Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 3, 535–546
DOI: 10.7546/nntdm.2025.31.3.535-546

On an analytical study of the generalized Fibonacci polynomials

Leandro Rocha [1], **Gabriel F. Pinheiro** [2] **and Elen V. P. Spreafico** [3]

[1] Faculty of Computing, Federal University of Mato Grosso do Sul - UFMS
Cidade Universitária, ave. Costa e Silva - Pioneiros - MS, Brazil
e-mail: `l.rocha@ufms.br`

[2] Institute of Mathematics, Statistics and Scientific Computing, State University of Campinas
Sérgio Buarque de Holanda St., 651, Cidade Universitária - Campinas - SP, Brazil
e-mail: `g232247@dac.unicamp.br`

[3] Institute of Mathematics, Federal University of Mato Grosso do Sul - UFMS
Cidade Universitária, ave. Costa e Silva - Pioneiros - MS, Brazil
e-mail: `elen.spreafico@ufms.br`

Received: 8 October 2024 **Revised:** 1 August 2025
Accepted: 4 August 2025 **Online First:** 16 August 2025

Abstract: In this work, we presented an analytical study of the generalized Fibonacci polynomial of order $r \geq 2$, by using properties of the fundamental system associated with the generalized Fibonacci polynomial. We established the generating function and provided the asymptotic behavior for each system sequence. Moreover, the properties are extended to any generalized Fibonacci type, given the general case's generating function and asymptotic behavior.

Keywords: Generalized Fibonacci polynomials, Fundamental system, Analytic representations, Asymptotic behavior, Generating functions.
2020 Mathematics Subject Classification: 11B39, 11B75, 11C20, 65Q10, 65Q30.

1 Introduction

The sequence of Fibonacci polynomials $(F_n(x))_{n\geq 0}$ was introduced in [4], and is defined by the initial conditions $F_0(x) = 0$, $F_1(x) = 1$, and the recurrence relation $F_{n+1}(x) = xF_n(x)+F_{n-1}(x)$ for $n \geq 1$. In the literature, there are generalizations of this well-known sequence (see, for example, [1, 5–8, 10]), as well as applications involving these generalizations.

One important generalization is given in [10], where the authors defined the generalized Fibonacci polynomial of order $r \geq 2$, and study this sequence by using the introduced fundamental system associated with the generalized Fibonacci polynomial of order $r \geq 2$.

The generalized Fibonacci polynomials of order $r \geq 2$, with initial conditions $F_0(x), F_1(x), \ldots, F_{r-1}(x)$, is defined by the following recurrence relation:

$$F_n(x) = xF_{n-1}(x) + \sum_{i=1}^{r-1} F_{n-i-1}(x),\ \forall n \geq r. \tag{1}$$

Thus, the generalized family of Fibonacci polynomials is defined by

$$F_r = \left\{ \left(F_n^{(s)}(x)\right)_{n\geq 0}, 1 \leq s \leq r \right\}, \tag{2}$$

where $F_n^{(s)}(x) = xF_{n-1}^{(s)}(x)+\sum_{i=1}^{r-1} F_{n-i-1}^{(s)}(x)$, with initial conditions $F_n^{(s)}(x)$ for $n = 0, 1, 2, \ldots, r-1$ given by $F_{s-1}^{(s)}(x) = 1$ and $F_n^{(s)}(x) = 0$ for $0 \leq n \neq s-1 \leq r-1$.

Let $(F_n(x))_{n\geq 0}$ be a generalized sequence of Fibonacci polynomials defined by the recursive relation (1) and initial conditions $F_0(x) = \alpha_1, \ldots, F_{r-1}(x) = \alpha_r$. It was established in [10] that $F_n(x)$ is given by $F_n(x) = \alpha_1 F_n^{(1)}(x) + \cdots + \alpha_r F_n^{(r)}(x)$, for all $n \geq 0$. Moreover, the family of Fibonacci polynomials F_r is linearly independent and form a fundamental system for the real vector space of solutions of Equation (1). There are several properties involving this fundamental system presented in [10]. Here we highlight some of these properties and new properties that form the basis for presenting generating functions involving this system and an analytical study. Moreover, the study of the asymptotic behavior of each sequence of the fundamental system permits us to establish the asymptotic behavior of the generalized Fibonacci polynomial of order $r \geq 2$.

The article is organized as follows. In Section 2, we give some preliminares results about the fundamental system of the Fibonacci polynomials. In Section 3, we establish the generating functions of the fundamental system and extend the results to the generalized Fibonacci polynomials of order $r \geq 2$. In Section 4, we stated an analytical study of the generalized Fibonacci polynomials of order $r \geq 2$ of type (1) in terms of the roots of characteristic polynomial associated with the recurrence relation of sequence of type (1). Moreover, we established an asymptotic behavior of the fundamental system and any sequence of type (1). Finally, some conclusions and remarks are stated.

2 The fundamental system of Fibonacci polynomials

In this section, we introduces the fundamental system of Fibonacci polynomials. To do this, we define the Casoratian matrix associated with the generalized family of Fibonacci polynomials (2) and, as discussed in [12], the determinant of the Casoratian matrix and its relationship with the companion matrix.

Definition 2.1. *The Casoratian matrix of the fundamental system of Fibonacci polynomials F_r is given by the following matrix:*

$$\hat{C}_F(n) = \begin{pmatrix} F_n^{(1)}(x) & F_n^{(2)}(x) & \cdots & F_n^{(r)}(x) \\ F_{n+1}^{(1)}(x) & F_{n+1}^{(2)}(x) & \cdots & F_{n+1}^{(r)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r-1}^{(1)}(x) & F_{n+r-1}^{(2)}(x) & \cdots & F_{n+r-1}^{(r)}(x) \end{pmatrix}.$$

The following lemma can be proven by induction.

Lemma 2.1. *For each $n \geq 0$, we have that $(A_F)^n = J_F \hat{C}_F(n) J_F$, where J_F is the antidiagonal matrix given by $J_F = (c_{ij})_{1\leq i,j,\leq r}$ with $c_{ij} = 1$ for $i + j = r + 1$ and $c_{ij} = 0$, otherwise. The companion matrix associated with the fundamental system of Fibonacci polynomials, A_F, is given by:*

$$A_F = \begin{pmatrix} x & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Note that the determinant of $(A_F)^n$ is equal to $(-1)^{n(r+1)} \neq 0$. Then the set F_r is linearly independent. Moreover, we can prove the following proposition.

Proposition 2.1. *Consider $(F_n(x))_{n\geq 0}$ to be any sequence of polynomials of type* (1)*, then, $F_n(x) = xF_{n-1}(x) + \sum_{i=1}^{r-1} F_{n-i-1}(x)$, with initial conditions $\alpha_1, \dots, \alpha_r$. Then,*

$$F_n(x) = \alpha_1 F_n^{(1)}(x) + \alpha_2 F_n^{(2)}(x) + \cdots + \alpha_r F_n^{(r)}(x). \tag{3}$$

Therefore, F_r is a basis for the real vector space of solutions of Equation (1), and we have the following definition.

Definition 2.2. *The fundamental system of Fibonacci polynomials is given by r copies of* (1)*, represented in compact form:*

$$\begin{cases} F_n^{(s)}(x) = xF_{n-1}^{(s)}(x) + F_{n-2}^{(s)}(x) + \cdots + F_{n-r}^{(s)}(x), & n \geq r, \\ F_n^{(s)}(x) = \delta_{s-1,n}, & 0 \leq n \leq r-1, \end{cases}$$

where $\delta_{s-1,n}$ is a Kronecker delta defined by 1 *if $n = s - 1$, and* 0 *otherwise.*

First, consider some properties of the fundamental system of the generalized Fibonacci polynomial of order $r \geq 2$. Let F_r be the fundamental system of Fibonacci polynomials. Then, in Proposition 1.6 [10], for each $n \geq 1$, it holds that $F_n^{(1)}(x) = F_{n-1}^{(r)}(x)$. In addition, for all $n \geq j$ and $1 \leq j \leq r - 1$, we have

$$F_n^{(j)}(x) = \sum_{i=1}^{j} F_{n-i}^{(r)}(x). \tag{4}$$

Thus, we can establish a relationship between $F_n(x)$, that is, any sequence of the type (1) and $F_n^{(r)}(x)$ as follows.

Proposition 2.2. *Let* $(F_n(x))_{n\geq 0}$ *be a sequence of generalized Fibonacci polynomials of type* (1), *with initial conditions* $\alpha_1, \alpha_2, \ldots, \alpha_r$. *Then, for all* $n \geq r$, *holds*

$$F_n(x) = \sum_{i=1}^{r-1} \left(\sum_{j=i}^{r-1} \alpha_j \right) F_{n-i}^{(r)}(x) + \alpha_r F_n^{(r)}(x).$$

3 The generating functions

In this section, we will provide the generating functions for each sequence of the fundamental system, and by using properties of this family of sequences we will derive a generating function for a generalized Fibonacci polynomial of order $r \geq 2$ of type (1).

In what follows, the generating function for each sequence of the fundamental system is given.

Theorem 3.1. *The generating function* $F^{(s)}(t,x)$ *for the generalized Fibonacci polynomials is given by*

$$(1 - xt - \sum_{n=2}^{r} t^n) F^{(s)}(t,x) = \sum_{n=0}^{r-1} F_n^{(s)}(x) t^n - xt \sum_{n=0}^{r-2} F_n^{(s)}(x) t^n - t^2 \sum_{n=0}^{r-3} F_n^{(s)}(x) t^n - \cdots - t^{r-1} F_0^{(s)}(x)$$

Proof. Consider the sequence $(F_n(x))_{n\geq 0}$ defined by the recurrence relation:

$$F_n^{(s)}(x) = x F_{n-1}^{(s)}(x) + F_{n-2}^{(s)}(x) + \cdots + F_{n-r}^{(s)}(x), \tag{5}$$

Denote $F^{(s)}(t,x) = \sum_{n=0}^{\infty} F_n^{(s)}(x) t^n$. We multiply both sides of the relation (5) and sum over all values of $n \geq r$, getting

$$\sum_{n=r}^{\infty} F_n^{(s)}(x) t^n = \sum_{n=r}^{\infty} (x F_{n-1}^{(s)}(x) + F_{n-2}^{(s)}(x) + \cdots + F_{n-r}^{(s)}(x)) t^n.$$

That is,

$$\sum_{n=r}^{\infty} F_n^{(s)}(x) t^n = x \sum_{n=r}^{\infty} F_{n-1}^{(s)}(x) t^n + \sum_{n=r}^{\infty} F_{n-2}^{(s)}(x) t^n + \cdots + \sum_{n=r}^{\infty} F_{n-r}^{(s)}(x) t^n.$$

Thus, adjusting the indices and isolating $F(t,x)$, we obtain

$$(1 - xt - \sum_{n=2}^{r} t^n) F^{(s)}(t,x) = \sum_{n=0}^{r-1} F_n^{(s)}(x) t^n - xt \sum_{n=0}^{r-2} F_n^{(s)}(x) t^n - t^2 \sum_{n=0}^{r-3} F_n^{(s)}(x) t^n - \cdots - t^{r-1} F_0^{(s)}(x). \qquad \square$$

Note that, according to Equation (4), we have that $F_n^{(j)}(x)$ can be expressed in terms of $F_n^{(r)}(x)$. Thus, we can take $s = r$ in the previous proposition and considering that $F_n^{(s)}(x) = \delta_{s-1,n}$, for $0 \leq n \leq r-1$, we obtain a generating function for the generalized Fibonacci polynomials $F_n^{(r)}(x)$, which will be used in the following results.

Corollary 3.1. *The generating function for* $(F_n^{(r)}(x))_{n\geq 0}$ *is*

$$F^{(r)}(t,x) = \frac{t^{r-1}}{1 - xt - \sum_{n=2}^{r} t^n}. \tag{6}$$

Given $(F_n(x))_{n\geq 0}$ a sequence of generalized Fibonacci polynomials of type (1), since $F_n(x)$ is given in terms of sequences of the fundamental system, then we obtain the following proposition.

Proposition 3.1. *Let* $(F_n(x))_{n\geq 0}$ *be a sequence of generalized Fibonacci polynomials of type* (1)*, with initial conditions* $\alpha_1, \alpha_2, \ldots, \alpha_r$*. Then, for every* $n \geq 0$*, the generating function is defined by*

$$F(y,x) = \sum_{n=1}^{r} \alpha_n y^n F^{(n)}(y,x).$$

Proof. Consider the sequence $(F_n(x))_{n\geq 0}$ defined by the recurrence relation:

$$F_n(x) = \alpha_1 F_n^{(1)}(x) + \alpha_2 F_n^{(2)}(x) + \cdots + \alpha_r F_n^{(r)}(x), \tag{7}$$

Multiply both sides of the relation (7) by y^n and sum over all values of $n \geq 0$:

$$\sum_{n=0}^{\infty} F_n(x) y^n = \alpha_1 \sum_{n=0}^{\infty} F_n^{(1)}(x) y^n + \alpha_2 \sum_{n=0}^{\infty} F_n^{(2)}(x) y^n + \cdots + \alpha_r \sum_{n=0}^{\infty} F_n^{(r)}(x) y^n.$$

Thus, we can adjust the indices and rewrite the expression in terms of each generating function. We have:

$$F(y,x) = \alpha_1 y F^{(1)}(y,x) + \alpha_2 y^2 F^{(2)}(y,x) + \cdots + \alpha_r y^r F^{(r)}(y,x) \tag{8}$$

$$= \sum_{n=1}^{r} \alpha_n y^n F^{(n)}(y,x). \tag{9}$$

This completes the proof. □

In general, by Proposition 2.2 and Corollary 3.1, we get the generating function for a generalized Fibonacci polynomial of order $r \geq 2$ of type (1) in terms of the generating function of $F^{(r)}(x)$.

Proposition 3.2. *Let* $(F_n(x))_{n\geq 0}$ *be a sequence of generalized Fibonacci polynomials of type* (1)*, with initial conditions* $\alpha_1, \alpha_2, \ldots, \alpha_r$*. Then, for every* $n \geq 0$*, the generating function is defined by*

$$F(y,x) = \left(\sum_{i=1}^{r-1} \Omega_i y^i + \alpha_r\right) F^{(r)}(y,x) + \sum_{i=1}^{r-1} \alpha_i y^{i-1},$$

where $F^{(r)}(y,x)$ *is the generating function of the sequence* $F_n^{(r)}(x)$ *and* $\Omega_i = \left(\sum_{j=i}^{r-1} \alpha_j\right)$.

Proof. Consider the sequence $(F_n(x))_{n\geq 0}$ defined by the recurrence relation:

$$F_n(x) = \sum_{i=1}^{r-1} \left(\sum_{j=i}^{r-1} \alpha_j\right) F_{n-i}^{(r)}(x) + \alpha_r F_n^{(r)}(x). \tag{10}$$

By multiplying both sides of the relation (10) by y^n and summing over all values of $n \geq r-1$, we obtain

$$\sum_{n=r-1}^{\infty} F_n(x)y^n = \sum_{n=r-1}^{\infty} \Omega_1 F_{n-1}^{(r)}(x)y^n + \cdots + \alpha_{r-1}F_{n-r+1}^{(r)}(x)y^n + \alpha_r F_n^{(r)}(x)y^n.$$

and we can rewrite the above expression in terms of $F^{(r)}(y,x)$ and $F(y,x)$:

$$F(y,x) - \sum_{n=0}^{r-2} F_n(x)y^n = \Omega_1 y(F^{(r)}(y,x) - \sum_{n=0}^{r-3} F_n^{(r)}(x)y^n) + \cdots$$
$$+ \alpha_{r-1}y^{r-1}F^{(r)}(y,x) + \alpha_r(F^{(r)}(y,x) - \sum_{n=0}^{r-2} F_n^{(r)}(x)y^n).$$

Removing the null terms, it follows that:

$$F(y,x) = \Omega_1 y F^{(r)}(y,x) + \Omega_2 y^2 F^{(r)}(y,x) + \cdots$$
$$+ \alpha_{r-1}y^{r-1}F^{(r)}(y,x) + \alpha_r F^{(r)}(y,x) + \sum_{n=0}^{r-2} F_n(x)y^n.$$

Therefore,

$$F(y,x) = \left(\sum_{i=1}^{r-1} \Omega_i y^i + \alpha_r\right) F^{(r)}(y,x) + \sum_{i=1}^{r-1} \alpha_i y^{i-1},$$

where $F^{(r)}(y,x)$ is the generating function of the sequence $(F_n^{(r)}(x))_{n\geq 0}$ and $\Omega_i = \left(\sum_{j=i}^{r-1} \alpha_j\right)$. □

4 Analytical study

It is well known that the analytical formula for linear recursive sequences is related to the roots of the associated characteristic polynomial (see, for example, [3,13]). The characteristic polynomial associated to the generalized Fibonacci polynomial $F_n(x) = xF_{n-1}(x) + \sum_{i=1}^{r-1} F_{n-i-1}(x)$ is given by $P(t) = t^r - xt^{r-1} - t^{r-2} - \cdots - t - 1$. It was shown in [10] that the roots of $P(t)$ are simple and that the number of real roots for $x \geq 1$ is given by

$$2^M, \text{ where } M = \begin{cases} 0, & \text{if } \deg(P) \text{ is odd,} \quad \text{where } \deg \text{ is the degree of } P. \\ 1, & \text{if } \deg(P) \text{ is even.} \end{cases}$$

For $r = 2$, $F_0(x) = 0$ and $F_1(x) = 1$, we have the well-known extension of the Binet formula given by: $F_n(x) = \dfrac{\alpha^n + \beta^n}{\alpha + \beta}$, where $\alpha = \dfrac{x + \sqrt{x^2+4}}{2}$ and $\beta = \dfrac{x - \sqrt{x^2+4}}{2}$ are roots of $P(t) = t^2 - xt - 1$, its associated characteristic equation.

Thus, we can state the following proposition.

Proposition 4.1. *Let F_r be the fundamental system of Fibonacci polynomials. Then for $r = 2$ and $n \geq r$, we have: $F_n^{(2)}(x) = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $F_n^{(1)}(x) = \dfrac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}$, where $\alpha = \dfrac{x + \sqrt{x^2+4}}{2}$ and $\beta = \dfrac{x - \sqrt{x^2+4}}{2}$ are roots of $P(t) = t^2 - xt - 1$.*

Consider the following results for the case of $r \geq 3$.

Lemma 4.1 ([10]). *Given the polynomial equation $P(t) = t^r - xt^{r-1} - t^{r-2} - \cdots - t - 1$, the number of real roots for $x \geq 1$ is given by:*

$$2^M, \text{ where } M = \begin{cases} 0, & \text{if } \deg(P) \text{ is odd,} \\ 1, & \text{if } \deg(P) \text{ is even.} \end{cases}$$

Proposition 4.2 ([10]). *Given the polynomial equation $P(t) = t^r - xt^{r-1} - t^{r-2} - \cdots - t - 1$, we have that its roots are simple.*

Before starting the next result, we highlight the following definition, as presented in [9].

Definition 4.1 ([9]). *A matrix $A \in \mathbb{R}^{n\times n}$ is called non-negative if each entry a_{ij} is non-negative, and we denote this by $A \geq 0$. The set of distinct eigenvalues of a matrix A, denoted by $\sigma(A)$, is called the spectrum of A. The spectral radius of a matrix A is defined by $\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$.*

Theorem 4.1 (Perron's Theorem [9]). *If $A \geq 0$, irreducible and primitive, then the following statements are true.*

- $\rho(A) > 0$;
- $\rho(A) \in \sigma(A)$;
- $\rho(A)$ *is the unique eigenvalue on the spectral radius of* A.

By applying Perron's Theorem for the characteristic polynomial $P(t) = \det(A_F - tI_r)$, where A_F is the companion matrix associated, we obtain the next result.

Lemma 4.2. *Given the polynomial equation $P(t) = t^r - xt^{r-1} - t^{r-2} - \cdots - t - 1$ with $x \geq 1$, then $x_1 > |x_j|$ for $j = 2, \ldots, r$, where x_1 is the unique positive real root of $P(t)$.*

Proof. Note that A_F has only real and non-negative entries, that is, $A_F \geq 0$ and is in fact primitive, using the Frobenius test, taking $n \geq r$. Furthermore, by construction, A_F, for $x > 0$, are irreducible. Thus, by Perron's Theorem, it follows that $\rho(A_F) \in \sigma(A_F)$, $\rho(A_F) > 0$ and $\rho(A_F)$ is the unique eigenvalue in the spectral radius of A_F, that is, $x_1 > |x_j|$, for $j = 2, \ldots, r$, where x_1 is the unique positive real root of $P(t)$. □

Now, under the previous results, we will study the asymptotic behavior of the sequences of the fundamental system generalized Fibonacci polynomials.

Proposition 4.3. *Let F_r be the fundamental system of generalized Fibonacci polynomials. Then for $r = 2$ and $n \geq r$, we have*

$$\lim_{n\to\infty} \frac{F_n^{(1)}(x)}{F_{n-1}^{(1)}(x)} = \lim_{n\to\infty} \frac{F_n^{(2)}(x)}{F_{n-1}^{(2)}(x)}.$$

Proof. For $r = 2$, the associated characteristic polynomial is $P(t) = t^2 - xt - 1 = 0$, with roots $z = \frac{x + \sqrt{x^2+4}}{2}$ and $y = \frac{x - \sqrt{x^2+4}}{2}$. Note that for any fixed $x \geq 1 \in \mathbb{R}$, $\frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)}$ is a positive real number, and

$$F_n^{(j)}(x) = A_j \left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + \ B_j \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n$$

is a solution of $F_n^{(j)}(x) - xF_{n-1}^{(j)}(x) - F_{n-2}^{(j)}(x) = 0$, with constants A_j and B_j.

Thus, the expression $\frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)}$ becomes:

$$\frac{Az^n + By^n}{Az^{n-1} + By^{n-1}}.$$

This can be rewritten as

$$\frac{Az + By\,(y/z)^{n-1}}{A + By\,(y/z)^{n-1}}.$$

Noting that $\left|\frac{y}{z}\right| < 1$, for any $x \geq 1$, $\lim_{n\to\infty} (y/z)^{n-1} = 0$. Therefore,

$$\lim_{n\to\infty} \frac{Az + By\left(\frac{y}{z}\right)^{n-1}}{A + By\left(\frac{y}{z}\right)^{n-1}} = \frac{Az}{A} = z = \frac{x+\sqrt{x^2+4}}{2}.$$

Thus, the limit of the expression is $\frac{x+\sqrt{x^2+4}}{2}$. □

Note that for $x = 1$, we have $\frac{x+\sqrt{x^2+4}}{2} = \Phi$, where Φ is the well-known golden ratio. For $x > 1$, we have $\frac{x+\sqrt{x^2+4}}{2} < x+1$. Therefore, we get:

$$\Phi \leq \lim_{n\to\infty} \frac{F_n^{(1)}(x)}{F_{n-1}^{(1)}(x)} = \lim_{n\to\infty} \frac{F_n^{(2)}(x)}{F_{n-1}^{(2)}(x)} < x+1.$$

In general, for $r > 2$, we have the following proposition.

Proposition 4.4. *Let F_r be the fundamental system of generalized Fibonacci polynomials. Then for each $r > 2$ and $n \geq r$, we have*

$$\lim_{n\to\infty} \frac{F_n^{(1)}(x)}{F_{n-1}^{(1)}(x)} = \cdots = \lim_{n\to\infty} \frac{F_n^{(r)}(x)}{F_{n-1}^{(r)}(x)}.$$

Proof. Similarly, for $r > 2$, the associated characteristic polynomial is

$$P(t) = t^r - xt^{r-1} - t^{r-2} - \cdots - t - 1.$$

Let $x_1, \ldots, x_r$ be the roots of $P(t)$. Then $F_n^{(j)}(x) = A_{j1}x_1^n + \cdots + A_{jr}x_r^n$ is a solution of $F_n^{(j)}(x) = xF_{n-1}^{(j)}(x) + \sum_{i=1}^{r-1} F_{n-i-1}^{(j)}(x)$, with constants $A_{j1}, \ldots, A_{jr}$.

Thus, the expression $\frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)}$ becomes

$$\frac{A_{j1}x_1^n + \cdots + A_{jr}x_r^n}{A_{j1}x_1^{n-1} + \cdots + A_{jr}x_r^{n-1}}.$$

Now, we can divide both the numerator and the denominator by x_1^{n-1}, where x_1 is the unique positive real root:

$$\frac{A_{j1}x_1 + \cdots + A_{jr}x_r \left(\frac{x_r}{x_1}\right)^{n-1}}{A_{j1} + \cdots + A_{jr}\left(\frac{x_r}{x_1}\right)^{n-1}}.$$

Observing that $\left|\frac{x_j}{x_1}\right| < 1$ for $j = 2, \ldots, r$, by Lemma 4.2, $\lim_{n\to\infty}\left(\frac{x_j}{x_1}\right)^{n-1} = 0$. Thus, as n increases, we get:

$$\lim_{n\to\infty} \frac{A_{j1}x_1 + \cdots + A_{jr}x_r \left(\frac{x_r}{x_1}\right)^{n-1}}{A_{j1} + \cdots + A_{jr}\left(\frac{x_r}{x_1}\right)^{n-1}} = \frac{A_{j1}x_1}{A_{j1}} = x_1. \qquad \square$$

Thus, we have the following theorem.

Theorem 4.2. *Let F_r be the fundamental system of Fibonacci polynomials. Then for each fixed $r \geq 2$, $1 \leq j \leq r$, $x \geq 1$ and $n > r$, we have*

$$\Phi \leq \lim_{n\to\infty} \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} < x + 1.$$

Proof. Let us now prove the inequality $\Phi < x_1$. Assume that $t = x_1$ and $r > 2$. Then we have: $t^r - xt^{r-1} - t^{r-2} - \cdots - t - 1 = 0$, and since the number t is positive, we have:

$$t^r - xt^{r-1} - t^{r-2} > 0.$$

Dividing both sides by t^{r-2}, we obtain:

$$t^2 - xt - 1 > 0.$$

This inequality has the solution $t > \frac{x + \sqrt{x^2+4}}{2}$. Therefore, we conclude that $x_1 > \frac{x + \sqrt{x^2+4}}{2}$, therefore $x_1 > \Phi$ and the equality occurs in the case $r = 2$.

Then, it remains only to prove the inequality $x_1 < x + 1$.

Here we will use Lagrange's Theorem: If t is a positive root of a polynomial with a leading coefficient 1, then $t < 1 + |L|^1/p$, where L is the first coefficient with negative sign in the polynomial, and p is the subsequent number of the position of this coefficient after the leading one. For our particular polynomial, we have $L = -x$ and $p = 1$. Then we obtain $t = x_1 < 1 + x$. $\square$

Corollary 4.1. *Let $\mathcal{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ be the fundamental system of Fibonacci numbers. Then for each $r \geq 2$, $1 \leq j \leq r$, and $n > r$, we have*

$$\Phi \leq \lim_{n\to\infty} \frac{F_n^{(j)}}{F_{n-1}^{(j)}} \leq 2.$$

Proposition 4.5. *Let F_r be the fundamental system of generalized Fibonacci polynomials. Then for each $r \geq 2$, $1 \leq j \leq r$, $x \geq 1$ and $n > r$, we have*

$$\lim_{x\to\infty}\left(\frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x\right) = 0.$$

Proof. Indeed,

$$\frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x = \frac{F_n^{(j)}(x) - xF_{n-1}^{(j)}(x)}{F_{n-1}^{(j)}(x)} = \frac{\sum_{i=1}^{r-1} F_{n-i-2}^{(j)}(x)}{xF_{n-2}^{(j)}(x) + \sum_{i=1}^{r-1} F_{n-i-2}^{(j)}(x)},$$

so, as the degree of $F_{n-2}^{(j)}(x)$ is greater than the degree of $F_{n-3}^{(j)}(x), \ldots, F_{n-r-1}^{(j)}(x)$, it follows that

$$\lim_{x\to\infty} \left(\frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x \right) = 0. \qquad \square$$

As seen in Figure 1, for $r = 2$, as we assign and increase the values of x, $\frac{F_n^{(1)}(x)}{F_{n-1}^{(1)}(x)} - x$ tends to zero. For this case, we fix $n = 10$.

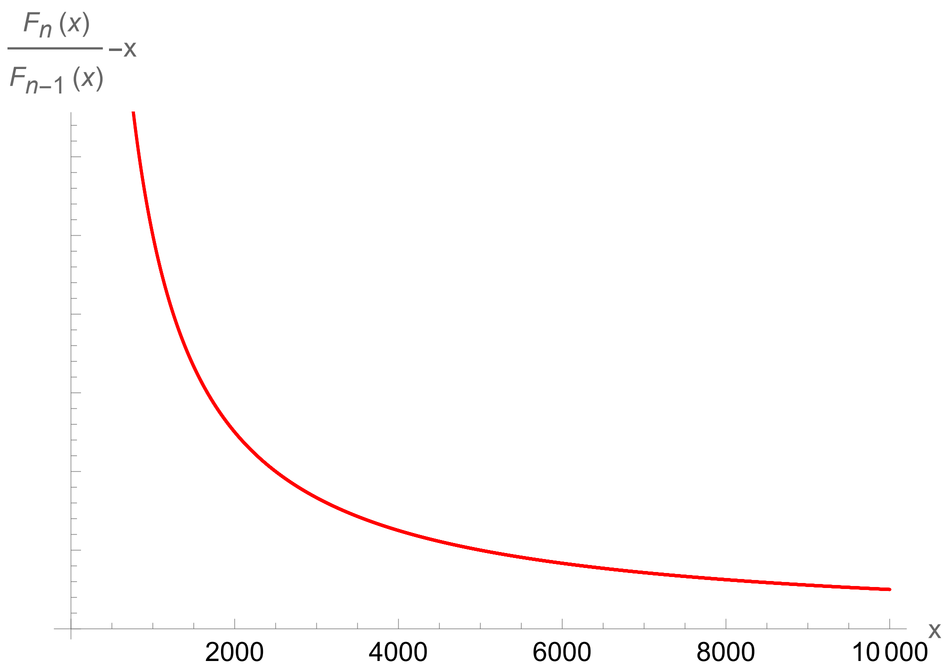

Figure 1. Convergence of Fibonacci polynomials

Note that by Proposition 4.4 given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\left| \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x_1 \right| < \frac{\varepsilon}{2}, \quad \text{for all } n > N_1,$$

and by Proposition 4.5, given $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\left| \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x \right| < \frac{\varepsilon}{2}, \quad \text{for all } x > N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n > N$ and $x > N$, we have

$$|x_1 - x| = \left| x_1 - \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} + \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x \right| \leq \left| x_1 - \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} \right| + \left| \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that as n and x increase, $|x_1 - x|$ decreases, meaning that x approaches the root of $P(t)$. Therefore, we have the following result.

Proposition 4.6. *Let F_r be the fundamental system of generalized Fibonacci polynomials. Then for $r \geq 2$, $1 \leq j \leq r$, $x \geq 1$ and $n > r$, we have*

$$\lim_{\substack{x\to\infty \\ n\to\infty}} \left| \frac{F_n^{(j)}(x)}{F_{n-1}^{(j)}(x)} - x \right| = 0.$$

In general, for any $(F_n(x))_{n\geq 0}$ of type (1) we have the following proposition.

Proposition 4.7. *Let $(F_n(x))_{n\geq 0}$ be a sequence of generalized Fibonacci polynomials of type* (1)*, with initial conditions $\alpha_1, \alpha_2, \ldots, \alpha_r$, then for all $n \geq 0$. Then for each fixed r, $j \geq 1$ and $n > r+1$, we have*

$$\lim_{x\to\infty} \left(\frac{F_n(x)}{F_{n-1}(x)} - x \right) = 0.$$

Proof. Indeed,

$$\begin{aligned} \frac{F_n(x)}{F_{n-1}(x)} - x &= \frac{\alpha_1 F_n^{(1)}(x) + \cdots + \alpha_r F_n^{(r)}(x)}{\alpha_1 F_{n-1}^{(1)}(x) + \cdots + \alpha_r F_{n-1}^{(r)}(x)} - x \\ &= \frac{\alpha_1(F_n^{(1)}(x) - xF_{n-1}^{(1)}(x)) + \cdots + \alpha_r(F_n^{(r)} - xF_{n-1}^{(r)}(x))(x)}{\alpha_1 F_{n-1}^{(1)}(x) + \cdots + \alpha_r F_{n-1}^{(r)}(x)} \\ &= \frac{\sum_{j=1}^{r}\sum_{i=1}^{r-1} \alpha_j F_{n-i-1}^{(j)}(x)}{\alpha_1 F_{n-1}^{(1)}(x) + \cdots + \alpha_r F_{n-1}^{(r)}(x)}, \end{aligned}$$

so, as the degree of $F_{n-1}^{(j)}(x)$ is greater than the degree of $F_{n-2}^{(j)}(x), \ldots, F_{n-r}^{(j)}(x)$, for $j = 1, \ldots, r$ it follows that

$$\lim_{x\to\infty} \left(\frac{F_n(x)}{F_{n-1}(x)} - x \right) = 0.$$ □

5 Conclusion

In this study, we explored additional properties of the generalized Fibonacci fundamental system, applying it to generalized Fibonacci polynomials of type (1), that defines generalized Fibonacci polynomials. We presented an analytical study of the fundamental system of the generalized Fibonacci polynomial of order $r \geq 2$. It established the generating function and provided the asymptotic behavior for each sequence of the system. Moreover, the properties are extended to any generalized Fibonacci of type (1). It seems to us that the results presented here are new in the literature.

Acknowledgements

The first author expresses gratitude to the Federal University of Mato Grosso do Sul (UFMS/MEC) and the State University of Campinas (UNICAMP/MEC) for their invaluable support. The second author sincerely thanks the Brazilian National Council for Scientific and Technological Development – CNPq – Brazil and the State University of Campinas – UNICAMP/MEC – Brazil for their valuable support. The third author expresses her sincere thanks to the Brazilian National

Council for Scientific and Technological Development – CNPq – Brazil and the Federal University of Mato Grosso do Sul – UFMS/MEC – Brazil for their valuable support.

References

[1] Alves, F. R. V., & Catarino, P. M. M. C. (2017). A classe dos polinômios bivariados de Fibonacci (PBF): Elementos recentes sobre a evolução de um modelo. *Revista Thema*, 14(1), 112–136.

[2] Ben Taher, R., & Rachidi, M. (2003). On the matrix powers and exponential by the r-generalized Fibonacci sequences methods: The companion matrix case. *Linear Algebra and Its Applications*, 370, 341–353.

[3] Ben Taher, R., & Rachidi, M. (2016). Solving some generalized Vandermonde systems and inverse of their associate matrices via new approaches for the Binet formula. *Applied Mathematics and Computation*, 290, 267–280.

[4] Falcón, S., & Plaza, Á. (2007). The k-Fibonacci sequence and the Pascal 2-triangle. *Chaos, Solutions & Fractals*, 33(1), 38–49.

[5] Flórez, R., McAnally, N., & Mukherjee, A. (2017). *Identities for the generalized Fibonacci polynomial.* ArXiv. Available online at: `https://arxiv.org/abs/1702.01855`.

[6] Hoggatt, V. E. Jr., & Long, C. T. (1974). Divisibility properties of generalized Fibonacci polynomials. *The Fibonacci Quarterly*, 12(2), 113–120.

[7] Koç, A. B., Çakmak, M., Kurnaz, A., & Uslu, K. (2013). A new Fibonacci type collocation procedure for boundary value problems. *Advances in Difference Equations*, 2013, Article 262.

[8] Kurt, A., Yalçınbaş, S., & Sezer, M. (2013). Fibonacci collocation method for solving high-order linear Fredholm integro-differential-difference equations. *International Journal of Mathematics and Mathematical Sciences*, 2013, Article ID 486013.

[9] Meyer, C. D. (2023). *Matrix Analysis and Applied Linear Algebra* (2nd ed.). Other Titles in Applied Mathematics, Vol. 188. Society for Industrial and Applied Mathematics, Philadelphia.

[10] Rocha, L., & Spreafico, E.V.P. (2025). On generalized Fibonacci polynomials. *International Journal of Advances in Applied Mathematics and Mechanics* (Submitted).

[11] Santos, J. P. O., Mello, M. P., & Murari, I. T. C. (2007). *Introdução à Análise Combinatória* (4th ed.). Editora Ciência Moderna, Rio de Janeiro.

[12] Spreafico, E. V. P., & Rachidi, M. (2019). Fibonacci fundamental system and generalized Cassini identity. *The Fibonacci Quarterly*, 57(2), 155–167.

[13] Stanley, R. P. (1999). *Enumerative Combinatorics: Volume 2*. Cambridge Studies in Advanced Mathematics, Vol. 62. Cambridge University Press, Cambridge.