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Gaps of size 2, 4, and (conditionally) 6 between successive odd composite numbers occur infinitely often

Joel E. Cohen [1,2,3] **and Dexter Senft** [4]

[1] Laboratory of Populations, The Rockefeller University
1230 York Avenue, Box 20, New York, NY 10065, USA
e-mail: `cohen@rockefeller.edu`

[2] Department of Statistics, Columbia University, New York, NY

[3] Department of Statistics, University of Chicago, Chicago, IL

[4] Asymptotic Systems
Saratoga Springs, NY, USA
e-mail: `dexter.senft@gmail.com`

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Abstract: The infinite sequence of gaps (first differences) between successive odd composite numbers contains only the numbers 2, 4, and 6. We prove that, for any natural number k, the sequence of gaps contains infinitely many k-tuplets of *consecutive* gaps all equal to 2. Infinitely many gaps equal 4. The sequence of gaps includes infinitely many gap pairs $(4, 4)$ if the sequence of positive primes has infinitely many pairs of successive primes that differ by 4 (cousin primes), which is unproved but holds under a conjecture of Hardy and Littlewood. Gap triplets $(4, 4, 4)$ never occur. Infinitely many gaps equal 6 if and only if there are infinitely many twin primes. Moreover, gap pairs $(6, 6)$ occur infinitely often if other conjectures of Hardy and Littlewood are true. Six of the 27 potential triplets of values of gaps between successive odd composite numbers never occur: $(4, 4, 4)$, $(6, 6, 6)$, $(6, 4, 4)$, $(4, 4, 6)$, $(6, 2, 6)$, and $(6, 4, 6)$.

Keywords: Composite number, Cousin primes, Gaps, Odd number, Odd composite number, Prime number, Twin primes.
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1 Introduction

One enduring attraction of number theory is that even its humblest objects, the pebbles that lie everywhere at our feet, are loaded with gold. Here we examine the infinite sequence of gaps (first differences) between successive odd composite numbers. We find surprises and unanswered questions.

The natural numbers $\mathbb{N} := \{1, 2, 3, 4, \ldots\}$ (sequence A000027 in the OEIS [5]) may be partitioned into the singleton set $\{1\}$, the prime numbers $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$ (sequence A000040 in the OEIS [5]), the even composite numbers $\mathbb{E} = \{4, 6, 8, 10, \ldots\}$ (a subsequence of sequence A005843 in the OEIS [5]), and the odd composite numbers $\mathbb{O} = \{o_1 = 9, o_2 = 15, o_3 = 21, o_4 = 25, \ldots\}$ (sequence A071904 in the OEIS [5]). The long-studied gaps between successive primes (sequence A001223 in the OEIS [5]) remain mysterious in many respects. For example, no one can prove whether there are infinitely many twin primes, i.e., gaps equal to 2 between successive primes [3] or infinitely many cousin primes, i.e., gaps equal to 4 between successive primes [3, 8–10]. Wolf [7, p. 336] first proposed calling a pair of primes with gap 4 "cousin" primes.

The gaps between successive odd composite numbers (sequence A164510 in the OEIS [5]) are less studied than the gaps between successive primes. Denote the gaps (first differences) between successive odd composite numbers by $g_n := o_{n+1} - o_n$ for all $n \in \mathbb{N}$. Let the infinite sequence $\mathbb{G} := (g_1 = 6, g_2 = 6, g_3 = 4, g_4 = 2, g_5 = 6, g_6 = 2, g_7 = 4, g_8 = 6, \ldots)$ contain the gaps in order of occurrence. It has been proved (OEIS A164510) and is easy to see that $g_n \in \{2, 4, 6\}$ for all $g_n \in \mathbb{G}$. For to have a gap greater than or equal to 8 would require a sequence of 3 or more primes $p, p+2, p+4, \ldots$ preceded and followed by odd composite numbers; but the only sequence of 3 primes of the form $p, p+2, p+4$ is $3, 5, 7$ and the preceding odd number 1 is not composite; so, as claimed, $g_n \in \{2, 4, 6\}$ for all $g_n \in \mathbb{G}$.

Here we analyze whether gaps of size $g_n = 2,\ 4,\ 6$ occur finitely or infinitely often in $\mathbb{G}$ and whether ordered pairs and ordered triplets of these gaps occur finitely often, infinitely often, or never in $\mathbb{G}$.

2 Preliminaries

If $f(x)$ and $g(x)$ are real-valued functions of real x and $g(x) > 0$ for all x sufficiently large, define $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to \infty$. If $f(x) \sim g(x)$, we say that $f(x)$ and $g(x)$ are asymptotically equivalent. Define the floor function $\lfloor x \rfloor$ of any positive real x to be the largest nonnegative integer less than or equal to x.

If $\mathbb{A} := (a_1, a_2, \ldots)$ is any sequence of positive integers and $x > 0$, the counting function $\#(\mathbb{A}, x)$ of $\mathbb{A}$ at x is the number of elements of $\mathbb{A}$ less than or equal to x. By tradition, the counting function of the sequence of positive prime numbers is written $\pi(x)$ and the number of positive primes $p \le x$ such that also $p+2$ is prime is written $\pi_2(x)$, the twin prime counting function. The prime number theorem (PNT) states:

$$\pi(x) \sim \frac{x}{\log x},\ x \to \infty. \tag{1}$$

As noted, $\mathbb{N}$ is the disjoint union of these subsets:

$$\mathbb{N} = \{1\} \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{O}. \tag{2}$$

For $n \in \mathbb{N}$, we have $\#(\mathbb{E}, n) = \lfloor n/2 \rfloor - 1$. We subtract 1 from $\lfloor n/2 \rfloor$ because $2 \in \mathbb{P}$, not $2 \in \mathbb{E}$. Counting the elements up to and including n in each subset of $\mathbb{N}$ gives

$$n = 1 + \pi(n) + (\lfloor n/2 \rfloor - 1) + \#(\mathbb{O}, n). \tag{3}$$

Hence

$$n - \lfloor n/2 \rfloor = \pi(n) + \#(\mathbb{O}, n) \tag{4}$$

and $n - \lfloor n/2 \rfloor \sim n/2$. By the PNT, $\pi(n)/(n/2) \to 0$ as $n \to \infty$. Thus

$$\#(\mathbb{O}, n) \sim n/2. \tag{5}$$

Hence the sequence $\mathbb{O}$ of odd composite numbers is infinite, and therefore the sequence $\mathbb{G}$ of the gaps (first differences) $g_n := o_{n+1} - o_n$ between successive odd composite numbers is also infinite.

3 Gaps equal to 2

Theorem 3.1. *For any natural number k, the sequence $\mathbb{G}$ of gaps between consecutive odd composite numbers contains infinitely many k-tuplets of* consecutive *gaps all equal to* 2*.*

Proof 1. According to Hardy and Wright [2, Theorem 5, p. 5], there are blocks of consecutive composite numbers whose length exceeds any given number N. Therefore there are infinitely many k-tuplets of consecutive gaps all equal to 2. □

Proof 2 by explicit construction. Fix $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, the $k+1$ numbers

$$o(j,n) := n \cdot 2 \left(\prod_{h=1}^{k+1} \left(3 + 2 \cdot (h-1)\right) \right) + 3 + 2 \cdot (j-1), \qquad j = 1, 2, \ldots, k+1, \tag{6}$$

are odd, composite (because $o(j,n)$ is divisible by $3 + 2 \cdot (j-1)$), and consecutive elements of the sequence $\mathbb{O}$ of odd composite numbers. The only term on the right that depends on j is the last, $2 \cdot (j-1)$. Letting n run through $\mathbb{N}$ yields infinitely many $(k+1)$-tuples of consecutive odd composite numbers. For $j = 1, \ldots, k$,

$$o(j+1,n) - o(j,n) = [2 \cdot j] - [2 \cdot (j-1)] = 2, \qquad j = 1, 2, \ldots, k+1. \tag{7}$$

This procedure constructs infinitely many k-tuples of successive gaps equal to 2 between consecutive odd composite numbers. □

For example, set $k = 5$. Then $2 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 = 270270$ and, by construction, for any $n \in \mathbb{N}$, each of $n \cdot 270270 + 3, n \cdot 270270 + 5, n \cdot 270270 + 7, n \cdot 270270 + 9, n \cdot 270270 + 11$, and $n \cdot 270270 + 13$ is composite. Thus there are infinitely many 6-tuplets of consecutive odd composite numbers with gap 2 and therefore $\mathbb{G}$ contains infinitely many 5-tuplets of consecutive gaps all equal to 2.

4 Gaps equal to 4

Theorem 4.1. *In the sequence $\mathbb{G}$ of gaps between successive odd composite numbers:*

1. *Infinitely many gaps equal* 4.
2. *$\mathbb{G}$ includes infinitely many gap pairs $(g_n, g_{n+1}) = (4, 4)$ if $\mathbb{P}$ has infinitely many cousin primes, which is an unproved consequence of the first conjecture* (12) *below of Hardy and Littlewood [1] [6, eq. 3].*
3. *There are zero gap triplets $(g_n, g_{n+1}, g_{n+2}) = (4, 4, 4)$.*

Proof.

1. Define a prime $p > 2$ to be a *solo* prime if and only if neither $p + 2$ is prime nor $p - 2$ is prime. The first two solo primes are 23 and 37. Then $g_n = 4$ if and only if the odd integer $o_n + 2$ between o_n and $o_{n+1} = o_n + 4$ is a solo prime. So there are infinitely many gaps equal to 4 if and only if there are infinitely many solo primes.

   Brun announced in 1919 and proved in 1920 [4, p. 194] that, for some effectively computable $x_0 \in \mathbb{N}$, if $x \geq x_0$, then

   $$\pi_2(x) < \frac{100x}{(\log x)^2}. \tag{8}$$

   Wu [11, p. 218, Theorem 3] proved that, for sufficiently large x,

   $$\pi_2(x) < \frac{Kx}{(\log x)^2}, \tag{9}$$

   where

   $$K \approx 4.48857 \approx 3.3996 \cdot C, \qquad C := 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \approx 1.3203236, \tag{10}$$

   C is the twin prime constant, and the product is over primes $p > 2$. It follows that, as $x \to \infty$,

   $$\lim_{x\to\infty} \frac{\pi_2(x)}{\pi(x)} < \lim_{x\to\infty} \left(\frac{Kx}{(\log x)^2} \Big/ \frac{x}{\log x}\right) = \lim_{x\to\infty} \frac{K}{\log x} = 0. \tag{11}$$

   Consequently, the ratio of the number of solo primes up to x divided by the number $\pi(x)$ of all primes up to x converges to 1 as $x \to \infty$. Since $\pi(x) \to \infty$ as $x \to \infty$ and the fraction of them that are solo primes converges to 1, the number of solo primes, and the number of gaps $g_n = 4$ between consecutive odd composite numbers, is infinite.

2. Two consecutive gaps are of size 4 if and only if, for some n, the three numbers $2n+1$, $2n+5$, $2n+9$ are odd composites and $2n+3$ and $2n+7$ are prime. If $p = 2n+3$ is the first prime, then the second prime is $p+4 = 2n+7$. Define the prime pair $(p, p+4)$ to be *solo* cousin primes if both $p-2$ and $p+6$ are composites, that is, if the cousin primes $(p, p+4)$ are neither the last two primes of a prime triplet $(p, p+2, p+6)$ nor the first two primes of a prime triplet $(p, p+4, p+6)$. (The cousin primes $(7, 11)$ do not correspond to $(g_n, g_{n+1}) = (4, 4)$ because 5 is prime and 13 is prime. The cousin primes $(37, 41)$ do not correspond to $(g_n, g_{n+1}) = (4, 4)$ because 43 is prime. The cousin primes $(43, 47)$ do not correspond to $(g_n, g_{n+1}) = (4, 4)$ because 41 is prime.)

   Under parts of the first Hardy–Littlewood conjecture [6, eqs. 3, 7, 9], we now prove that solo cousin primes occur infinitely often. For any finite strictly increasing sequence of nonnegative numbers $k_1 := 0, \ldots, k_n$ with $n < \infty$, and for any real $x > 0$, let the number of prime constellations of the form $(p, p+k_2, \ldots, p+k_n)$ such that $p < x$ be $P_x(p, p+k_2, \ldots, p+k_n)$.

   Then, according to the first Hardy–Littlewood conjecture [6, eqs. 3, 7, 9], as $x \to \infty$, the numbers of prime pairs $(p, p+4)$, the numbers of prime triplets $(p, p+2, p+6)$, and the numbers of prime triplets $(p, p+4, p+6)$ such that $p \le x$ are asymptotically equivalent to, respectively,

$$P_x(p, p+4) := 2 \prod_{p \ge 3} \frac{p(p-2)}{(p-1)^2} \int_2^x \frac{dt}{(\log t)^2} \approx 1.32032\ldots \int_2^x \frac{dt}{(\log t)^2}, \tag{12}$$

$$P_x(p, p+2, p+6) := \frac{9}{2} \prod_{p \ge 5} \frac{p^2(p-3)}{(p-1)^3} \int_2^x \frac{dt}{(\log t)^3} \approx 2.85825\ldots \int_2^x \frac{dt}{(\log t)^3}, \tag{13}$$

$$P_x(p, p+4, p+6) = P_x(p, p+2, p+6). \tag{14}$$

   As $x \to \infty$ the logarithmic integrals in (12)–(14) all go to infinity and $P_x(p, p+2, p+6)/P_x(p, p+4) \to 0$ and $P_x(p, p+4, p+6)/P_x(p, p+4) \to 0$. Hence solo cousin primes occur infinitely often and $(g_m, g_{m+1}) = (4, 4)$ occurs infinitely often, given parts of the first Hardy–Littlewood conjecture [6, eqs. 3, 7, 9].

3. Three consecutive gaps of size 4 occur if and only if, for some $n \in \mathbb{N}$, all of $2n+1$, $2n+5, 2n+9, 2n+13$ are odd composites and all of $2n+3, 2n+7, 2n+11$ are prime. But $2n+3, 2n+7, 2n+11$ modulo 3 give three different residues, so one of them must be 0, that is, divisible by 3. Hence not all of $2n+3, 2n+7, 2n+11$ can be prime and therefore no triplet $(g_n, g_{n+1}, g_{n+2}) = (4, 4, 4)$ can exist. □

The first pair $(g_n, g_{n+1}) = (4, 4)$ of consecutive gaps equal to 4 between consecutive odd composite numbers occurs when $o_{18} = 77$, $o_{19} = 81$, and $o_{20} = 85$. The second such pair occurs when $o_{33} = 125$, $o_{34} = 129$, and $o_{35} = 133$.

5 Gaps equal to 6 and mixed triplets of gaps

Theorem 5.1. *In the sequence $\mathbb{G}$ of gaps between successive odd composite numbers:*

1. $g_n = 6$ *infinitely often if and only if there are infinitely many twin primes (if and only if the twin prime conjecture is true).*

2. $(g_n, g_{n+1}) = (6, 6)$ *infinitely often if there are infinitely many prime quadruplets of the form* $(p, p + 2, p + 6, p + 8)$*, which is an unproved consequence of the first Hardy–Littlewood conjecture [6, eq. 11].*

3. *There are zero triplets of gaps* $(g_n, g_{n+1}, g_{n+2}) = (6, 6, 6), (6, 4, 4), (4, 4, 6), (6, 2, 6)$*, and* $(6, 4, 6)$ *between successive odd composite numbers.*

Proof.

1. A gap of size 6 occurs if for some $n \in \mathbb{N}$, the numbers $2n + 1,\ 2n + 7$ are composite and $p = 2n + 3,\ p + 2 = 2n + 5$ are twin primes. Since the first odd composite number is $o_1 = 9$, we must have $p > 7$.

   Conversely, if $(p, p + 2)$ are twin primes and $p > 7$, then both $p - 2$ and $p + 4$ must be odd composite numbers because $(3, 5, 7)$ is the only triple of primes of the form $(p, p+2, p+4)$.

   Under the conjecture that there are infinitely many twin primes $(p, p + 2)$ with $p > 7$, there are infinitely many $g_n = 6$.

2. A pair $(6, 6)$ of consecutive gaps equal to 6 between consecutive odd composite numbers exists if, for some $n \in \mathbb{N}$, $2n + 1,\ 2n + 7,\ 2n + 13$ are odd composites and $2n + 3,\ 2n + 5,\ 2n + 9,\ 2n + 11 = (p,\ p + 2,\ p + 6,\ p + 8)$ are prime. Because $p > 7$ and because $(3, 5, 7)$ is the only triple of primes of the form $(p, p + 2, p + 4)$, it is not possible that $p - 2$ be prime nor that $p + 10$ be prime, so odd numbers that immediately precede and follow a prime quadruplet $(p,\ p + 2,\ p + 6,\ p + 8)$ must be composite. Part of the first Hardy–Littlewood conjecture [6, eq. 11] states that the number of such prime quadruplets with $p < x$ is asymptotic as $x \to \infty$ to

$$P_x(0, 2, 6, 8) := \frac{27}{2} \prod_{p \geq 5} \frac{p^3(p - 4)}{(p - 1)^4} \int_2^x \frac{dt}{(\log t)^4} \approx 4.15118 \ldots \int_2^x \frac{dt}{(\log t)^4}. \qquad (15)$$

   Since the logarithmic integral in (15) goes to infinity as $x \to \infty$, $(g_n, g_{n+1}) = (6, 6)$ occurs infinitely often.

3. A triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 6, 6)$ of gaps between successive odd composite numbers exists if and only if there exist three pairs of twin primes of the form $p + (0, 2, 6, 8, 12, 14)$ while $p - 2,\ p + 4,\ p + 10$, and $p + 16$ are odd composite numbers.

   We now show that no such 6-tuple of primes can exist.

   Define $v := (0, 2, 6, 8, 12, 14)$. We cannot have $p \equiv 0 \pmod 5$ because p is a prime, not divisible by 5. If $p \equiv 1 \pmod 5$, then $(1+v) \equiv (1, 3, 2, 4, 3, 0) \pmod 5$, so $p+14$ is divisible by 5, not prime. If $p \equiv 2 \pmod 5$, then $(2+v) \equiv (2, 4, 3, 0, 4, 1) \pmod 5$, so $p+8$ is divisible by 5, not prime. If $p \equiv 3 \pmod 5$, then $(3+v) \equiv (3, 0, 4, 1, 0, 2) \pmod 5$, so $p+2$ and $p+12$ are divisible by 5, not prime. If $p_h \equiv 4 \pmod 5$, then $(4+v) \equiv (4, 6, 0, 2, 1, 3) \pmod 5$, so $p+6$ is divisible by 5, not prime. So every possible positive integer for p makes one of the six numbers in v divisible by 5, not prime, so no such 6-tuple of consecutive primes exists, and therefore no triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 6, 6)$ of gaps between consecutive composite numbers.

Similarly, a triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 4)$ of gaps between successive odd composite numbers exists if and only there exists a quadruplet of primes of the form $p+(0, 2, 6, 10) := p + w$. We cannot have $p \equiv 0 \pmod 3$ because p is a prime, not divisible by 3. If $p \equiv 1 \pmod 3$, then $(1 + w) \equiv (1, 0, 1, 2) \pmod 3$, so $p + 2$ is divisible by 5, not prime. If $p \equiv 2 \pmod 3$, then $(1 + w) \equiv (2, 1, 2, 0) \pmod 3$, so $p + 10$ is divisible by 3, not prime. So every possible positive integer for p makes one of the four numbers $p + w$ divisible by 3, not prime, so no such subsequence of consecutive primes exists. Therefore there exists no triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 4)$ of gaps between consecutive composite numbers.

The same method may be used to prove the non-existence of triplets $(g_n, g_{n+1}, g_{n+2}) = (4, 4, 6),\ (6, 2, 6)$, and $(6, 4, 6)$. □

The first pair $(g_n, g_{n+1}) = (6, 6)$ of consecutive gaps equal to 6 between consecutive odd composite numbers occurs when $o_1 = 9$, $o_2 = 15$, and $o_3 = 21$. The second such pair occurs when $o_{25} = 99$, $o_{26} = 105$, and $o_{27} = 111$.

Theorem 5.2. *The triplets $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ and $(6, 4, 2)$ occur with asymptotically equivalent frequency if parts of the first Hardy–Littlewood conjecture [6, eqs. 9, 11] are true.*

Proof. The triplet $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ occurs if and only if $p + (0, 4, 6)$ are all primes and $p-2$ and $p-4$ and $p+8$ are all odd composite numbers. It is impossible that $(p+4, p+6, p+8)$ all be primes because $(3, 5, 7)$ is the only triple of primes of the form $(p', p' + 2, p' + 4)$. It is also impossible that $(p - 4, p - 2, p)$ all be primes for exactly the same reason. It is also impossible that $(p - 4, p, p + 4, p + 6)$ all be primes with $p - 2$ odd composite because then $(p-4, p, p+4) = (p', p'+4, p'+8)$ would have all three residue classes $0, 1, 2 \ (\mathrm{mod}\ 3)$. However, it is possible that $(p - 2, p, p + 4, p + 6) = (p', p' + 2, p' + 6, p' + 8)$ with $p' = p - 2$ could all be primes.

According to part of the first Hardy–Littlewood conjecture [6, eq. 9], as $x \to \infty$, the number of prime triplets $p + (0, 4, 6)$ such that $p \le x$ is asymptotic to $P_x(p, p + 4, p + 6)$ in (14). The conjectured asymptotic form of $P_x(p, p + 2, p + 6, p + 8)$ from (15) implies that $P_x(p, p + 2, p + 6, p + 8)/P_x(p, p + 4, p + 6) \to 0$ as $x \to \infty$. Hence for large x almost all prime triplets of the form $(p, p + 4, p + 6)$ are *not* the last three primes of a prime quadruplet $(p, p + 2, p + 6, p + 8)$ and hence are associated with a triplet $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ of gaps between consecutive odd composite numbers.

On the other hand, the triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 2)$ occurs if and only if $(p, p+2, p+6)$ are all primes and $p - 2$ and $p + 4$ and $p + 8$ are all odd composite numbers. When p and $p + 2$ are primes and $p > 7$, it is impossible that $p - 2$ also be prime, because $(3, 5, 7)$ is the only triple of primes of the form $(p', p'+2, p'+4)$, so $p-2$ is guaranteed to be an odd composite number. When $p+2$ and $p+6$ are primes, it is impossible that $p+4$ be prime, for exactly the same reason. So $p+4$ is guaranteed to be an odd composite number.

The conjectured asymptotic counting function $P_x(p, p+2, p+6, p+8)$ in (15) [6, eq. 11] implies that $P_x(p, p+2, p+6, p+8)/P_x(p, p+2, p+6) \to 0$ as $x \to \infty$. So for large x almost all prime triplets of the form $(p, p+2, p+6)$ are *not* the first three primes of a prime quadruplet $(p, p+2, p+6, p+8)$ and hence are associated with a triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 2)$ of gaps between consecutive odd composite numbers.

We have shown that, for large x, almost all prime triplets of the form $(p, p+4, p+6)$ are associated with a triplet $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ and almost all prime triplets of the form $(p, p+2, p+6)$ are associated with a triplet $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 2)$, and these triplets of gaps between consecutive odd composite numbers are not associated with other prime configurations. Since the asymptotic counting function $P_x(p, p+2, p+6)$ in (13) of prime triplets $p+(0, 2, 6)$ is conjectured in (14) to equal the asymptotic counting function $P_x(p, p+4, p+6)$ of prime triplets $p+(0, 4, 6)$, the counting functions of $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ and $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 2)$ are (under these Hardy–Littlewood conjectures) asymptotically equivalent. □

6 Numerical example and conjecture

We listed the 4 544 947 488 gaps (first differences) between the 4 544 947 489 odd composite numbers less than 10^{10}. Check: 5×10^9 (even numbers up to and including 10^{10}, including 2) + 4 544 947 489 (odd composite numbers less than 10^{10}, excluding 1) + 455 052 511 ($= \pi(10^{10})$, the number of primes less than 10^{10}, including 2) $= 10^{10}$ exactly. The exclusion of 1 from the odd composite numbers balances the double counting of 2 in the primes and the even numbers.

As all gaps equal 2, 4, or 6, we counted the frequency of each of the $27 = 3 \times 3 \times 3$ possible (in principle) triplets of values of the successive 4 544 947 486 overlapping triplets of gaps (Table 1). For example, the triplet $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ occurs 2 531 697 times; $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 2)$ occurs 2 532 818 times; $(g_n, g_{n+1}, g_{n+2}) = (6, 2, 4)$ occurs 2 535 001 times; and $(g_n, g_{n+1}, g_{n+2}) = (4, 2, 6)$ occurs 2 535 136 times. The close similarity of these four frequencies suggests that these four triplets may occur with asymptotically equivalent frequency. The sum of all frequencies in the table equals 4 544 947 486, which is the number of overlapping triplets in the list of 4 544 947 488 gaps between consecutive composite numbers less than 10^{10}. As noted above, six of the 27 potential triplets of values of gaps between successive odd composite numbers cannot occur. The forbidden triplets of gaps are $(g_n, g_{n+1}, g_{n+2}) = (4, 4, 4)$, $(6, 6, 6)$, $(6, 4, 4)$, $(4, 4, 6)$, $(6, 2, 6)$, and $(6, 4, 6)$.

For $x = 10^{10}$, from (14) $P_{10^{10}}(p, p+4, p+6) \approx 2715284.5779\ldots$ and $P_{10^{10}}(p, p+2, p+6, p+8) \approx 181074.8947\ldots$. Hence $P_{10^{10}}(p, p+2, p+6, p+8)/P_{10^{10}}(p, p+4, p+6) \approx 0.0667\ldots$. The estimated number of triplets $(g_n, g_{n+1}, g_{n+2}) = (2, 4, 6)$ of gaps between consecutive odd composite numbers less than 10^{10} is $P_{10^{10}}(p, p+4, p+6) - P_{10^{10}}(p, p+2, p+6, p+8) \approx 2534209.6832\ldots$ whereas the exact number of such triplets of gaps is 2 531 697, according to Table 1. The estimated number deviates from the counted number by less than 0.1 percent of the

Table 1. Counts of the triplets of gaps (first differences) between the successive odd composite numbers less than 10^{10}.

	$g_{n+2} = 2$		
$g_n \downarrow$	$g_{n+1} \rightarrow 2$	4	6
2	3366240715	350832730	22348479
4	328672780	22164954	2190526
6	22164330	2532818	160326
	$g_{n+2} = 4$		
$g_n \downarrow$	$g_{n+1} \rightarrow 2$	4	6
2	328671794	22164954	2191655
4	44322586	0	320961
6	2535001	0	20202
	$g_{n+2} = 6$		
$g_n \downarrow$	$g_{n+1} \rightarrow 2$	4	6
2	22165315	2531697	160317
4	2535136	0	20210
6	0	0	0

counted number. Similarly, the exact number of triplets of gaps $(g_n, g_{n+1}, g_{n+2}) = (6, 4, 2)$ between consecutive odd composite numbers less than 10^{10} is 2 532 818, also very close to the asymptotic estimate.

We conjecture that the frequency of $(g_n, g_{n+1}, g_{n+2}) = (4, 2, 6)$ is asymptotically equivalent to the frequency of $(g_n, g_{n+1}, g_{n+2}) = (6, 2, 4)$. The counts in Table 1 are 2 535 136 and 2 535 001, respectively.

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