Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 3, 471–480
DOI: 10.7546/nntdm.2025.31.3.471-480

Recursive sufficiency for the Collatz conjecture and computational verification

Mohammad Ansari

Department of Mathematics, Azad University of Gachsaran
Gachsaran, Iran
e-mail: `mo.ansari@iau.ac.ir`

Received: 13 December 2024 **Revised:** 24 July 2025
Accepted: 28 July 2025 **Online First:** 4 August 2025

Abstract: We define the notion of recursive sufficiency for the Collatz conjecture and we use it to present some results concerning the computational verification of the conjecture. For any integer $N \geq 1$ and any recursively sufficient set F, it is proved that all integers in the interval $[1, N]$ satisfy the conjecture if and only if $F \cap [1, N]$ satisfies the conjecture. We offer a sequence of sieves for which the corresponding sequence of elimination percentages tends to 100%, and as a result, for any integer P arbitrarily close to 100, we give a sieve whose elimination percentage is at least $P\%$. Also, we prove that if $N = 2(3^n) + 1$ is the largest known integer for which all integers $1, 2, \ldots, N$ satisfy the conjecture, then all integers $N + 1, N + 2, \ldots, 2N$ will satisfy the conjecture as well, and hence, they can be eliminated from the verification process.
Keywords: Collatz conjecture, Recursive sufficiency, Computational verification.
2020 Mathematics Subject Classification: 11A99, 11Y55.

1 Introduction

The Collatz conjecture, also known as the $3x + 1$ problem, is one of the most famous open problems in mathematics which says that repeating two simple arithmetic operations will eventually transform every positive integer into 1. More precisely, if $\mathbb{N}$ is the set of all positive integers, then the Collatz conjecture can be formulated as follows: If $C : \mathbb{N} \to \mathbb{N}$ is the map defined by

$$C(n) = \begin{cases} 3n+1 & \text{for odd } n, \\ n/2 & \text{for even } n, \end{cases}$$

then there is some $k \in \mathbb{N}$ such that $C^k(n) = 1$. For example, let us try the integers 1, 3 and 14:

- $1: \quad 1, 4, 2, 1.$
- $3: \quad 3, 10, 5, 16, 8, 4, 2, 1.$
- $14: \quad 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$

While computational verification has shown that the conjecture holds true for all positive integers up to 1.5×2^{70} [2], no mathematical proof has been offered. We refer the reader to [1,4–6] for more information on the studies done on the problem. The problem is named after the mathematician Lothar Collatz, who introduced the idea in 1937.

Since $C(n)$ is even for any odd integer n, the problem is often reformulated by using the map T on $\mathbb{N}$ defined by

$$T(n) = \begin{cases} (3n+1)/2 & \text{for odd } n, \\ n/2 & \text{for even } n. \end{cases}$$

Then the conjecture says that, for each $n \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $T^k(n) = 1$.

For a pair of positive integers n, m, we say that n *merges* with m and we write $n \leftrightarrow m$, whenever there are some $i, j \geq 0$ for which $T^i(n) = T^j(m)$ (note that $T^0(p) = p$ for all $p \in \mathbb{N}$). It is clear that if $n \leftrightarrow m$ then n satisfies the conjecture if and only if m satisfies it.

In [7], the authors show that, for all pairs of integers $B > 0, A \geq 0$, the set $B\mathbb{N}_0 + A$ is *sufficient* for the Collatz conjecture, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, that is, for any $n \in \mathbb{N}$, there is some $m \in B\mathbb{N}_0 + A$ such that n merges with m. We believe that we had better call this property the *merge sufficiency* and offer the notion of *sufficiency* in another way. We bring the two notions in the following definition. Note that, when we say a subset E of $\mathbb{N}$ satisfies the conjecture, we mean that all members of E satisfy it.

Definition 1.1. *A nonempty subset E of $\mathbb{N}$ is called sufficient for the Collatz conjecture provided that, E satisfies the conjecture if and only if the conjecture is true. A nonempty set E is called merge sufficient whenever, for any $n \in \mathbb{N}$, there is some $m \in E$ such that n merges with m ($n \leftrightarrow m$).*

It is clear that merge sufficiency implies sufficiency, but we will see later that the converse is true if and only if the conjecture is true.

In the computational verification of the Collatz conjecture, the relevant algorithms do not need to check all integers, for example, since the conjecture is true if and only if it is true for all odd integers, it is not necessary to consider even integers as *starting values* (input integers which should be checked). Thus 50% of all integers are *eliminated* from the verification process. As another instance, since for any $n \in \mathbb{N}_0$, we have that $2n+1 \leftrightarrow 3n+2$ ($T(2n+1) = 3n+2$), the algorithms can skip all integers of the form $3n+2$ and only evaluate all odd integers having the forms $3n$ or $3n+1$, that is all integers of the forms $6n+3$ and $6n+1$. Hence, for any block of 6 consecutive integers, it suffices to evaluate only 2 of them. In other words, we can eliminate at least 66.6% of all starting values (the exact percentage is $\frac{4}{6} \times 100$). Thus the notion of a *sieve* which is used in computational verification of the conjecture, is in fact a synonym for the notion of a sufficient set for the conjecture. It may come together with the notion of *elimination percentage* which is defined as follows.

Definition 1.2. *Let S be a sieve for the Collatz conjecture. If there are positive integers m, n such that, for any block of n consecutive integers, exactly m of them should be checked (or equivalently, $n - m$ of them could be eliminated), we say that the elimination percentage of S is* $\mathrm{ep}(S) = 100(\frac{n-m}{n})\%$.

In this article, we define the notion of *recursive sufficiency* for the Collatz conjecture and use it to offer a sequence of sieves for which the corresponding sequence of elimination percentages tends to 100%. Thus, for any integer P arbitrarily close to 100, we can offer a sieve whose elimination percentage is at least $P\%$.

Definition 1.3. *A positive integer $n > 1$ is said to be recursive if there is some positive integer $m < n$ such that $m \leftrightarrow n$. A subset R of $\mathbb{N}$ is called recursive whenever n is recursive for every $1 < n \in R$. Finally, a proper subset F of $\mathbb{N}$ is said to be recursively sufficient if $\mathbb{N} \backslash F$ is recursive.*

Example 1.1. *It is easily seen that the sufficient set $F = 2\mathbb{N}_0 + 1$ is in fact recursively sufficient. Indeed, if $1 < n \notin F$ and we take $m = \frac{n}{2}$, then it is clear that $m < n$ and $m \leftrightarrow n$. As another sample, we show that $F_0 = 4\mathbb{N}_0 + 3$ is recursively sufficient. Suppose $1 < n \notin F$. If n is even then it merges with $m = \frac{n}{2} < n$. If n is odd then $n = 4k + 1$ for some $k \geq 0$. Then, for $m = 3k + 1$, we have that $m < n$ and $m \leftrightarrow n$ ($T^2(n) = m$).*

We see that the recursive sufficiency of the sufficient set $F = 2\mathbb{N}$ is equivalent to the truth of the Collatz conjecture.

Proposition 1.1. *The sufficient set $F = 2\mathbb{N}$ is recursively sufficient if and only if the Collatz conjecture is true.*

Proof. Assume that the Collatz conjecture is true, i.e., for every $n \in \mathbb{N}$ we have that $n \leftrightarrow 1$. Now let n be an arbitrary integer satisfying $1 < n \notin F$. Then, by our assumption, $n \leftrightarrow 1$. Thus, if we take $m = 1$, then $m < n$ and $m \leftrightarrow n$. This shows that F is recursively sufficient.

To prove the converse, assume that F is recursively sufficient. To show that the Collatz conjecture is true, since $2\mathbb{N}+1$ is a sufficient set, we only need to show that all integers $n \in 2\mathbb{N}+1$ satisfy the conjecture. We use that complete mathematical induction to achieve our goal. For $n = 3$ the assertion of the conjecture holds true. Suppose that $3 < n \in 2\mathbb{N} + 1$ and the conjecture is true for all odd integers i satisfying $1 \leq i < n$. We show that the conjecture is true for n. Since $n \notin F$ and F is recursively sufficient (by our assumption), there is some positive integer $m < n$ such that $m \leftrightarrow n$. Now, if m is odd, then $m \leftrightarrow 1$ by our inductive assumption, and so, $n \leftrightarrow 1$. Otherwise, there is an integer $s \geq 1$ and an odd integer $p \geq 1$ such that $m = 2^s p$. But then $p < n$ and since $p \leftrightarrow m$, we have that $p \leftrightarrow n$. On the other hand, $p \leftrightarrow 1$ by our inductive assumption, and hence, $n \leftrightarrow 1$. $\square$

It is natural to talk about the relationship between the properties of sufficiency and recursive sufficiency.

Proposition 1.2. *Every nonempty recursively sufficient set is sufficient. The converse is true if and only if the Collatz conjecture is true.*

Proof. Suppose $F \subsetneq \mathbb{N}$ is recursively sufficient. To show that F is sufficient, assume that all members of F satisfy the Collatz conjecture. We need to prove that all positive integers satisfy the conjecture. We use the complete mathematical induction to prove this assertion. It is clear that $n = 1$ satisfies the conjecture. Assume that $n > 1$ and for all integers $1 \leq j < n$ we have that $j \leftrightarrow 1$. Now we need to show that $n \leftrightarrow 1$. If $n \in F$ then $n \leftrightarrow 1$ by our assumption. On the other hand, if $n \in \mathbb{N}\backslash F$ then, by the recursive sufficiency of F, there is some $m \in \mathbb{N}$ such that $m < n$ and $m \leftrightarrow n$. But $m < n$ implies that $m \leftrightarrow 1$ by our inductive assumption, and hence, we have that $n \leftrightarrow 1$.

To prove the second assertion of the proposition, first assume that the Collatz conjecture is true. Then it is easily seen that, not only every sufficient set, but also every subset of $\mathbb{N}$ is recursively sufficient. Now assume that the Collatz conjecture is not true. Then the sufficient set $2\mathbb{N}$ is not recursively sufficient by Proposition 1.1, and we are done. □

We need the following lemma to talk about the implication "sufficiency ⇒ merge sufficiency".

Lemma 1.1. *The intersection of any family of recursively sufficient sets is recursively sufficient.*

Proof. Let $\mathcal{J}$ be an arbitrary index set and let $\{F_\alpha : \alpha \in \mathcal{J}\}$ be a family of recursively sufficient sets. To show that $\mathcal{F} = \bigcap_{\alpha \in \mathcal{J}} F_\alpha$ is recursively sufficient, suppose that $1 < n \notin \mathcal{F}$. Then there is some $\beta \in \mathcal{J}$ such that $n \notin F_\beta$, and since F_β is recursively sufficient by our assumption, there is some $m < n$ such that $m \leftrightarrow n$, and hence, $\mathcal{F}$ is recursively sufficient. □

Proposition 1.3. *Sufficiency implies merge sufficiency if and only if the conjecture is true.*

Proof. If the conjecture is true then $n \leftrightarrow m$ for any pair $n, m \in \mathbb{N}$, and hence, every nonempty subset of $\mathbb{N}$ is merge sufficient. To prove the converse, assume that sufficiency implies merge sufficiency. Let $\mathcal{F}$ be the intersection of all recursively sufficient sets. Then $\mathcal{F}$ is recursively sufficient by Lemma 1.1. We show that $\mathcal{F} = \varnothing$. To get a contradiction, assume that $\mathcal{F} \neq \varnothing$. Then $\mathcal{F}$ is sufficient by Proposition 1.2, and so, it is merge sufficient by our assumption. Hence there is some $m \in \mathcal{F}$ such that $1 \leftrightarrow m$. Now it is clear that $F = \mathcal{F}\backslash\{m\}$ is also recursively sufficient. But then the definition of $\mathcal{F}$ shows that $\mathcal{F} \subseteq F$ which says that $m \notin \mathcal{F}$. This contradiction shows that we must have $\mathcal{F} = \varnothing$.

Now we show that the Collatz conjecture is true. By the definition of recursive sufficiency we have that $\mathbb{N}\backslash\mathcal{F} = \mathbb{N}$ is recursive, i.e., for any $1 < n \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $m < n$ and $m \leftrightarrow n$. Now, to get a contradiction, assume that the Collatz conjecture is not true and let n_1 be the smallest positive integer which does not satisfy the conjecture. Then it is clear that $n_1 > 1$, and so, there is some $m_1 \in \mathbb{N}$ such that $m_1 < n_1$ and $m_1 \leftrightarrow n_1$. But m_1 satisfies the conjecture, and hence, n_1 should satisfy the conjecture as well, which is a contradiction. Therefore, the Collatz conjecture is true. □

We see that, with an extra weak assumption, recursive sufficiency implies merge sufficiency.

Proposition 1.4. *Let F be a recursively sufficient set. Then F is merge sufficient if and only if there is some $p \in F$ which satisfies the Collatz conjecture.*

Proof. Suppose that there is some $p \in F$ which satisfies the conjecture. To get a contradiction, assume that F is not merge sufficient and let n be the smallest integer which does not merge with any member of F. Then it is obvious that $n > 1$ (because $1 \leftrightarrow p$), and since F is recursively sufficient by our hypothesis, there is some $m < n$ such that $m \leftrightarrow n$. But the choice of n shows that there is some $k \in F$ such that $k \leftrightarrow m$. Then we have that $n \leftrightarrow k$ which is a contradiction. Thus F must be merge sufficient.

To prove the converse, suppose that F is merge sufficient. Then any positive integer merges with some element in F. In particular, there is some $p \in F$ such that $1 \leftrightarrow p$, and so, p satisfies the conjecture. □

As a direct consequence of the above proposition, we give the following corollary.

Corollary 1.1. *If F is a recursively sufficient set and $p \in \mathbb{N}$ satisfies the Collatz conjecture then $F \cup \{p\}$ is merge sufficient.*

In Section 2, we prove that if F is recursively sufficient then, for any $N \in \mathbb{N}$, the set

$$\mathbb{S}_N = \{1, 2, 3, \ldots, N\}$$

satisfies the Collatz conjecture if and only if $F \cap \mathbb{S}_N$ satisfies the conjecture. In Section 3, we offer a sequence $(F_n)_{n=0}^{\infty}$ of recursively sufficient sets for which the corresponding sequence of elimination percentages tends to 100%. Then we use the recursively sufficient set $F = \bigcap_{n=0}^{\infty} F_n$ to show that if $N = 2(3^n) + 1$ is the largest known integer for which the set $\mathbb{S}_N$ satisfies the conjecture, then all integers in the interval $(N, 2N]$ will satisfy the conjecture as well, and hence, they can be eliminated from the verification process.

2 Recursive sufficiency and computational verification of the conjecture

Assume that for some $N \in \mathbb{N}$, all members of $\mathbb{S}_N$ have successfully passed the computational verification process of the Collatz conjecture and let F be a recursively sufficient set. If we want to apply the verification process to all integers in the interval $(N, M]$, where $M > N$ is any integer, we claim that we only need to apply the process to all members of $F \cap (N, M]$. This result is offered as Corollary 2.1 which is a direct consequence of the following theorem.

Theorem 2.1. *Let F be a recursively sufficient set and let $N \geq 1$ be any integer. Then $\mathbb{S}_N$ satisfies the Collatz conjecture if and only if $F \cap \mathbb{S}_N$ satisfies it. The assertion may not remain valid if we replace recursive sufficiency with merge sufficiency.*

Proof. The necessity is obvious, and so, we only prove the sufficiency. Assume that $F \cap \mathbb{S}_N$ satisfies the conjecture. We prove that $\mathbb{S}_N$ satisfies the conjecture by applying the complete mathematical induction. We know that $n = 1$ satisfies the conjecture. Suppose $1 < n \in \mathbb{S}_N$ and for all $1 \le j < n$ we have that $j \leftrightarrow 1$, i.e., j satisfies the conjecture. We show that n satisfies it as well. If $n \in F \cap \mathbb{S}_N$ then n satisfies the conjecture by our assumption. If $n \notin F$ then, since F is recursively sufficient, there exists some $m < n$ such that $m \leftrightarrow n$. But $m < n$ implies that $m \leftrightarrow 1$ by our inductive assumption, and hence, $n \leftrightarrow 1$.

To prove the second assertion of the theorem, we show that if the Collatz conjecture is not true then there is some integer N such that $2\mathbb{N} \cap \mathbb{S}_N$ satisfies the conjecture but $\mathbb{S}_N$ does not. To this end, assume that the Collatz conjecture is not true. Then, in view of Proposition 1.1, the merge sufficient set $2\mathbb{N}$ is not recursively sufficient. Now let $n_1 = \min C'$, where C' is the set of all positive integers which do not satisfy the conjecture. Then it is clear that n_1 is an odd integer, because otherwise, $1 \leftrightarrow \frac{n_1}{2} \leftrightarrow n_1$ which is a contradiction. Then $N = n_1 + 1$ is even, and so, $1 \leftrightarrow \frac{N}{2} \leftrightarrow N$ because $\frac{N}{2} < n_1$. Now it is readily seen that $2\mathbb{N} \cap \mathbb{S}_N$ satisfies the conjecture while S_N does not. □

Corollary 2.1. *Let F be a recursively sufficient set and let $1 < N < M$ be any integers. If both sets $\mathbb{S}_N$ and $F \cap (N, M]$ satisfy the conjecture then all integers in the interval $(N, M]$ satisfy the conjecture.*

Proof. In view of Theorem 2.1, $F \cap \mathbb{S}_M$ satisfies the conjecture if and only if $\mathbb{S}_M$ satisfies it. But, since by our assumption, $\mathbb{S}_N$ satisfies the conjecture, we see that $F \cap (N, M]$ satisfies the conjecture if and only if all integers in the interval $(N, M]$ satisfy it. □

A particular case for the above corollary can be presented as follows.

Corollary 2.2. *Let F be a recursively sufficient set and let $1 < N < M$ be any integers. If $\mathbb{S}_N$ satisfies the conjecture and $F \cap (N, M] = \varnothing$, then $\mathbb{S}_M$ satisfies the conjecture.*

Remark 2.1. *It is worth mentioning that the assertion of Theorem 2.1 may not remain valid if we replace the recursively sufficient set F with the merge sufficient set $E = B\mathbb{N} + A$ ($B > 0$, $A \ge 0$) because, in view of [7], no order-friendly argument is guaranteed in merging a given integer with some members of E. As a result, we cannot use the mathematical induction for the set $\mathbb{S}_N$.*

We finish this section by highlighting the fact that, in view of Corollary 2.1, if N is the largest known integer for which $\mathbb{S}_N$ satisfies the Collatz conjecture, F is a recursively sufficient set, and $M > N$ is any integer, then, to verify the conjecture for all integers in the interval $(N, M]$, it suffices to apply the computational verification process only to all members of $F \cap (N, M]$.

3 Making a sieve finer and finer

It is easily seen that, if F is recursively sufficient and $A \subseteq F$, then $F_1 = F \backslash A$ is recursively sufficient if and only if A is recursive. For example, for the recursively sufficient set $F_0 = 4\mathbb{N}_0+3$, we can write $F_0 = \bigcup_{a_0=0}^{2} \big(4(3\mathbb{N}_0 + a_0) + 3\big)$, and since $A = 12\mathbb{N}_0 + 11$ is a recursive subset of F_0 ($8n + 7 \leftrightarrow 12n + 11$), we have that the set

$$F_1 = F_0 \backslash A = (12\mathbb{N}_0 + 3) \cup (12\mathbb{N}_0 + 7) = \bigcup_{a_0=0}^{1} \big(4(3\mathbb{N}_0 + a_0) + 3\big)$$

is recursively sufficient.

Now, it is clear that if we consider F_0 and $F_1 \subsetneq F_0$ as sieves for the computational verification of the Collatz conjecture, then F_1 is *finer* than F_0, that is, $\mathrm{ep}(F_1) > \mathrm{ep}(F_0)$. Thus we have made a finer sieve F_1 by removing the recursive subset $A = 12\mathbb{N}_0 + 11$ from F_0. We will use this pattern to obtain other recursively sufficient sets F_n $(n \geq 2)$ such that $F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq \cdots$. In other words, we will repeatedly make the sieve F_0 finer and finer.

Let us define F_n $(n \geq 0)$ as follows: $F_0 = 4\mathbb{N}_0 + 3$ and, for $n \geq 1$,

$$F_n = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-1}=0}^{1} \big(4(3^n\mathbb{N}_0) + 4(3^{n-1}a_{n-1}) + \cdots + 4a_0 + 3\big). \tag{1}$$

Then, it is obvious that $F_1 = (12\mathbb{N}_0 + 3) \cup (12\mathbb{N}_0 + 7)$.

Note that in (1) we had better write $a_{j,n}$ for a_j $(0 \leq j \leq n-1)$ to show the dependence of these scalars to n in F_n, but, for simplicity in notations and since we will not face any ambiguity in our computations, we write a_j instead of $a_{j,n}$.

Now we prove that $F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq \cdots$, and that each F_n $(n \geq 0)$ is a recursively sufficient set.

Lemma 3.1. *For the sets F_n $(n \geq 0)$ defined in* (1), *we have that $F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq \cdots$. Moreover, each F_n $(n \geq 0)$ is a recursively sufficient set.*

Proof. It is clear that $\mathbb{N}_0 = \bigcup_{i=0}^{2}(3\mathbb{N}_0 + i)$. Thus, if we use this fact in (1), then we have that

$$F_n = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-1}=0}^{1} \bigcup_{a_n=0}^{2} \big(4(3^n(3\mathbb{N}_0 + a_n)) + 4(3^{n-1}a_{n-1}) + \cdots + 4a_0 + 3\big),$$

which can also be written as

$$F_n = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-1}=0}^{1} \bigcup_{a_n=0}^{2} \big(4(3^{n+1}\mathbb{N}_0) + 4(3^n a_n) + 4(3^{n-1}a_{n-1}) + \cdots + 4a_0 + 3\big). \tag{2}$$

Now, in view of (1) (with $n+1$ instead of n) and (2), it is clear that $F_{n+1} \subsetneq F_n$, and hence, the first assertion of the lemma is proved.

Now we prove that each F_n $(n \geq 0)$ is recursively sufficient. We use the complete mathematical induction to achieve our goal. For $n = 0$ and $n = 1$ we have that $F_0 = 4\mathbb{N}_0 + 3$ and $F_1 = (12\mathbb{N}_0 + 3) \cup (12\mathbb{N}_0 + 7)$, and we have already shown that these two sets are recursively sufficient. Now suppose that $n \geq 1$ and F_j is recursively sufficient for all $1 \leq j \leq n$. We show that then F_{n+1} would also be recursively sufficient. Let us replace n with $n-1$ in (2) to obtain the following identity:

$$F_{n-1} = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-2}=0}^{1} \bigcup_{a_{n-1}=0}^{2} \big(4(3^n\mathbb{N}_0) + 4(3^{n-1}a_{n-1}) + 4(3^{n-2}a_{n-2}) + \cdots + 4a_0 + 3\big). \tag{3}$$

Now, if we compare (1) and (3), we will observe that $F_n = F_{n-1} \backslash A$ where A is the following subset of F_{n-1}:

$$A = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-2}=0}^{1} \big(4(3^n\mathbb{N}_0) + 8(3^{n-1}) + 4(3^{n-2}a_{n-2}) + \cdots + 4a_0 + 3\big).$$

In fact, A is the yield of choosing $a_{n-1} = 2$ in (3). Now, since both F_n and F_{n-1} are recursively sufficient by our inductive assumption, we see that A is a recursive set. Thus, for any typical element $m \in A$ with the form

$$m = 4(3^n k) + 8(3^{n-1}) + 4(3^{n-2}a_{n-2}) + \cdots + 4a_0 + 3 \ \ (k \in \mathbb{N}_0),$$

there is some positive integer $f(m) < m$ such that $f(m) \leftrightarrow m$. Now, let us consider a particular subset A' of A comprising of all integers $m \in A$ for which $k = 3p + 2$ ($p \in \mathbb{N}_0$). Then any $m \in A'$ has the form

$$m = 4(3^{n+1}p) + 8(3^n) + 8(3^{n-1}) + 4(3^{n-2}a_{n-2}) + \cdots + 4a_0 + 3.$$

Now it is clear that A' is defined by

$$A' = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-2}=0}^{1} \big(4(3^{n+1}\mathbb{N}_0) + 8(3^n) + 8(3^{n-1}) + 4(3^{n-2}a_{n-2}) + \cdots + 4a_0 + 3\big),$$

and evidently, A' is also a recursive set. But A' is a subset of F'_n, where

$$F'_n = \bigcup_{a_0=0}^{1} \cdots \bigcup_{a_{n-2}=0}^{1} \bigcup_{a_{n-1}=0}^{2} \bigcup_{a_n=0}^{2} \big(4(3^{n+1}\mathbb{N}_0) + 4(3^n a_n) + 4(3^{n-1}a_{n-1}) + 4(3^{n-2}a_{n-2}) + \cdots + 4a_0 + 3\big).$$

In fact, A' is obtained by taking $a_{n-1} = a_n = 2$ in the above identity. On the other hand, F'_n is recursively sufficient because $F_n \subseteq F'_n$ (see (2)) and F_n is recursively sufficient (by our inductive assumption). Therefore, $F'_n \backslash A'$ is also recursively sufficient. But, in view of (1) with $n+1$ instead of n, it is clear that $F_{n+1} = F'_n \backslash A'$, and hence, F_{n+1} is recursively sufficient. $\square$

Now we compute the elimination percentages of the sieves F_n ($n \geq 0$).

Proposition 3.1. *For any* $n \geq 0$ *we have that* $\mathrm{ep}(F_n) = 100(\frac{4(3^n)-2^n}{4(3^n)})\%$.

Proof. We claim that, for any fixed non-negative integer n, any block of $4(3^n)$ consecutive positive integers contains exactly 2^n elements of F_n. In fact, we use the mathematical induction to prove this claim. For $n = 0$, any block of $4 = 4(3^0)$ consecutive integers contains $1 = 2^0$ integer of the form $4k + 3$. Assume that the assertion is true for $n > 0$. We show that then it is also true for $n + 1$. Let B be a block of $4(3^{n+1})$ consecutive positive integers. Since $4(3^{n+1}) = 3(4(3^n))$, the block B contains $3(2^n)$ members of F_n by our inductive assumption. On the other hand, one-third of these members are removed in the process of obtaining F_{n+1} from F_n (see (3) and the three lines following it in the proof of Lemma 3.1). Therefore, two-third of them are members of F_{n+1}, and so, F_{n+1} has 2^{n+1} members among the integers in B and the assertion is proved. Now, in view of Definition 1.2, the elimination percentage of F_n is $\mathrm{ep}(F_n) = 100(\frac{4(3^n)-2^n}{4(3^n)})\%$, and hence, we are done. $\square$

It is clear that (the sequence $(\mathrm{ep}(F_n))_n$ is strictly increasing and) $\mathrm{ep}(F_n) \to 100\%$ as $n \to \infty$. This means that, for any integer P arbitrarily close to 100, we have a sieve F_N for which $\mathrm{ep}(F_n) \geq P\%$ for all $n \geq N$. For example, if $P = 99.999$, it is easily seen that $\mathrm{ep}(F_n) > P\%$ for any $n \geq 25$. So, in the computational verification of the conjecture, if we only check the elements of F_{25} then we can claim that we are checking less than 0.001% of all integers.

We will use the following lemma in the proof of Proposition 3.2.

Lemma 3.2. *Let F_n ($n \geq 0$) be the recursively sufficient sets obtained in Lemma* 3.1*. Then we have that*

$$\bigcap_{n=0}^{\infty} F_n = \{4(3^n) + 4(3^{n-1})a_{n-1} + \cdots + 4(3a_1) + 4a_0 + 3 : n \geq 0, a_i \in \{0,1\}, 0 \leq i \leq n-1\} \tag{4}$$

Proof. Let $F = \bigcap_{n=0}^{\infty} F_n$ and E be the right hand side of (4). First we show that $E \subseteq F$. Let $e \in E$ be arbitrary. Then there is some $N \geq 0$ such that

$$e = 4(3^N) + 4(3^{N-1})a_{N-1} + \cdots + 4(3a_1) + 4a_0 + 3.$$

It is clear that $e \in F_n$ for all $0 \leq n \leq N$ (remember that $F_0 \supseteq F_1 \supseteq \cdots$). Now, for any $n > N$, we can write

$$e = 4(3^n)(0) + \cdots + 4(3^{N+1})(0) + 4(3^N)(1) + 4(3^{N-1})a_{N-1} + \cdots + 4(3a_1) + 4a_0 + 3,$$

which shows that $e \in F_n$ (see (1)), and since $n > N$ was arbitrary, we deduce that $e \in F$.

Now we show that $F \subseteq E$. Pick an arbitrary element $f \in F$. Then $f \in F_n$ for all $n \geq 0$, and hence, for any $n \geq 0$, there exists a non-negative integer k_n such that

$$f = 4(3^n)k_n + 4(3^{n-1})a_{n-1} + \cdots + 4(3a_1) + 4a_0 + 3 > 4(3^n)k_n$$

Now it is clear that there must exist some $N > 0$ such that $k_n = 0$ for all $n \geq N$. Then $f \in F_N$ and $k_N = 0$ imply that

$$f = 4(3^{N-1})a_{n-1} + \cdots + 4(3a_1) + 4a_0 + 3.$$

Now assume that $0 \leq m \leq N-1$ is the largest integer for which $a_m \neq 0$. Then $a_m = 1$, and so,

$$f = 4(3^m) + 4(3^{m-1})a_{m-1} + \cdots + 4a_0 + 3.$$

This shows that $f \in E$ and we are done. □

As a consequence of Corollary 2.2 and Lemmas 1.1, 3.1, and 3.2, we offer the following result.

Proposition 3.2. *Let $N = 2(3^n) + 1$ be the largest known integer for which $\mathbb{S}_N$ satisfies the Collatz conjecture. Then all integers in the interval $(N, 2N]$ will satisfy the conjecture as well.*

Proof. By Lemmas 1.1, 3.1, and 3.2, the set

$$
\begin{aligned}
F &= \bigcap_{n=0}^{\infty} F_n \\
&= \{4(3^n) + 4(3^{n-1})a_{n-1} + \cdots + 4(3a_1) + 4a_0 + 3 : n \geq 0, a_i \in \{0,1\}, 0 \leq i \leq n-1\}
\end{aligned}
$$

is recursively sufficient. Now, since

$$N = 4(3^{n-1}) + 4(3^{n-2}) + \cdots + 4(3) + 4 + 3,$$

we observe that $N \in F$. Then the smallest member of F which is greater than N would be $4(3^n) + 3$, and hence, we have that $(N, 2N] \cap F = \varnothing$. Now, by Corollary 2.2, all integers in the interval $(N, 2N]$ will satisfy the conjecture. □

Remark 3.1. *Before the last stage of preparing this paper for publication, we noticed that the website in [2] has been updated recently and the new verification limit* 2^{71} *has been achieved (see also [3]). Now, since* $2(3^{44}) + 1 < 2^{71} < 4(3^{44}) + 2$*, a superficial glance at the statement of Proposition* 3.2 *shows that all integers in the interval* $[2^{71}, 4(3^{44}) + 2]$ *will satisfy the conjecture, and hence, the number* $4(3^{44}) + 2 \approx 1.66 \times 2^{71}$ *would be an upgraded verification limit.*

Acknowledgements

We are very grateful to the reviewers and editors who carefully read this paper.

References

[1] Barina, D. (2021). Convergence verification of the Collatz problem. *The Journal of Supercomputing*, 77(3), 2681–2688.

[2] Barina, D. (2025). Convergence verification of the Collatz problem. Available online at: `https://pcbarina.fit.vutbr.cz/`

[3] Barina, D. (2025). Improved verification limit for the convergence of the Collatz conjecture. *The Journal of Supercomputing*, 81, Article ID 810.

[4] Lagarias, J. C. (1985). The $3x + 1$ problem and its generalizations. *The American Mathematical Monthly*, 92(1), 3–23.

[5] Lagarias, J. C. (2009). *The* $3x + 1$ *problem: An Annotated bibliography, II (2000-2009).* ArXiv. Available online at: `https://arxiv.org/abs/math/0608208v5`

[6] Lagarias, J. C. (2021). *The* $3x + 1$ *problem: An overview*. ArXiv. Available online at: `https://arxiv.org/abs/2111.02635`

[7] Monks, K. M. (2006). The sufficiency of arithmetic progressions for the $3x + 1$ conjecture. *Proceedings of the American Mathematical Society*, 134(10), 2861–2872.