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The Bombieri–Vinogradov theorem for exponential sums over primes

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Abstract: In this paper, we revisit Lemma 18 from [2], which concerns a Bombieri–Vinogradov type theorem for exponential sums over primes. We provide a corrected version of the lemma, clarify the original arguments, and address certain inaccuracies present in the initial proof.
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1 Introduction and statement of the result

This note revisits Lemma 18 from [2], originally published in *Ramanujan Journal* in 2022. The celebrated Bombieri–Vinogradov theorem is extremely important result and plays a significant role in analytic number theory. It concerns the distribution of primes in arithmetic progressions, averaged over a range of moduli. It asserts that when $A > 0$ is fixed and $X > 2$, then:

$$\sum_{d\le\sqrt{X}/(\log X)^{A+5}}\max_{y\le X}\max_{(a,\,d)=1}\left|\sum_{\substack{p\le y\\ p\equiv a\ (\mathrm{mod}\ d)}}\log p-\frac{y}{\varphi(d)}\right|\ll\frac{X}{\log^A X}\,,$$

where $\varphi(n)$ is Euler's function. An interesting problem is the proof of the Bombieri–Vinogradov theorem for prime numbers of a special form. In this connection Peneva [14], Wang and Cai [22], Lu [11] and J. Li, M. Zhang and F. Xue [10] proved such result for Piatetski-Shapiro [16] primes $p=[n^c]$. Recently Nath [13] obtained a Bombieri-Vinogradov type theorems for Maynard [12] primes which are primes with a missing digit. In [3] the author established a Bombieri–Vinogradov type result for prime numbers of the form $p=x^2+y^2+1$. A lot of articles are devoted to problems of this type. We point out the papers of Baier and Zhao [1], Fouvry and Iwaniec [5], Granville and Shao [6], Huxley and Iwaniec [7], Perelli, Pintz and Salerno [15] and Tao [17], but many other similar results can be found in literature.

In this paper we establish a Bombieri–Vinogradov type result for exponential sums over primes. More precisely, we prove the following theorem.

Theorem 1.1. *Let $1<c<3$, $c\neq 2$, $0<\mu<1$, $|t|\le X^{\frac{1}{4}-c}$, $A>0$ and $X>2$. Then the following inequality holds*

$$\sum_{d\le\sqrt{X}/(\log X)^{2A+10}}\max_{y\le X}\max_{(a,\,d)=1}\left|\sum_{\substack{\mu y<p\le y\\ p\equiv a\ (d)}}e(tp^c)\log p-\frac{1}{\varphi(d)}\int\limits_{\mu y}^{y}e(tx^c)\,dx\right|\ll\frac{X}{\log^A X}\,.$$

2 Notations

Assume that $X>2$. The letter p with or without subscript will always denote prime numbers. The notation $m\sim M$ means that m runs through the interval $(M,2M]$. Moreover $e(t)=\exp(2\pi it)$. Let (a,d) be the greatest common divisor of a and d. As usual $\varphi(n)$ is Euler's function, $\mu(n)$ is Möbius' function, $\tau(n)$ denotes the number of positive divisors of n and $\Lambda(n)$ is von Mangoldt's function. We shall use the convention that a congruence, $m\equiv n \pmod d$ will be written as $m\equiv n\,(d)$. The letter χ denotes a Dirichlet character to a given modulus. The sums $\sum_{\chi(d)}$ and $\sum^*_{\chi(d)}$ denotes respectively summation over all characters and all primitive characters modulo d. Throughout this paper we suppose that $0<\mu<1$ and $1<c<3$, $c\neq 2$. Denote

$$\Psi(y,\chi,t)=\sum_{\mu y<n\le y}\Lambda(n)\chi(n)e(tn^c)\,;\tag{1}$$

$$E(y,t,d,a)=\sum_{\substack{\mu y<n\le y\\ n\equiv a\ (d)}}\Lambda(n)e(tn^c)-\frac{1}{\varphi(d)}\int\limits_{\mu y}^{y}e(tx^c)\,dx\,.\tag{2}$$

3 Preliminary lemmas

Lemma 3.1. *Let $0<\mu<1$, $c>1$, $B>0$, $C>0$ and $|t|\le X^{\frac{2}{3}-c-\delta}$ for a sufficiently small $\delta>0$. Then*

$$\sum_{1<q\le\log^C X}\frac{1}{\varphi(q)}\sum_{\chi(q)}{}^{*}\left|\sum_{\mu X<n\le X}\Lambda(n)\chi(n)e(tn^c)\right|\ll\frac{X}{\log^B X}\,.$$

Proof. See [4, Lemma 12]. □

Lemma 3.2. *Let* $1 < c < 3$, $c \neq 2$, $0 < \mu < 1$ *and* $|t| \leq X^{1-c-\varepsilon}$. *Then*

$$\sum_{\mu X < p \leq X} e(tp^c) \log p = \int\limits_{\mu X}^{X} e(ty^c)\, dy + \mathcal{O}\left(\frac{X}{e^{(\log X)^{1/5}}} \right) .$$

Proof. See [19, Lemma 14]. □

Lemma 3.3. *(Pólya – Vinogradov inequality) Suppose that* M, N *are positive integers and* χ *is a non-principal character modulo* q. *Then*

$$\left| \sum_{M < n \leq M+N} \chi(n) \right| \leq 6\sqrt{q} \log q \, .$$

Proof. See [8, Theorem 12.5]. □

Lemma 3.4. *(Perron's formula) Let*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be Dirichlet series with abscissa of absolute convergence σ_a. *Then for* $\varkappa > \max(0, \sigma_a) - 1$, $T \geq 1$ *and* $x \geq 1$ *we have*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi\imath} \int\limits_{1+\varkappa-\imath T}^{1+\varkappa+\imath T} F(s) \frac{x^s}{s} ds + \mathcal{O}\left(x^{1+\varkappa} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{1+\varkappa} \left(1 + T \left|\log \frac{x}{n}\right|\right)} \right) .$$

Proof. See [18, Ch. II.2, Theorem 2]. □

Lemma 3.5. *We have*

$$\int\limits_{\mu X}^{X} y^{\beta-1+i\gamma} e(ty^c)\, dy \ll \begin{cases} \frac{X^\beta}{|t| X^c} & \textit{for} \quad |\gamma| \leq \pi c \mu^c |t| X^c \, , \\ \frac{X^\beta}{\sqrt{|t| X^c}} & \textit{for} \quad \pi c \mu^c |t| X^c < |\gamma| < 4\pi c |t| X^c \, , \\ \frac{X^\beta}{|\gamma|} & \textit{for} \quad |\gamma| \geq 4\pi c |t| X^c \, . \end{cases}$$

Proof. See [20, Lemma 10, pp. 257–258]. □

Lemma 3.6. *Assume that* $F(x)$ *and* $\varphi(x)$ *are real functions defined on* $[a, b]$ *and satisfying the following conditions:*

(i) there exist numbers $H > 0$, $U \gg 1$, $A \gg 1$, *such that*

$$A^{-1} \ll F''(x) \ll A^{-1} , \quad F'''(x) \ll A^{-1} U^{-1} ,$$

$$\varphi(x) \ll H , \quad \varphi'(x) \ll H U^{-1} , \quad \varphi''(x) \ll H U^{-2} ;$$

(ii) for some $c \in [a, b]$ *is fulfilled*

$$F'(c) = 0 ;$$

(iii) the function

$$G(x) = \frac{\varphi(x+c)}{F'(x+c)} - \frac{\varphi(c)}{\sqrt{2(F(x+c)-F(c))F''(c)}}$$

has a finite number intervals of monotony.

Then the asymptotic formula

$$\int_a^b \varphi(x)e\big(F(x)\big)\,dx = \frac{1+i}{\sqrt{2}} \cdot \frac{\varphi(c)e\big(F(c)\big)}{\sqrt{F''(c)}} + \mathcal{O}\big(HU^{-1}A\big)$$
$$+ \mathcal{O}\left(H \min\left(\frac{1}{|F'(a)|}, \sqrt{A}\right)\right) + \mathcal{O}\left(H \min\left(\frac{1}{|F'(b)|}, \sqrt{A}\right)\right)$$

holds.

Proof. See [9, Ch. 1, §3, Lemma 2]. □

Remark 1. *If $c \notin [a,b]$ then only the remainder terms remain in the above asymptotic formula.*

Lemma 3.7. (Large Sieve) *For any complex numbers a_n and positive integers M, N, Q we have*

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi(q)} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \ll \big(N+Q^2\big) \sum_{n=M+1}^{M+N} |a_n|^2$$

Proof. See [8, Theorem 7.13]. □

4 Proof of the Theorem

We will use the formula

$$\sum_{\substack{\mu y<p\le y\\ p\equiv a\,(d)}} e(tp^c)\log p = \sum_{\substack{\mu y<n\le y\\ n\equiv a\,(d)}} \Lambda(n)e(tn^c) + \mathcal{O}\left(\frac{y^{\frac{1}{2}+\varepsilon}}{d}\right) \tag{3}$$

for $d \le y^{\frac{1}{2}}$. Define

$$\delta(\chi) = \begin{cases} 1, & \text{if } \chi \text{ is principal,} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

By the orthogonality of characters we have

$$\sum_{\substack{\mu y<n\le y\\ n\equiv a\,(d)}} \Lambda(n)e(tn^c) - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c)\,dx$$
$$= \sum_{\mu y<n\le y} \Lambda(n)e(tn^c) \frac{1}{\varphi(d)} \sum_{\chi(d)} \chi(n)\overline{\chi}(a) - \frac{1}{\varphi(d)} \int_{\mu y}^{y} e(tx^c)\,dx$$
$$= \frac{1}{\varphi(d)} \sum_{\chi(d)} \left(\overline{\chi}(a) \sum_{\mu y<n\le y} \Lambda(n)\chi(n)e(tn^c) - \delta(\chi) \int_{\mu y}^{y} e(tx^c)\,dx \right)$$

and therefore

$$\begin{aligned}&\max_{(a,\,d)=1}\Bigg|\sum_{\substack{\mu y<n\le y\\ n\equiv a\,(d)}}\Lambda(n)e(tn^c)-\frac{1}{\varphi(d)}\int\limits_{\mu y}^{y}e(tx^c)\,dx\Bigg|\\ &\le\frac{1}{\varphi(d)}\sum_{\chi(d)}\Bigg|\Psi(y,\chi,t)-\delta(\chi)\int\limits_{\mu y}^{y}e(tx^c)\,dx\Bigg|,\end{aligned}\tag{5}$$

where $\Psi(y,\chi,t)$ is defined by (1). Denote

$$\Sigma=\sum_{d\le\sqrt{X}/(\log X)^{2A+10}}\max_{y\le X}\max_{(a,\,d)=1}\big|E(y,t,d,a)\big|\,.\tag{6}$$

From (2), (4), (5) and (6) we obtain

$$\Sigma\le\Sigma'+\Sigma''\,,\tag{7}$$

where

$$\Sigma'=\sum_{d\le\sqrt{X}/(\log X)^{2A+10}}\frac{1}{\varphi(d)}\max_{y\le X}\Bigg|\sum_{\mu y<n\le y}\Lambda(n)e(tn^c)-\int\limits_{\mu y}^{y}e(tx^c)\,dx+\mathcal{O}\Big(\log^2 y\Big)\Bigg|,\tag{8}$$

$$\Sigma''=\sum_{d\le\sqrt{X}/(\log X)^{2A+10}}\frac{1}{\varphi(d)}\sum_{\substack{\chi(d)\\ \chi\neq\chi_0}}\max_{y\le X}\big|\Psi(y,\chi,t)\big|\,.\tag{9}$$

By (3), (8) and Lemma 3.2 we find

$$\Sigma'\ll\frac{X}{e^{(\log X)^{1/5}}}\sum_{d\le\sqrt{X}/(\log X)^{2A+10}}\frac{1}{\varphi(d)}\ll\frac{X}{\log^A X}\,.\tag{10}$$

Next we consider Σ''. Moving to primitive characters from (9) we deduce

$$\begin{aligned}\Sigma''&\ll\sum_{d\le\sqrt{X}/(\log X)^{2A+10}}\frac{1}{\varphi(d)}\sum_{\substack{r|d\\ r>1}}\sideset{}{^*}\sum_{\chi(r)}\max_{y\le X}\big|\Psi(y,\chi,t)\big|+\frac{\sqrt{X}}{(\log X)^{2A+8}}\\ &\ll\sum_{r\le\sqrt{X}/(\log X)^{2A+10}}\left(\sum_{\substack{d\le\sqrt{X}/(\log X)^{2A+10}\\ d\equiv 0\,(r)}}\frac{1}{\varphi(d)}\right)\sideset{}{^*}\sum_{\chi(r)}\max_{y\le X}\big|\Psi(y,\chi,t)\big|+\frac{\sqrt{X}}{(\log X)^{2A+8}}\\ &\ll(\log X)\sum_{1<r\le\sqrt{X}/(\log X)^{2A+10}}\frac{1}{\varphi(r)}\sideset{}{^*}\sum_{\chi(r)}\max_{y\le X}\big|\Psi(y,\chi,t)\big|+\frac{\sqrt{X}}{(\log X)^{2A+8}}\\ &=(\Omega_1+\Omega_2)\log X+\frac{\sqrt{X}}{(\log X)^{2A+8}}\,,\end{aligned}\tag{11}$$

where

$$\Omega_1=\sum_{r\le R_0}\frac{1}{\varphi(r)}\sideset{}{^*}\sum_{\chi(r)}\max_{y\le X}\big|\Psi(y,\chi,t)\big|\,,\tag{12}$$

$$\Omega_2=\sum_{R_0<r\le R}\frac{1}{\varphi(r)}\sideset{}{^*}\sum_{\chi(r)}\max_{y\le X}\big|\Psi(y,\chi,t)\big|\,,\tag{13}$$

$$R_0=(\log X)^{A+5}\,,\quad R=\frac{\sqrt{X}}{(\log X)^{2A+10}}\,.\tag{14}$$

Taking into account (1), (12), (14) and Lemma 3.1 with $B = A + 1$ we obtain

$$\Omega_1 \ll \frac{X}{(\log X)^{A+1}} \,. \tag{15}$$

Next we consider Ω_2. Let $1 < u \le \mu y$ be a parameter that we will choose later. Using (1) and Vaughan's identity (see [21]), we get

$$\Psi(y, \chi, t) = U_1(y, \chi, t) - U_2(y, \chi, t) - U_3(y, \chi, t) - U_4(y, \chi, t) \,, \tag{16}$$

where

$$U_1(y, \chi, t) = \sum_{d \le u} \mu(d) \sum_{\mu y < dl \le y} \chi(dl) e(t d^c l^c) \log l \,, \tag{17}$$

$$U_2(y, \chi, t) = \sum_{d \le u} c(d) \sum_{\mu y < dl \le y} \chi(dl) e(t d^c l^c) \,, \tag{18}$$

$$U_3(y, \chi, t) = \sum_{u < d \le u^2} c(d) \sum_{\mu y < dl \le y} \chi(dl) e(t d^c l^c) \,, \tag{19}$$

$$U_4(y, \chi, t) = \sum_{\substack{d > u,\, l > u \\ \mu y < dl \le y}} \sum a(d) \Lambda(l) \chi(dl) e(t d^c l^c) \,, \tag{20}$$

and where

$$|c(d)| \le \log d \,, \quad |a(d)| \le \tau(d) \,. \tag{21}$$

Now (13), (16)–(20) give us

$$\Omega_2 \ll \Omega_2^{(1)} + \Omega_2^{(2)} + \Omega_2^{(3)} + \Omega_2^{(4)} \,, \tag{22}$$

where

$$\Omega_2^{(j)} = \sum_{R_0 < r \le R} \frac{1}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \max_{y \le X} \big| U_j(y, \chi, t) \big| \,, \quad j = 1,\, 2,\, 3,\, 4 \,. \tag{23}$$

Estimation of $\mathbf{\Omega_2^{(1)}}$ and $\mathbf{\Omega_2^{(2)}}$

From (14), (17), (23), Abel's summation formula and Lemma 3.3 it follows that

$$\begin{aligned}
\Omega_2^{(1)} &\ll \sum_{R_0 < r \le R} \frac{1}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \max_{y \le X} \sum_{d \le u} \Bigg| \sum_{\mu y/d < l \le y/d} \chi(dl) e(t d^c l^c) \log l \Bigg| \\
&\ll X^{\frac{1}{4}} (\log X) \sum_{R_0 < r \le R} \frac{1}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \sum_{d \le u} \max_{y \le X} \max_{\mu y/d < x \le y/d} \Bigg| \sum_{\mu y/d < l \le x} \chi(l) \Bigg| \\
&\ll X^{\frac{1}{4}} (\log X) u \sum_{R_0 < r \le R} r^{\frac{1}{2}} \log r \ll X^{\frac{1}{4}} u R^{\frac{3}{2}} \log^2 X \,.
\end{aligned} \tag{24}$$

Working in a similar way, we deduce

$$\Omega_2^{(2)} \ll X^{\frac{1}{4}} u R^{\frac{3}{2}} \log X \,. \tag{25}$$

Estimation of $\mathbf{\Omega_2^{(3)}}$ and $\mathbf{\Omega_2^{(4)}}$

We split the range of l of the exponential sum (20) into dyadic subintervals of the form $L < l \leq 2L$, where $u < L \leq y/2u$. Further we use (21), Abel's summation formula, Lemma 3.4 with parameters

$$\varkappa = \frac{1}{\log X}\,, \quad T = X^2 \tag{26}$$

and partial integration to find

$$\begin{aligned} U_4(y,\chi,t) &\ll (\log X)\bigg|\sum_{l\sim L}\sum_{\mu y/l<d\leq y/l}\Lambda(l)a(d)\chi(dl)e(td^cl^c)\bigg| \\ &= (\log X)\bigg|e(ty^c)\sum_{l\sim L}\sum_{\mu y/l<d\leq y/l}\Lambda(l)a(d)\chi(dl) \\ &\quad - \int\limits_{\mu y/l}^{y/l}\Bigg(\sum_{l\sim L}\sum_{\mu y/l<d\leq x}\Lambda(l)a(d)\chi(dl)\Bigg)\,de(tx^cl^c)\bigg| \\ &= (\log X)\big|e(ty^c)\mathfrak{X}_1 - \mathfrak{X}_2\big|\,, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \mathfrak{X}_1 = \frac{1}{2\pi i}\int\limits_{1-\varkappa-iT}^{1+\varkappa+iT}\sum_{l\sim L}\sum_{\mu y/2L<d\leq X/L}\frac{\Lambda(l)a(d)\chi(dl)}{(dl)^s}\frac{y^s}{s}\,ds \\ + \mathcal{O}\Bigg(\sum_{l\sim L}\sum_{\mu y/2L<d\leq y/L}\frac{y^{1+\varkappa}\Lambda(l)\tau(d)}{(dl)^{1+\varkappa}\big(1+T\left|\log\frac{y}{dl}\right|\big)}\Bigg)\,, \end{aligned} \tag{28}$$

$$\begin{aligned} \mathfrak{X}_2 = \int\limits_{\mu y/l}^{y/l}\Bigg(\frac{1}{2\pi i}\int\limits_{1-\varkappa-iT}^{1+\varkappa+iT}\sum_{l\sim L}\sum_{\mu y/2L<d\leq X/L}\frac{\Lambda(l)\chi(dl)a(d)}{(dl)^s}\frac{x^s}{s}\,ds \\ +\mathcal{O}\Bigg(\sum_{l\sim L}\sum_{\mu y/2L<d\leq y/L}\frac{x^{1+\varkappa}\Lambda(l)\tau(d)}{(dl)^{1+\varkappa}\left(1+T\left|\log\frac{x}{dl}\right|\right)}\Bigg)\Bigg)\,de(tx^cl^c)\,. \end{aligned} \tag{29}$$

If we assume, as we may, that $\|y\| = \frac{1}{2}$, we have $|\log\frac{y}{dl}| \geq \frac{1}{y}$. Now (26), (28), (29), partial integration and the well-known inequalities

$$\sum_{n\leq X}\Lambda(n) \ll X\,, \quad \sum_{n\leq X}\tau(n) \ll X\log X \tag{30}$$

imply

$$\mathfrak{X}_1 = \frac{1}{2\pi i}I_1 + \mathcal{O}\big(\log X\big)\,, \tag{31}$$

$$\mathfrak{X}_2 = \frac{1}{2\pi i}\big(e(t\mu^cy^c)I_2 - e(ty^c)I_3 + I_4\big) + \mathcal{O}\Big(X^{\frac{1}{4}}u^{-1}\log X\Big)\,, \tag{32}$$

where

$$I_1 = \int\limits_{1-\varkappa-iT}^{1+\varkappa+iT} \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl)^s} \frac{y^s}{s}\, ds\,, \tag{33}$$

$$I_2 = \int\limits_{1-\varkappa-iT}^{1+\varkappa+iT} \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^s} \frac{\mu^s y^s}{s}\, ds\,, \tag{34}$$

$$I_3 = \int\limits_{1-\varkappa-iT}^{1+\varkappa+iT} \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^s} \frac{y^s}{s}\, ds\,, \tag{35}$$

$$I_4 = \int\limits_{1-\varkappa-iT}^{1+\varkappa+iT} \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^s} \left(\int\limits_{\mu y}^{y} x^{s-1} e(tx^c)\, dx \right) ds\,. \tag{36}$$

Put $s = \beta + i\gamma$. Using (26), (30), (33)–(36), Lemma 3.5 and Cauchy's integral theorem for the rectangle

$$\{\varkappa \le \beta \le 1+\varkappa\,,\ -T \le \gamma \le T\},$$

we derive

$$I_1 \ll \left| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl)^{\varkappa+i\gamma_1}} \right| \log X + \log X\,, \tag{37}$$

$$I_2, I_3 \ll \left| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_2}} \right| \log X + \log X\,, \tag{38}$$

$$I_4 \ll |I_5| + \log X \tag{39}$$

for some $|\gamma_1|, |\gamma_2| \le T$, where

$$I_5 = \int\limits_{-T}^{T} \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma}} \left(\int\limits_{\mu y}^{y} x^{\varkappa-1+i\gamma} e(tx^c)\, dx \right) d\gamma\,. \tag{40}$$

Firstly, consider the case

$$4\pi c|t|y^c \le \log y\,. \tag{41}$$

By (26), (40), (41) and Lemma 3.5 we obtain

$$\begin{aligned} I_5 &\ll \left| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_3}} \right| \left(\int\limits_{0}^{\log y} y^{\varkappa}\, d\gamma + \int\limits_{\log y}^{T} \frac{y^{\varkappa}}{\gamma}\, d\gamma \right) \\ &\ll \left| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_3}} \right| \log X\,, \end{aligned} \tag{42}$$

for some $|\gamma_3| \le T$. Consider now the case

$$4\pi c|t|y^c > \log y\,. \tag{43}$$

From (26), (40), (43) and Lemma 3.5 we deduce

$$
\begin{aligned}
I_5 &\ll \left| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_4}} \right| \left(\int\limits_0^{\pi c\mu^c|t|y^c} \frac{y^{\varkappa}}{|t|y^c}\, d\gamma + \int\limits_{4\pi c|t|y^c}^{T} \frac{y^{\varkappa}}{\gamma}\, d\gamma \right) + |I_6| \\
&\ll \left| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_4}} \right| \log X + |I_6| \,, \qquad (44)
\end{aligned}
$$

for some $|\gamma_4| \le T$, where

$$
I_6 = \int\limits_{\pi c\mu^c|t|y^c<|\gamma|<4\pi c|t|y^c} \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma}} \left(\int\limits_{\mu y}^{y} x^{\varkappa-1+i\gamma} e(tx^c)\, dx \right) d\gamma \,. \qquad (45)
$$

With a change of variables we can write the integral with respect to x in the form

$$
\int\limits_{\mu y}^{y} x^{\varkappa-1+i\gamma} e(tx^c)\, dx = \frac{1}{c} \int\limits_{(\mu y)^c}^{y^c} \varphi(u) e\big(F(u)\big)\, du \,, \qquad (46)
$$

where

$$
F(u) = \frac{\gamma}{2\pi c} \log u + tu \,, \quad \varphi(u) = u^{\frac{1}{c}\varkappa-1} \,. \qquad (47)
$$

We have

$$
F'(u) = \frac{\gamma}{2\pi c u} + t \,, \quad F''(u) = -\frac{\gamma}{2\pi c u^2} \,, \quad F'''(u) = \frac{\gamma}{\pi c u^3} \,, \qquad (48)
$$

$$
\varphi'(u) = \Big(\frac{\varkappa}{c} - 1\Big) u^{\frac{1}{c}\varkappa-2} \,, \quad \varphi''(u) = \Big(\frac{\varkappa}{c} - 1\Big)\Big(\frac{\varkappa}{c} - 2\Big) u^{\frac{1}{c}\varkappa-3} \,. \qquad (49)
$$

Bearing in mind that $|\gamma| \asymp |t|y^c$, $u \asymp y^c$, by (47)–(49) we get

$$
|t| \ll F'(u) \ll |t| \,, \quad A^{-1} \ll F''(u) \ll A^{-1} \,, \quad F'''(u) \ll A^{-1}U^{-1} \,, \qquad (50)
$$

$$
\varphi(u) \ll H \,, \quad \varphi'(u) \ll HU^{-1} \,, \quad \varphi''(u) \ll HU^{-2} \,, \qquad (51)
$$

where

$$
H = y^{\varkappa-c} \,, \quad U = y^c \,, \quad A = y^c|t|^{-1} \,. \qquad (52)
$$

Put

$$
u_0 = -\frac{\gamma}{2\pi c t} \,. \qquad (53)
$$

From (48) it follows that

$$
F'(u_0) = 0 \,. \qquad (54)
$$

Set

$$
a = \pi c\mu^c|t|y^c| \,, \quad b = 4\pi c|t|y^c)| \,. \qquad (55)
$$

Taking into account Remark 1, we can consider that $u_0 \in [(\mu y)^c, y^c]$. Now (26), (43), (45)–(48), (50)–(55), Abel's summation formula and Lemma 3.6 yield

$$\begin{aligned}
I_6 &= \int\limits_a^b \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma}} \Bigg[\frac{1+i}{c\sqrt{2}} \cdot \frac{\varphi(u_0)e\big(F(u_0)\big)}{\sqrt{F''(u_0)}} + \mathcal{O}\big(HU^{-1}A\big) \\
&\quad + \mathcal{O}\Bigg(H\min\Bigg(\frac{1}{|F'(a)|}, \sqrt{A}\Bigg)\Bigg) + \mathcal{O}\Bigg(H\min\Bigg(\frac{1}{|F'(b)|}, \sqrt{A}\Bigg)\Bigg)\Bigg] d\gamma \\
&\ll \Bigg| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_5}\log(dl)} \Bigg| + \Bigg| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_6}} \Bigg| \\
&\ll \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_5}\log\left(\frac{lX}{L}\right)} \Bigg| \Bigg| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_5}} \Bigg| \\
&\quad + \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_5}\log^2(z_0 l)} \Bigg| \Bigg| \sum_{\mu y/2L<d\le z_0} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_5}} \Bigg| \log X \\
&\quad + \Bigg| \sum_{l\sim L} \sum_{\mu y/2L<d\le X/L} \frac{\Lambda(l)a(d)\chi(dl)}{(dl^2)^{\varkappa+i\gamma_6}} \Bigg| ,
\end{aligned} \tag{56}$$

for some $|\gamma_5|, |\gamma_6| \le T$ and for some $z_0 \in [\mu y/2L, X/L]$. Summarizing (27), (31), (32), (37), (38), (39), (42), (44) and (56) we find

$$\begin{aligned}
U_4(y,\chi,t) &\ll \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{\varkappa+i\gamma_1}} \Bigg| \Bigg| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_1}} \Bigg| \log^2 X + X^{\frac14}u^{-1}\log^2 X + \log^2 X \\
&\ll \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_5}\log\left(\frac{lX}{L}\right)} \Bigg| \Bigg| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_5}} \Bigg| \log X \\
&\quad + \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_5}\log^2(z_0 l)} \Bigg| \Bigg| \sum_{\mu y/2L<d\le z_0} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_5}} \Bigg| \log^2 X \\
&\quad + \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_7}} \Bigg| \Bigg| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_7}} \Bigg| \log^2 X
\end{aligned} \tag{57}$$

for some $|\gamma_1|, |\gamma_5|, |\gamma_7| \le T$. Now (23) and (57) lead to

$$\Omega_2^{(4)} \ll \big(\Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + RX^{\frac14}u^{-1} + R\big)\log^2 X\,, \tag{58}$$

where

$$\Xi_1 = \sum_{R_0<r\le R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{*} \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{\varkappa+i\gamma_1}} \Bigg| \Bigg| \sum_{\mu y/l<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_1}} \Bigg| , \tag{59}$$

$$\Xi_2 = \sum_{R_0<r\le R} \frac{1}{\varphi(r)} \sum_{\chi(r)}^{*} \Bigg| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_5}\log\left(\frac{lX}{L}\right)} \Bigg| \Bigg| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_5}} \Bigg| , \tag{60}$$

$$\Xi_3 = \sum_{R_0<r\le R} \frac{1}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \left| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_5}\log^2(z_0 l)} \right| \left| \sum_{\mu y/2L<d\le z_0} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_5}} \right| , \tag{61}$$

$$\Xi_4 = \sum_{R_0<r\le R} \frac{1}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \left| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{2\varkappa+i2\gamma_7}} \right| \left| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_7}} \right| . \tag{62}$$

First we estimate Ξ_1. By (14), (21), (26), Cauchy's inequality, Lemma 3.7 and the well known inequalities

$$\sum_{n\le X} \Lambda^2(n) \ll X\log X\,, \quad \sum_{n\le X} \tau^2(n) \ll X\log^3 X$$

we derive

$$\begin{aligned}
&\sum_{R_0<r\le R} \frac{r}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \left| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{\varkappa+i\gamma_1}} \right| \left| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_1}} \right| \\
&\ll \left(\sum_{R_0<r\le R} \frac{r}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \left| \sum_{l\sim L} \frac{\Lambda(l)\chi(l)}{l^{\varkappa+i\gamma_1}} \right|^2 \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{R_0<r\le R} \frac{r}{\varphi(r)} \sideset{}{^*}\sum_{\chi(r)} \left| \sum_{\mu y/2L<d\le X/L} \frac{a(d)\chi(d)}{d^{\varkappa+i\gamma_1}} \right|^2 \right)^{\frac{1}{2}} \\
&\ll \left(L+R^2\right)^{\frac{1}{2}} \left(\frac{X}{L}+R^2\right)^{\frac{1}{2}} \left(\sum_{l\sim L} \Lambda^2(l) \right)^{\frac{1}{2}} \left(\sum_{\mu y/2L<d\le X/L} \tau^2(d) \right)^{\frac{1}{2}} \\
&\ll \left(X + XRu^{-\frac{1}{2}} + X^{\frac{1}{2}}R^2\right)\log^2 X\,.
\end{aligned} \tag{63}$$

Now (14), (59), (63) and Abel's summation formula give us

$$\Xi_1 \ll \left(XR_0^{-1} + Xu^{-\frac{1}{2}}\log X + X^{\frac{1}{2}}R\right)\log^2 X\,. \tag{64}$$

Proceeding in the same way for the sums (60)–(62) we deduce

$$\Xi_2,\, \Xi_3,\, \Xi_4 \ll \left(XR_0^{-1} + Xu^{-\frac{1}{2}}\log X + X^{\frac{1}{2}}R\right)\log^2 X\,. \tag{65}$$

Bearing in mind (58), (64) and (65) we obtain

$$\Omega_2^{(4)} \ll \left(XR_0^{-1} + Xu^{-\frac{1}{2}}\log X + X^{\frac{1}{2}}R\right)\log^4 X\,. \tag{66}$$

Arguing as with $\Omega_2^{(4)}$ we get

$$\Omega_2^{(3)} \ll \left(XR_0^{-1} + Xu^{-\frac{1}{2}}\log X + X^{\frac{1}{2}}R\right)\log^4 X\,. \tag{67}$$

From (22), (24), (25), (66) and (67) we find

$$\Omega_2 \ll X^{\frac{1}{4}}uR^{\frac{3}{2}}\log^2 X + \left(XR_0^{-1} + Xu^{-\frac{1}{2}}\log X + X^{\frac{1}{2}}R\right)\log^4 X\,. \tag{68}$$

Using (11), (15) and (68) we derive

$$\Sigma'' \ll X^{\frac{1}{4}} u R^{\frac{3}{2}} \log^3 X + \left(X R_0^{-1} + X u^{-\frac{1}{2}} \log X + X^{\frac{1}{2}} R\right) \log^5 X + \frac{X}{\log^A X} \,. \tag{69}$$

Apparently for each $B > 0$, the theorem is true for $y = \log^B X$. Therefore we can choose

$$u = (\log X)^{2A+12} \tag{70}$$

Taking into account (7), (10), (14), (69) and (70) we deduce

$$\Sigma \ll \frac{X}{\log^A X} \,. \tag{71}$$

Summarizing (2), (3), (6) and (71) we establish Theorem 1.1.

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