$ and for integers m and s with $k > m$, it holds that:*

$$\sum_{r=0}^{n} f_{mr+k} = \frac{(x-y+1)(f_{mn+k} - f_{k-m}) - f_{mn+m+k} + f_k}{(x-y+1) - (\alpha^m - \beta^m) + 1}.$$

Proof. Using the relevant Binet formula and the equation (4), the following is obtained

$$
\begin{aligned}
\sum_{r=0}^{n} f_{mr+k} &= \sum_{r=0}^{n} \kappa\alpha^{mr+k} + \tau\beta^{mr+k} \\
&= \kappa\alpha^k \sum_{k=0}^{n} \alpha^{mr} + \tau\beta^k \sum_{k=0}^{n} \beta^{mr} \\
&= \kappa\alpha^k \left(\frac{(\alpha^m)^{n+1} - 1}{\alpha^m - 1} \right) + \tau\beta^k \left(\frac{(\beta^m)^{n+1} - 1}{\beta^m - 1} \right) \\
&= \frac{\kappa(\alpha^{mn+m+k} - \alpha^k)(\beta^m - 1)}{(\alpha^m\beta^m - \alpha^m - \beta^m + 1)} \\
&\quad + \frac{\tau(\beta^{mn+m+k} - \beta^k)(\alpha^m - 1)}{(\alpha^m\beta^m - \alpha^m - \beta^m + 1)} \\
&= \frac{\kappa(\alpha^{mn+k}(\alpha\beta)^m - \alpha^k\beta^m - \alpha^{mn+m+k} + \alpha^k)}{(\alpha^m\beta^m - \alpha^m - \beta^m + 1)} \\
&\quad + \frac{\tau(\beta^{mn+k}(\alpha\beta)^m - \alpha^m\beta^k - \beta^{mn+m+k} + \beta^k)}{(\alpha^m\beta^m - \alpha^m - \beta^m + 1)}
\end{aligned}
$$

$$= \frac{\begin{pmatrix} (\alpha\beta)^m\left(\kappa\alpha^{mn+k}+\tau\beta^{mn+k}\right)-(\alpha\beta)^m\left(\kappa\alpha^{k-m}+\tau\beta^{k-m}\right) \\ -\left(\kappa\alpha^{mn+m+k}+\tau\beta^{mn+m+k}\right)+\left(\kappa\alpha^{k}+\tau\beta^{k}\right) \end{pmatrix}}{(\alpha\beta)^m-(\alpha^m+\beta^m)+1}$$
$$= \frac{(x-y+1)(f_{mn+k}-f_{k-m})-f_{mn+m+k}+f_k}{(x-y+1)-(\alpha^m+\beta^m)+1}.$$

This completes the proof. □

Theorem 2.6. *Let f_n represent the binomial transform of the bivariate Fibonacci quaternion polynomial sequence. Then*

$$\sum_{n=0}^{\infty} f_{mn+k}t^n = \frac{f_k-(x-y+1)f_{k-m}t}{1-(\alpha^m+\beta^m)+(x-y+1)t^2}$$

for all natural numbers n and integers m and k with k being greater than m.

Proof. If we start by using the Binet formula in (5), we get the following

$$\begin{aligned} \sum_{n=0}^{\infty} f_{mn+k}t^n &= \sum_{n=0}^{\infty}\frac{\left(\kappa\alpha^{mn+k}+\tau\beta^{mn+k}\right)}{\alpha-\beta}t^n \\ &= \kappa\alpha^k\sum_{n=0}^{\infty}(\alpha^m t)^n+\tau\alpha^k\sum_{n=0}^{\infty}(\beta^m t)^n. \end{aligned}$$

Here, by utilizing the geometric series and (4) with the necessary adjustments, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} f_{mn+k}t^n &= \frac{\kappa\alpha^k}{1-\alpha^m t}+\frac{\tau\beta^k}{1-\beta^m t} \\ &= \frac{\kappa\alpha^k(1-\beta^m t)+\tau\beta^k(1-\alpha^m t)}{(1-\alpha^m t)(1-\beta^m t)} \\ &= \frac{(\kappa\alpha^k+B\beta^k)-(\kappa\alpha^k\beta^m+\tau\beta^k\alpha^m)t}{1-(\alpha^m+\beta^m)+(\alpha\beta)^m t^2} \\ &= \frac{\kappa\alpha^k+\tau\beta^k-(\alpha\beta)^m\left(\kappa\alpha^{k-m}+\tau\beta^{k-m}\right)t}{1-(\alpha^m+\beta^m)+(\alpha\beta)^m t^2} \\ &= \frac{f_k-(x-y+1)f_{k-m}t}{1-(\alpha^m+\beta^m)+(x-y+1)t^2}. \end{aligned}$$

Thus, the conclusion is evident. □

We can derive several well-known identities from the literature, by applying the Binet formula to our relevant binomial transform.

Theorem 2.7. *Let f_n represent the binomial transform of the bivariate Fibonacci quaternion polynomial sequence. Then*

(i) $$f_{n-k}f_{n+k}-f_n^2=(x-y+1)^n(\alpha^k-\beta^k)(\tau\kappa\alpha^k-\kappa\tau\beta^k)$$

for natural numbers n and k with $n \geq k$. (The Catalan identity),

(ii)
$$f_{n-1}f_{n+1}-f_n^2=(x-y+1)^n(\tau\kappa\alpha-\kappa\tau\beta)$$

for $n\in\mathbb{N}$ (The Cassini identity),

(iii)
$$f_kf_{n+1}-f_{k+1}f_n=\tau\kappa\beta^{k-n}-\kappa\tau\alpha^{k-n}$$

for $n\geq k$ (The d'Ocagne identity).

Proof. (i) First of all, let us start with our first option.

$$\begin{aligned}
f_{n-k}f_{n+k}-f_n^2 &= \left(\kappa\alpha^{n-k}+\tau\beta^{n-k}\right)\left(\kappa\alpha^{n+k}+\tau\beta^{n+k}\right)-\left(\kappa\alpha^n+\tau\beta^n\right)\left(\kappa\alpha^n+\tau\beta^n\right)\\
&= \kappa\kappa^{2n}+\kappa\tau\alpha^{n-k}\beta^{n+k}+\tau\kappa\beta^{n-k}\alpha^{n+k}+\tau\tau\beta^{2n}\\
&\quad -\left(\kappa\kappa^{2n}+\kappa\tau\alpha^n\beta^n+\tau\kappa\beta^n\alpha^n+\tau\tau\beta^{2n}\right)\\
&= \kappa\tau(\alpha\beta)^n\frac{\beta^k}{\alpha^k}+\tau\kappa(\alpha\beta)^n\frac{\alpha^k}{\beta^k}-\kappa\tau(\alpha\beta)^n-\tau\kappa(\alpha\beta)^n\\
&= (\alpha\beta)^n\left(\kappa\tau\left(\frac{\beta^k-\alpha^k}{\alpha^k}\right)+\tau\kappa\left(\frac{\alpha^k-\beta^k}{\beta^k}\right)\right)\\
&= (\alpha\beta)^n\left(\kappa\tau(\beta^k-\alpha^k\beta^k+\tau\kappa(\alpha^k-\beta^k)\alpha^k\right)\\
&= (\alpha\beta)^n(\alpha^k-\beta^k)(\tau\kappa\alpha^k-\kappa\tau\beta^k)\\
&= (x-y+1)^n(\alpha^k-\beta^k)(\tau\kappa\alpha^k-\kappa\tau\beta^k).
\end{aligned}$$

(ii) To get what we want, it is enough to take $k=1$ in (i).
(iii) Let us use the Binet formula in (5), we get the following

$$\begin{aligned}
f_kf_{n+1}-f_{k+1}f_n &= \left(\kappa\alpha^k+\tau\beta^k\right)\left(\kappa\alpha^{n+1}+\tau\beta^{n+1}\right)-\left(\kappa\alpha^{k+1}+\tau\beta^{k+1}\right)\left(\kappa\alpha^n+\tau\beta^n\right)\\
&= \kappa\kappa\alpha^{k+n+1}+\kappa\tau\alpha^k\beta^{n+1}+\tau\kappa\beta^k\alpha^{n+1}+\tau\tau\beta^{k+n+1}\\
&\quad -\kappa\kappa\alpha^{k+1+n}-\kappa\tau\alpha^{k+1}\beta^n-\tau\kappa\beta^{k+1}\alpha^n-\tau\tau\beta^{k+1+n}\\
&= \kappa\tau\alpha^k\beta^{n+1}+\tau\kappa\beta^k\alpha^{n+1}-\kappa\tau\alpha^{k+1}\beta^n-\tau\kappa\beta^{k+1}\alpha^n\\
&= -\kappa\tau\alpha^k\beta^n(\alpha-\beta)+\tau\kappa\beta^k\alpha^n(\alpha-\beta)\\
&= (\alpha-\beta)(\tau\kappa\beta^k\alpha^n-\kappa\tau\alpha^k\beta^n)\\
&= \tau\kappa\beta^k\alpha^n-\kappa\tau\alpha^k\beta^n\\
&= \tau\kappa\beta^{k-n}-\kappa\tau\alpha^{k-n}.
\end{aligned}$$

Thus, the proof is concluded. □

Theorem 2.8. *Let f_n represent the n-th binomial transform of the bivariate Fibonacci quaternion polynomial sequence. Then*

$$\sum_{r=0}^{n}\binom{n}{r}(y-x-1)^{n-r}(x+2)^r f_r=f_{2n},$$

where for each natural number n.

Proof. Let us begin the proof by starting from the Binet formula

$$\begin{aligned}
& \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} f_{r} \\
= & \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} (\kappa\alpha^{r} + \tau\beta^{r}) \\
= & \kappa \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} ((x+2)\alpha)^{r} + \tau \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} ((x+2)\beta)^{r}.
\end{aligned}$$

Considering (4) and continuing with elementary operations

$$\begin{aligned}
& \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} f_{r} \\
= & \kappa ((y-x-1) + \alpha (x+2))^{n} + \tau ((y-x-1) + \beta (x+2))^{n} \\
= & \kappa\alpha^{2n} + \tau\beta^{2n} \\
= & f_{2n}.
\end{aligned}$$

Thus, the desired result has been achieved. □

Theorem 2.9. *Let f_n represent the n-th binomial transform of the bivariate Fibonacci quaternion polynomial. In that case*

$$\sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} f_{r+1} = f_{2n+1},$$

where for each natural number r and integer n.

Proof. Let us begin the proof by starting from the Binet formula

$$\begin{aligned}
& \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} f_{r+1} \\
= & \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} \left(\kappa\alpha^{r+1} + \tau\beta^{r+1}\right) \\
= & \kappa\alpha \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} ((x+2)\alpha)^{r} + \tau\beta \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} ((x+2)\beta)^{r}.
\end{aligned}$$

Taking (4) into account and proceeding with basic operations

$$\begin{aligned}
& \sum_{r=0}^{n} \binom{n}{r} (y-x-1)^{n-r} (x+2)^{r} f_{r+1} \\
= & \kappa\alpha ((y-x-1) + \alpha (x+2))^{n} + \tau\beta ((y-x-1) + \beta (x+2))^{n} \\
= & \kappa\alpha\alpha^{2n} + \tau\beta\beta^{2n} \\
= & f_{2n+1}.
\end{aligned}$$

Therefore, the intended outcome has been accomplished. □

Theorem 2.10. *For all $n \in \mathbb{N}$, f_n represents the n-th binomial transform of the bivariate Fibonacci quaternion polynomial sequence. Then the following is valid*

$$(y-x-1)\, f_n^2 + f_{n+1}^2 = \Delta\left(\kappa^2\alpha^{2n+1} - \tau^2\beta^{2n+1}\right).$$

Proof. From the Binet formula and (4), we get

$$\begin{aligned}
& (y-x-1)\, f_n^2 + f_{n+1}^2 \\
= \;& (y-x-1)\left(\kappa\alpha^n + \tau\beta^n\right)\left(\kappa\alpha^n + \tau\beta^n\right) + \left(\kappa\alpha^{n+1} + \tau\beta^{n+1}\right)\left(\kappa\alpha^{n+1} + \tau\beta^{n+1}\right) \\
= \;& (y-x-1)\left(\kappa^2\alpha^{2n} + \tau^2\beta^{2n}\right) + \left(\kappa^2\alpha^{2n+2} + \tau^2\beta^{2n+2}\right) \\
= \;& \kappa^2\alpha^{2n}\alpha\,(\alpha-\beta) - \tau^2\beta^{2n}\beta\,(\alpha-\beta) \\
= \;& (\alpha-\beta)\left(\kappa^2\alpha^{2n+1} - \tau^2\beta^{2n+1}\right).
\end{aligned}$$

So this result verifies the proof. □

Theorem 2.11. *Let f_n represent the n-th binomial transform of the bivariate Fibonacci quaternion polynomial. In that case, for $n \geq 0$, we have*

$$(i) \qquad \sum_{r=0}^{n}\binom{n}{r} f_{2r+s}\,(y-x-1)^{n-r} = \begin{cases} f_{n+s}\Delta^n, & \textit{for even } n \\ (\kappa\alpha^{n+s} + \tau\beta^{n+s})\Delta^n, & \textit{for odd } n \end{cases}$$

$$(ii) \qquad \sum_{r=0}^{n}\binom{n}{r}(-1)^r f_{2r+s}\,(y-x-1)^{n-r} = \begin{cases} (x+2)^n\, f_{n+s}, & \textit{for even } n \\ -(x+2)^n\, f_{n+s}, & \textit{for odd } n \end{cases}$$

$$(iii) \qquad \sum_{r=0}^{n}\binom{n}{r} f_r f_{r+s}\,(y-x-1)^{n-r} = \begin{cases} (\kappa^2\alpha^{n+s} + \tau^2\beta^{n+s})\Delta^n, & \textit{for even } n \\ (\kappa^2\alpha^{n+s} - \tau^2\beta^{n+s})\Delta^n, & \textit{for odd } n \end{cases}$$

$$(iv) \qquad \sum_{r=0}^{n}\binom{n}{r} f_r^2\,(y-x-1)^{n-r} = \begin{cases} (\kappa^2\alpha^k + \tau^2\beta^k)\Delta^n, & \textit{for even } n \\ (\kappa^2\alpha^k - \tau^2\beta^k)\Delta^n, & \textit{for odd } n \end{cases}$$

Proof. Using the related Binet formula and (4), we get

$$\begin{aligned}
\sum_{r=0}^{n}\binom{n}{r} f_{2r+s}\,(y-x-1)^{n-r} &= \sum_{r=0}^{n}\binom{n}{r}\left(\kappa\alpha^{2r+s} + \tau\beta^{2r+s}\right)(y-x-1)^{n-r} \\
&= \kappa\alpha^s\sum_{r=0}^{n}\binom{n}{r}(\alpha^2)^k\,(y-x-1)^{n-r} \\
&\quad + \tau\beta^s\sum_{r=0}^{n}\binom{n}{r}(\beta^2)^k\,(y-x-1)^{n-r} \\
&= \kappa\alpha^s(\alpha^2+y-x-1)^n + \tau\beta^s(\beta^2+y-x-1)^n \\
&= \kappa\alpha^s\,(\alpha\Delta)^n + \tau\beta^s\,(-\beta\Delta)^n \\
&= \kappa\alpha^s\alpha^n\Delta^n + (-1)^n\tau\beta^s\beta^n\Delta^n \\
&= \left(\kappa\alpha^{n+s} + (-1)^n\tau\beta^{n+s}\right)\Delta^n \\
&= \begin{cases} f_{n+s}\Delta^n, & \text{for even } n \\ (\kappa\alpha^{n+s} - \tau\beta^{n+s})\Delta^{n-1}, & \text{for odd } n. \end{cases}
\end{aligned}$$

The proof has been completed. The remaining parts (ii), (iii) and (iv) can be handled in a manner similar to (i). □

Theorem 2.12. *Let f_n represent the n-th binomial transform of the bivariate Fibonacci quaternion polynomial. In that case for all $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, we have*

$$\sum_{r=0}^{n} \binom{n}{r} (-y+x+1)^{n-r} f_{2r+s} = (x+2)^n f_{n+s}.$$

Proof. If we start again with the relevant Binet formula, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} (-y+x+1)^{n-r} f_{2r+s} \right] \frac{t^n}{n!} \\
= & \left(\sum_{n=0}^{\infty} (-y+x+1)^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} f_{2n+s} \frac{t^n}{n!} \right) \\
= & \left(\sum_{n=0}^{\infty} (-y+x+1)^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \left(\kappa\alpha^{2n+s} - \tau\beta^{2n+s} \right) \frac{t^n}{n!} \right).
\end{aligned}$$

Considering the binomial formula

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} (-y+x+1)^{n-r} f_{2r+s} \right] \frac{t^n}{n!} \\
= & \left(\sum_{n=0}^{\infty} (-y+x+1)^n \frac{t^n}{n!} \right) \left(\kappa\alpha^s \sum_{n=0}^{\infty} \frac{(\alpha^2 t)^n}{n!} + \tau\beta^s \sum_{n=0}^{\infty} \frac{(\beta^2 t)^n}{n!} \right) \\
= & e^{-(y-x-1)t} \left(\kappa\alpha^s e^{\alpha^2 t} + \tau\beta^s e^{\beta^2 t} \right) \\
= & \kappa\alpha^s e^{(x+2)\alpha t} + \tau\beta^s e^{(x+2)\beta t} \\
= & \kappa\alpha^s \sum_{n=0}^{\infty} \frac{((x+2)\,\alpha t)^n}{n!} + \tau\beta^s \sum_{n=0}^{\infty} \frac{((x+2)\,\beta t)^n}{n!} \\
= & \sum_{n=0}^{\infty} (x+2)^n \left(\kappa\alpha^{n+s} - \tau\beta^{n+s} \right) \frac{x^n}{n!},
\end{aligned}$$

thus, we arrive at the desired result, as follows

$$\sum_{r=0}^{n} \binom{n}{r} (-y+x+1)^{n-r} f_{2r+s} = (x+2)^n f_{n+s}. \qquad \square$$

Theorem 2.13. *For $r, n \in \mathbb{N}$, the following equality holds:*

$$f_r f_{n+1} + (y-x-1) f_{r-1} f_n = \left(\kappa^2 \alpha^{n+r} - \tau^2 \beta^{n+r} \right) \Delta,$$

where f_n is the n-th binomial transform of the bivariate Fibonacci quaternion polynomial sequence.

Proof. If the proof is started using the relevant Binet formula, we have

$$\begin{aligned}
& f_r f_{n+1} + (y-x-1) f_{r-1} f_n \\
= & \left(\kappa\alpha^r + \tau\beta^r \right) \left(\kappa\alpha^{n+1} + \tau\beta^{n+1} \right) + (y-x-1) \left(\kappa\alpha^{r-1} + \tau\beta^{r-1} \right) \left(\kappa\alpha^n + \tau\beta^n \right) \\
= & \kappa^2 \alpha^{n+r+1} + \kappa\tau\alpha^r \beta^{n+1} + \tau\kappa\beta^r \alpha^{n+1} + \tau^2 \beta^{n+r+1} \\
& + (y-x-1)\, \kappa^2 \alpha^{n+r-1} + (y-x-1)\, \kappa\tau\alpha^{r-1} \beta^n \\
& + (y-x-1)\, \tau\kappa\beta^{r-1} \alpha^n + (y-x-1)\, \tau^2 \beta^{n+r-1}.
\end{aligned}$$

Taking into account (4), we obtain:

$$\begin{aligned}
& f_r f_{n+1} + (y-x-1)\, f_{r-1} f_n \\
= \;& \kappa^2 \alpha^{n+r} (\alpha + \frac{y-x-1}{\alpha}) + \kappa\tau\alpha^{r-1}\beta^n (\alpha\beta + y - x - 1) \\
& + \kappa\tau\beta^{r-1}\alpha^n (\alpha\beta + y - x - 1) + \tau^2 \beta^{n+r} (\beta + \frac{q}{\beta}) \\
= \;& \kappa^2 \alpha^{n+r} (\alpha - \beta) - \tau^2 \beta^{n+r} (\alpha - \beta) \\
= \;& \left(\kappa^2 \alpha^{n+r} - \tau^2 \beta^{n+r}\right) \Delta.
\end{aligned}$$

Thus, the proof is concluded. □

3 Matrix representation

A binomial transform matrix is a matrix whose entries are binomial transform of bivariate Fibonacci quaternion polynomials, extending the concept of a matrix to the binomial transform system. Here is an overview of matrices of the binomial transform of bivariate Fibonacci quaternion polynomials.

Matrix methods are useful for deriving results related to different identities as well as for algebraic representations in the examination of recurrence relations. Charles H. King discovered a novel method for getting Fibonacci numbers in 1960. He found that by raising a 2×2 matrix, known as the Fibonacci Q-matrix, to various powers, each element of the resulting matrix corresponds to a Fibonacci number dependent on the power applied. King described this approach in his master thesis, as follows:

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

where Q is also known as the companion matrix [6]. The matrix that plays the role of Q-matrix is defined by in [1]. We present matrix M as a second order matrix whose entries are the binomial transform of bivariate Fibonacci quaternion polynomial, as follows:

$$M = \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix} \quad \text{and} \quad M^n = \begin{bmatrix} f_{n+1} & f_n \\ (y-x-1)f_n & (y-x-1)f_{n-1} \end{bmatrix}.$$

Theorem 3.1. *Let f_n represent the n-th binomial transform of the bivariate Fibonacci quaternion polynomial sequence. For an integer $n \geq 1$, we have*

$$\begin{bmatrix} f_{n+1} & f_n \\ (y-x-1)f_n & (y-x-1)f_{n-1} \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ (y-x-1)f_1 & (y-x-1)f_0 \end{bmatrix} \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix}^{n-1}.$$

Proof. The induction method can be employed to prove this. For $n = 1$, the base case is evident. Next, let us suppose that the equation is

$$\begin{aligned}
&\begin{bmatrix} f_k & f_{k-1} \\ (y-x-1)f_{k-1} & (y-x-1)f_{k-2} \end{bmatrix} \\
&= \begin{bmatrix} f_2 & f_1 \\ (y-x-1)f_1 & (y-x-1)f_0 \end{bmatrix} \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix}^{k-2}
\end{aligned}$$

for $n = k-1$. For $n = k$, it becomes

$$\begin{aligned}
&\begin{bmatrix} f_2 & f_1 \\ (y-x-1)f_1 & (y-x-1)f_0 \end{bmatrix} \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix}^{k-1} \\
&= \begin{bmatrix} f_2 & f_1 \\ (y-x-1)f_1 & (y-x-1)f_0 \end{bmatrix} \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix}^{k-2} \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} f_k & f_{k-1} \\ (y-x-1)f_{k-1} & (y-x-1)f_{k-2} \end{bmatrix} \begin{bmatrix} x+2 & 1 \\ y-x-1 & 0 \end{bmatrix}.
\end{aligned}$$

This completes the proof. □

The Honsberger identity for f_n can also be derived using the companion M matrix. The following result illustrates this approach.

Theorem 3.2. *Let f_n present the binomial transform of the bivariate Fibonacci quaternion polynomial sequence. In this case, the Honsberger identity for f_n is*

$$f_{n+k} = (x-y+1)f_n f_{k-1} + f_k f_{n+1}$$

for natural numbers n and k with $n \geq k$.

Proof. We know that

$$M^n = \begin{bmatrix} f_{n+1} & f_n \\ (y-x-1)f_n & (y-x-1)f_{n-1} \end{bmatrix}.$$

Here if we take $n+k$ instead of n in M^n, we have

$$M^{n+k} = \begin{bmatrix} f_{n+k+1} & f_{n+k} \\ (y-x-1)f_{n+k} & (y-x-1)f_{n+k-1} \end{bmatrix}.$$ □

On the other hand, we have

$$M^n M^k = \begin{bmatrix} f_{n+1}f_{k+1} + (y-x-1)f_n f_k & f_{n+1}f_k + (y-x-1)f_n f_{k-1} \\ (y-x-1)f_n f_{k+1} + (y-x-1)^2 f_{n-1}f_k & (y-x-1)f_n f_k + (y-x-1)^2 f_{n-1}f_{k-1} \end{bmatrix}.$$

Now, from the properties of matrices, we obtain what is known in the literature as Honsberger identity

$$f_{n+k} = (x-y+1)f_n f_{k-1} + f_k f_{n+1}.$$

Corollary 3.1. *When we take $n+1$ instead of k, we obtain*

$$f_{2n+1} = (x-y+1)f_n^2 + f_{n+1}^2.$$

from the Honsberger identity.

4 Conclusion

The aim of this research is to focus on the binomial transform of quaternion polynomials, which has not been studied before in the literature. In the most general sense, what is intended to be given is to examine a binomial transform, but the technique used for this is realized with the help of the two-variable Fibonacci quaternion polynomial. The literature review about the sequence was given in the first part and the binomial transform of the sequence is represented in the second part. Again in the second part, the Binet formula of the relevant sequence, the generating function, identities recognized in the literature (such as Cassini) and some binomial identities are derived. In the last section, for the relevant sequence matrix form is given. Our next article will focus on different types of binomial transforms—such as rising, falling transforms—for the relevant sequences and their fundamental properties.

References

[1] Catalani, M. (2004). *Generalized bivariate Fibonacci polynomials*. Preprint. Available online at: `https://arxiv.org/abs/math/0211366`.

[2] Catalani, M. (2004). *Identities for Fibonacci and Lucas polynomials derived from a book of Gould*. Preprint. Available online at: `https://arxiv.org/abs/math/0407105`.

[3] Catalani, M. (2004). *Some formulae for bivariate Fibonacci and Lucas polynomials*. Preprint. Available online at: `https://arxiv.org/abs/math/0406323`.

[4] Chen, K.-W. (2007) Identities from the binomial transform. *Journal of Number Theory*, 124(1), 142–150.

[5] Falcón, S., & Plaza, Á. (2009). Binomial transforms of the k-Fibonacci sequences. *International Journal of Nonlinear Sciences and Numerical Simulation*, 10(11–12), 1527–1538.

[6] Gould, H. W. (1981). A history of the Fibonacci Q-matrix and a higher dimensional problem. *The Fibonacci Quarterly*, 19(3), 250–257.

[7] Gould, H. W. (1990). Series transformations for finding recurrences for sequences. *The Fibonacci Quarterly*, 28(2), 166-171.

[8] Guan, H., Khan, W. A., & Kızılateş, C. (2023). On generalized bivariate (p, q)-Bernoulli–Fibonacci polynomials and generalized bivariate (p, q)-Bernoulli–Lucas polynomials. *Symmetry*, 15(4), Article ID 943.

[9] Kaplan, F., & Özkoç Öztürk, A. (2022). On the binomial transforms of the Horadam quaternion sequences. *Mathematical Methods in the Applied Sciences*, 45(18), 12009–12022.

[10] Koçer, E. G., & Tuncez, S. (2016). Bivariate Fibonacci and Lucas like polynomials. *Gazi University Journal of Science*, 29(1), 109–113.

[11] Özkoç Öztürk, A., & Gündüz, E. (2022). Binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion. *Universal Journal of Mathematics and Applications*, 5(4), 145–155.

[12] Özkoç Öztürk, A., & Kaplan, F. (2020). Some properties of bivariate Fibonacci and Lucas quaternions polynomials. *Facta Universitatis, Series: Mathematics and Informatics*, 35(1), 73–87.

[13] Panwar, Y. K., Gupta, V. K., & Bhandari, J. (2020). Generalized identities of bivariate Fibonacci and bivariate Lucas polynomials. *Journal of Amasya University the Institute of Sciences and Technology*, 1(2), 146–154.

[14] Prodinger, H. (1994). Some information about the binomial transform. *The Fibonacci Quarterly*, 32(5), 412–415.

[15] Ribenboim, P. (2000). *My Numbers, My Friends: Popular Lectures on Number Theory*. Springer-Verlag, New York, Inc.