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On sums of k-generalized Fibonacci and k-generalized Lucas numbers as first and second kinds of Thabit numbers

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Abstract: Let $(F_r^{(k)})_{r\geq 2-k}$ and $(L_r^{(k)})_{r\geq 2-k}$ be generalizations of the Fibonacci and Lucas sequences, where $k \geq 2$. For these sequences the initial k terms are $0, 0, \ldots, 0, 1$ and $0, 0, \ldots, 2, 1$, and each subsequent term is the sum of the preceding k terms. In this paper, we determined all first and second kinds of Thabit numbers that can be expressed as the sums of k-Fibonacci and k-Lucas numbers. We employed the theory of linear forms in logarithms of algebraic numbers and a reduction method based on the continued fraction.

Keywords: Diophantine equations, Linear forms in logarithms, Generalized Fibonacci numbers, Generalized Lucas numbers, Reduction method.
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1 Introduction

Let $(T_r^{(k)})_{r\geq 2-k}$ be a k-order linear recurrence sequence, where $k \geq 2$ and defined as

$$T_r^{(k)} = T_{r-1}^{(k)} + T_{r-2}^{(k)} + \cdots + T_{r-k}^{(k)}, \quad \text{for all } r \geq 2,$$

with initial conditions $T^{(k)}_{-(k-2)} = T^{(k)}_{-(k-3)} = \cdots = T^{(k)}_{-1} = 0, T^{(k)}_0 = a, T^{(k)}_1 = b$. If $(a, b) = (0, 1)$, the sequence $(T^{(k)}_r)_{r\geq 2-k}$ becomes k-generalized Fibonacci and is denoted by $(F^{(k)}_r)_{r\geq 2-k}$. If $(a, b) = (2, 1)$, we obtain the k-generalized Lucas sequence and denote it as $(L^{(k)}_r)_{r\geq 2-k}$. In both cases, if $k = 2$ we obtain the Fibonacci and Lucas sequences.

The Thabit number refers to the Thâbit ibn Qurra number, which is also known as the 321 number. The first and second kinds of Thabit numbers are of the forms $3 \cdot 2^d - 1$ and $3 \cdot 2^d + 1$ for a non-negative integer d, respectively. The first and second kinds of Thabit numbers are represented by sequences A055010 and A181565 in the Online Encyclopedia of Integer Sequences (OEIS), respectively.

Several investigations have been conducted on perfect powers, Fermat, and Mersenne numbers, which can be expressed as generalized linear recurrence sequences. Bravo and Luca [7] showed that all k-generalized Fibonacci numbers can be written as powers of two. Later, Bravo *et al.* [3] determined the sums of two k-generalized Fibonacci numbers, which are powers of two. Furthermore, Bravo and Gómez [2] found all the common terms between Mersenne numbers and k-Fibonacci numbers. Bravo and Herrera [5] proved the connection between Fermat numbers and k-Fibonacci or k-Lucas numbers. In addition, they investigated generalized Pell sequences expressed as even perfect numbers [6]. Normenyo *et al.* [15] showed the Fermat and Mersenne numbers in k-Pell sequences.

Recently, Hernane *et al.* [13] studied the Fermat and Mersenne numbers expressible as a product of two k-Fibonacci numbers. Gueye *et al.* [12] determined the coincidence between k-Fibonacci numbers and the products of two Fermat numbers. Şiar and Keskin [18] worked on perfect powers, which can be written as a k-generalized Pell–Lucas sequence. Altassan and Alan [1] showed that all Mersenne numbers that can be written as generalized Lucas sequences. Şiar *et al.* [19] identified the common values of two k-generalized Pell sequences. This paper examines the first and second types of Thabit numbers expressed as sums of k-generalised Fibonacci and k-generalised Lucas sequences. Finally, we present the following results.

Theorem 1.1. *The only solutions of the Diophantine equation*

$$F^{(k)}_r + L^{(k)}_s = 3 \cdot 2^d \pm 1, \tag{1}$$

for non-negative integers r, s and d with $r > s \geq 2$ and $k \geq 2$, for the first kind are

$$(r, s, d, k) \in \left\{ \begin{array}{c} (3, 2, 1, \geq 2), (6, 2, 2, 2), (12, 8, 6, 2), (6, 4, 3, 3), \\ (8, 2, 4, 3), (5, 2, 2, \geq 4), (9, 7, 6, 4) \end{array} \right\},$$

and for the second kind are

$$(r, s, d, k) \in \{(4, 3, 1, 2), (8, 3, 3, 2), (4, 2, 1, \geq 3), (5, 3, 2, 3)\}.$$

In Theorem 1.1, we examined the case in which $r > s \geq 2$. Moreover, assuming that $r > s$ with $s \in \{0, 1\}$, Eq. (1) becomes $F^{(k)}_r + 2 = 3 \cdot 2^d \pm 1$ and $F^{(k)}_r + 1 = 3 \cdot 2^d \pm 1$. Consequently, instead of computing both equations, we derive the following theorem.

Theorem 1.2. *The only solutions of the Diophantine equation*

$$F_r^{(k)} = 3 \cdot 2^d - y, \tag{2}$$

in non-negative integers r, d *and* y *with* $y \in \{0, 1, 2, 3\}$ *and* $k \geq 2$ *are given by*

$$(r, d, y, k) \in \left\{ \begin{array}{c} (4,0,0,2), (7,3,0,3), (3,0,1,\geq 2), (5,1,1,2), (1,0,2,\geq 2), \\ (2,0,2,\geq 2), (4,1,2,\geq 3), (0,0,3,\geq 2), (4,1,3,2), (8,3,3,2) \end{array} \right\}.$$

2 Auxiliary results

In this section, we commence with several fundamental results from algebraic number theory.

2.1 Properties of k-Fibonacci and k-Lucas sequences

In this part, we examine the specific details and characteristics of both sequences, which will be used in subsequent sections. The characteristic polynomial of both k-order sequences is $\Psi(x) = x^k - x^{k-1} - \cdots - x - 1$, which constitutes an irreducible polynomial over $\mathbb{Q}[x]$. The polynomial $\Psi(x)$ possesses precisely one real root $\varphi(k)$ outside the unit circle, whilst the remaining roots are strictly contained within a unit circle. Moreover, we have $\varphi \in (2(1 - 2^{-k}), 2)$ for all $k \geq 2$. Bravo *et al.* [4] proved that the inequalities $1/2 < f_k(\varphi) < 3/4$ and $|f_k(\varphi_i)| < 1$ hold for all $2 \leq i \leq k$, where $f_k(x) = (x-1)/(2 + (k+1)(x-2))$ and φ_i represents the zeros of $\Psi_k(x)$. In addition, the number $f_k(\varphi)$ is not an algebraic integer. They proved that the inequality

$$h\left(f_k(\varphi)\right) < 3 \log k, \tag{3}$$

holds for all $k \geq 2$. In [8, 11] the authors presented the following results:

$$F_r^{(k)} = f_k(\varphi)\varphi^{r-1} + e_k(r), \quad \text{where} \quad |e_k(r)| < \frac{1}{2}, \tag{4}$$

and

$$L_r^{(k)} = (2\varphi - 1) f_k(\varphi)\varphi^{r-1} + e_k(r), \quad \text{where} \quad |e_k(r)| < \frac{3}{2}, \tag{5}$$

hold for all $r \geq 2 - k$ and $k \geq 2$. Furthermore, the authors [7, 8] determined that the inequalities

$$\varphi^{r-2} \leq F_r^{(k)} \leq \varphi^{r-1}, \tag{6}$$

and

$$\varphi^{r-1} \leq L_r^{(k)} \leq 2\varphi^r, \tag{7}$$

hold for $r \geq 1$ and $k \geq 2$. In addition, in [7, 8], we can find the following equations

$$F_r^{(k)} = 2^{r-2}, \quad \text{for all} \quad 2 \leq r \leq k+1, \tag{8}$$

and

$$L_r^{(k)} = 3 \cdot 2^{r-2}, \quad \text{for all} \quad 2 \leq r \leq k. \tag{9}$$

2.2 Linear forms in logarithms

Definition 2.1. *The absolute logarithmic height is denoted by $h(\Upsilon)$ and is defined by*

$$h(\Upsilon) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log \max\left\{\left|\Upsilon^{(i)}\right|, 1\right\}\right), \tag{10}$$

where $\Upsilon^{(i)}$ denotes the conjugates of the algebraic number Υ of degree d, and $c_0 > 0$ is the leading coefficient of the minimal polynomial of Υ given by $f(X) = c_0 \prod_{i=1}^{d}\left(X - \Upsilon^{(i)}\right) \in \mathbb{Z}[X]$.

The absolute logarithmic height of a rational number $\Upsilon = \frac{a}{b}$ is given by $h(\Upsilon) = \log(\max\{|a|, b\})$, where $b > 0$ and $\gcd(a, b) = 1$.

In the subsequent sections of this paper, the following properties of the logarithmic height function will be utilized:

$$h(\Upsilon \pm \Phi) \leq h(\Upsilon) + h(\Phi) + \log 2, \quad h\left(\Upsilon\Phi^{\pm 1}\right) \leq h(\Upsilon) + h(\Phi), \quad \text{and} \quad h\left(\Upsilon^{t}\right) = |t| h(\Upsilon).$$

The following theorem, which is a modified version of Matveev's result [14], was presented by Bugeaud *et al.* [9, Theorem 9.4].

Theorem 2.1. *Let $\Upsilon_1, \ldots, \Upsilon_t$ be positive real algebraic numbers in the number field $\mathbb{L}$ of degree D over $\mathbb{Q}$, and let $b_1, b_2, \ldots, b_t$ be nonzero integers. Let A_i be a positive real number that satisfies*

$$A_i \geq \max\left\{Dh\left(\Upsilon_i\right), \left|\log\left(\Upsilon_i\right)\right|, 0.16\right\}, \quad 1 \leq i \leq t,$$

and

$$B := \max\left\{\left|b_1\right|, \ldots, \left|b_t\right|\right\}.$$

If $\Lambda := \Upsilon_1^{b_1} \cdots \Upsilon_t^{b_t} - 1 \neq 0$, then

$$\log|\Lambda| > (-1.4)\left(30^{t+3}\right)\left(t^{4.5}\right)\left(D^2\right)\left(A_1 \cdots A_t\right)(1 + \log D)(1 + \log B).$$

2.3 The de Weger reduction method

We present a variation of Baker and Davenport's reduction method developed by de Weger [10] to reduce the upper bound. Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given and $x_1, x_2 \in \mathbb{Z}$ be unknowns.
Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{11}$$

Let c, δ be positive constants. We set $X = \max\left\{\left|x_1\right|, \left|x_2\right|\right\}$ and let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{12}$$

$$Y \leq X \leq X_0. \tag{13}$$

If $\beta \neq 0$ in Eq. (11), then, we get $\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2$, where $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Let p/q be a convergent of ϑ with $q > X_0$. The distance between a real number m and the nearest integer is denoted by $\|m\| = \min\{|m - n| : n \in \mathbb{Z}\}$. The following lemma is presented.

Lemma 2.1. ([10, Lemma 3.3]) *Suppose that* $\|q\psi\| > \frac{2X_0}{q}$*. Then the solutions of* (12) *and* (13) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2|\, X_0}\right).$$

In the subsequent, we present several lemmas that are fundamental to our calculations.

Lemma 2.2. ([17, Lemma 7]) *If* $c \geq 1$*,* $S > (4c^2)^c$*, and* $L/(\log L)^c < S$*, then* $L < 2^c S(\log S)^c$*.*

Lemma 2.3. ([10, Lemma 2.2]) *Let* v*,* $x \in \mathbb{R}$ *and* $0 < v < 1$*. If* $|x| < v$*, then*

$$|\log(1+x)| < \frac{-\log(1-v)}{v}|x|.$$

Lemma 2.4. ([4, Lemma 3]) *If* $r < 2^{\frac{k}{2}}$ *where* $r \geq k+2$*, then* $F_r^{(k)} = 2^{r-2}(1+\zeta)$*, where* $|\zeta| < \frac{2}{2^{k/2}}$*.*

Lemma 2.5. ([16, Lemma 2.6]) *If* $r < 2^{\frac{k}{2}}$ *where* $r \geq k+1$*, then* $L_r^{(k)} = 3\cdot 2^{r-2}(1+\zeta)$*, where* $|\zeta| < \frac{1}{2^{k/2}}$*.*

3 Main proof of Theorem 1.1

The proof of Theorem 1.1 is executed in the following subsection.

3.1 Case $r \leq k+1$, where $k \geq 2$

In this case, the following results were obtained.

Lemma 3.1. *The non-trivial solutions of the Diophantine equation*(1) *where* $2 \leq s < r \leq k+1$ *with* $k \geq 2$*, for the first kind are* $(r,s,d) \in \{(3,2,1),(5,2,2)\}$*, and for the second kind only is* $(r,s,d) \in \{(4,2,1)\}$*.*

Proof. Given that $2 \leq s < r \leq k+1$, by combining (8) and (9) with Eq. (1), we obtain $2^{r-2} + 2^{s-2} = 3\cdot 2^d \pm 1$. We shall consider the following two cases.

Case 1: For the first kind of Thabit number, the equation $2^{r-2} + 3\cdot 2^{s-2} = 3\cdot 2^d - 1$ is obtained. Given that $r > s$, it follows that $2^{s-2}(2^{r-s}+3) = 3\cdot 2^d - 1$. Consequently, $2^{s-2} = 1$ implies that $s = 2$. Thus, the equation $2^{r-2} = 3\cdot 2^d - 4$ is solvable only when $(r,d) \in \{(3,1),(5,2)\}$.

Case 2: For the second Thabit number, employing the same argument as previously discussed, we obtain $s = 2$ and $2^{r-2} = 3\cdot 2^d - 2$. The only possible solution for the equation $2^{r-2} = 3\cdot 2^d - 2$ is $(r,d) = (4,1)$. □

3.2 Case $r \geq k+2$, where $k \geq 2$

In this case, we determine the following results.

3.2.1 The connection between r and d

We have $r \geq k+2$ and $s < r$, combining inequalities (6) and (7) with Eq. (1) yields

$$2^{d+1} \leq 3 \cdot 2^d \pm 1 = F_r^{(k)} + L_s^{(k)} \leq \varphi^{r-1} + 2\varphi^s \leq \varphi^{r-1} + 2\varphi^{r-1} < 4 \cdot 2^{r-1} \leq 2^{r+1}.$$

This leads to $d < r$.

3.2.2 Finding an upper bound of r in terms of k

To determine this case, we obtain the following lemma.

Lemma 3.2. *If (r,s,d,k) is a solution in non-negative integers of Eq.* (1) *where $r \geq k+2$ with $k \geq 2$, then the following inequality holds*

$$r < 8.2 \cdot 10^{28} k^7 (\log k)^5. \tag{14}$$

Proof. Using (4) in Eq. (1), we obtain $f_k(\varphi)\varphi^{r-1} - 3 \cdot 2^d = -L_s^{(k)} - e_k(r) \pm 1$. Taking the absolute value, we get $\left| f_k(\varphi)\varphi^{r-1} - 3 \cdot 2^d \right| < 2\varphi^s + 1.5$. Dividing both sides by $f_k(\varphi)\varphi^{r-1}$ and using $1/2 < f_k(\varphi) < 3/4$, we obtain

$$|\Lambda_1| = \left| \frac{3 \cdot 2^d \varphi^{-(r-1)}}{f_k(\varphi)} - 1 \right| < \varphi^{s-r} \left(\frac{2\varphi}{f_k(\varphi)} + \frac{1.5}{f_k(\varphi)\varphi^{s-1}} \right) < \frac{10}{\varphi^{r-s}}. \tag{15}$$

We apply Theorem 2.1 with parameters

$$t := 3 \quad (\Upsilon_1, b_1) := (3/f_k(\varphi), 1), \quad (\Upsilon_2, b_2) := (\varphi, -(r-1)), \quad (\Upsilon_3, b_3) := (2, d).$$

It is clear that $\Upsilon_1, \Upsilon_2, \Upsilon_3$ are elements of $\mathbb{L} := \mathbb{Q}(\varphi)$ and $D := [\mathbb{L}, \mathbb{Q}] = k$. To demonstrate that $\Lambda_1 \neq 0$, we assume that $\Lambda_1 = 0$. Applying the conjugate by $\mathbb{Q}$-automorphism of the Galois extension $\mathbb{L} := \mathbb{Q}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$ and taking the absolute values gives $3 \leq 3 \cdot 2^d = |f_k(\varphi_i)||\varphi_i|^{r-1}| \leq 1$, which is contradictory. Hence, $\Lambda_1 \neq 0$. Furthermore, using the definition and properties of the logarithmic height with (3), we can estimate that

$$h(\Upsilon_1) \leq \log 3 + 3\log k < 5\log k, \quad \text{for all} \quad k \geq 2, \quad h(\Upsilon_2) = \log\varphi/k, \quad h(\Upsilon_3) = \log 2.$$

Thus, $A_1 := 5k\log k$, $A_2 := \log\varphi$ and $A_3 := k\log 2$.
Then, we take $B := r > \max\{1, |-(r-1)|, |d|\}$. According to Theorem 2.1, we get

$$10/\varphi^{r-s} > |\Lambda_1| > \exp\{-G(1+\log(r)(5k\log k)(\log\varphi)(k\log 2)\},$$

where $G = 1.4\,(30^6)\,(3^{4.5})\,(k^2)(1+\log(k))$, used the fact $1+\log k \leq 3\log k$ for all $k \geq 2$ and $1+\log r \leq 2\log r$ for all $r \geq 4$, we obtain

$$(r-s)\log\varphi < 2.1 \cdot 10^{12} k^4 (\log k)^2 (\log r). \tag{16}$$

Again, using (4) and (5), we can write Eq. (1) as

$$f_k(\varphi)\varphi^{r-1}(1 + (2\varphi - 1)\varphi^{s-r}) - 3 \cdot 2^d = -e_k(r) - e_k(s) \pm 1.$$

Taking the absolute value and dividing both sides by $f_k(\varphi)\varphi^{r-1}(1+(2\varphi-1)\varphi^{s-r})$, we obtain

$$|\Lambda_2| = \left|\frac{3\cdot 2^d\varphi^{-(r-1)}}{f_k(\varphi)(1+(2\varphi-1)\varphi^{s-r})} - 1\right| < \frac{3}{f_k(\varphi)\varphi^{r-1}(1+(2\varphi-1)\varphi^{s-r})} < \frac{12}{\varphi^r}. \tag{17}$$

As in the previous case, it can be seen that $t = 3$. We take

$$(\Upsilon_1, b_1) := \left(3/f_k(\varphi)(1+(2\varphi-1)\varphi^{s-r}), 1\right), \quad (\Upsilon_2, b_2) := (\varphi, -(r-1)), \quad (\Upsilon_3, b_3) := (2, d).$$

Thus, we have $D := [\mathbb{L}, \mathbb{Q}] = k$, $B := r$, $A_2 := \log\varphi$, $A_3 := k\log 2$. We shall estimate $h(\Upsilon_1)$ to determine A_1. From (16), (3), and the properties of the logarithmic height, we deduce the following:

$$h(\Upsilon_1) \leq \log 18 + 3\log k + (r-s)\frac{\log\varphi}{k} < 2.5\cdot 10^{12}k^3(\log k)^2(\log r).$$

Therefore, we can take $A_1 := 2.5\cdot 10^{12}k^4(\log k)^2(\log r)$. Thus, similar to the previous argument, it is clear that $\Lambda_2 \neq 0$. By Theorem 2.1, we get

$$12/\varphi^r > |\Lambda_2| > \exp\{-G(1+\log(r)(2.5\cdot 10^{12}k^4(\log k)^2(\log r))(\log\varphi)(k\log 2)\},$$

where $G = 1.4\,(30^6)\,(3^{4.5})\,(k^2)(1+\log k)$. It can be observed that

$$r\log\varphi < 1.1\cdot 10^{24}k^7(\log k)^3(\log r)^2.$$

After simplifying the calculation, we obtain $r/(\log r)^2 < 2.7\cdot 10^{24}k^7(\log k)^3$. Applying Lemma 2.2, we can put $S = 2.7\cdot 10^{24}k^7(\log k)^3$, $L = r$ and $c = 2$, and we get

$$r < 2^2(2.7\cdot 10^{24}k^7(\log k)^3)(\log(2.7\cdot 10^{24}k^7(\log k)^3))^2 < 8.2\cdot 10^{28}k^7(\log k)^5.$$

In the above calculation, we used the fact that $56.3 + 7\log k + 3\log\log k < 87\log k$ for all $k \geq 2$. □

3.2.3 Absolute upper bound of *r* on *k*

In this case, we aim to find the following lemma.

Lemma 3.3. *The Eq.* (1) *has no non-negative integer solutions, where* $r \geq k+2$ *with* $k > 340$.

Proof. Now, assume that $k > 340$, thus we have $s < r < 8.2\cdot 10^{28}k^7(\log k)^5 < 2^{k/2}$. From Lemmas 2.4 and 2.5 together with Eq. (1), we deduce that

$$\left|2^{r-2}(1+3\cdot 2^{s-r}) - 3\cdot 2^d\right| \leq 2^{r-2}|\zeta_r| + 3\cdot 2^{s-2}|\zeta_s| + 1.$$

Dividing by 2^{r-2}, we get

$$\left|(1+3\cdot 2^{s-r}) - 3\cdot 2^{d-(r-2)}\right| < \frac{2}{2^{k/2}} + \frac{3\cdot 2^{s-r}}{2^{k/2}} + \frac{1}{2^{r-2}} < \frac{4.5}{2^{k/2}},$$

where we used the fact that $r \geq k+2$ implies that $k \leq r-2$. It can be seen that the above inequality is not possible for $d < r$ since we have $0.125 < \left|(1+3\cdot 2^{s-r}) - 3\cdot 2^{d-(r-2)}\right|$. Nevertheless, the right-hand side is very small because $k > 340$. Therefore, Eq. (1) has no solutions where $k > 340$. □

3.2.4 Reducing the bound of r on k

Let $\chi_1 := \log(\Lambda_1+1) = -(r-1)\log\varphi + d\log 2 + \log\left(\frac{3}{f_k(\varphi)}\right)$. In (15), we assume that $r-s \geq 6$; consequently, $|\Lambda_1| < 0.9$. By Lemma 2.3, we obtain

$$|\chi_1| = -(\log(1-0.9)/0.9)\cdot(10/\varphi^{r-s}) < 25.6/\varphi^{r-s}.$$

It can be seen that

$$0 < |-(r-1)\log\varphi + d\log 2 + \log(3/f_k(\varphi))| < 25.6\exp(-(r-s)\log\varphi).$$

Applying Lemma 2.1, we have $\beta := \log(3/f_k(\varphi))$ and putting

$$c := 25.6, \quad \delta := \log\varphi, \quad \psi := \frac{\log(3/f_k(\varphi))}{\log 2},$$
$$\vartheta := \frac{\log\varphi}{\log 2}, \quad \vartheta_1 := -\log\varphi, \quad \vartheta_2 := \log 2.$$

From (14), we obtain $X_0 := 2.9\cdot 10^{50}$ for all $k \in [2, 340]$, which constitutes the upper bound for r and s. Analysis utilizing the *Maple* program demonstrated that the maximum value of $\frac{1}{\delta}\log(\frac{q^2c}{|\vartheta_2|X_0})$ is 321. Consequently, we determine that $r-s \leq 321$.

Again, we let $\chi_2 := -(r-1)\log\varphi + d\log 2 + \log\left(3/f_k(\varphi)(1+(2\varphi-1)\varphi^{-(r-s)})\right)$. Assuming $r \geq 7$ in (17), we obtain $|\Lambda_2| < 0.7$. According to Lemma 2.3, we get $|\chi_2| < 20.6/\varphi^r$. Therefore, we have

$$0 < \left|-(r-1)\log\varphi + d\log 2 + \log\left(3/f_k(\varphi)(1+(2\varphi-1)\varphi^{-(r-s)})\right)\right| < 20.6\exp(-r\log\varphi).$$

Again, by Lemma 2.1, we have $\beta := \log\left(3/f_k(\varphi)(1+(2\varphi-1)\varphi^{-(r-s)})\right)$ and putting

$$c := 20.6, \quad \delta := \log\varphi, \quad \psi := \frac{\log\left(\frac{3}{f_k(\varphi)(1+(2\varphi-1)\varphi^{-(r-s)})}\right)}{\log 2},$$
$$\vartheta := \frac{\log\varphi}{\log 2}, \quad \vartheta_1 := -\log\varphi, \quad \vartheta_2 := \log 2.$$

As in the previous argument, we have $X_0 := 2.9\cdot 10^{50}$. Using the *Maple* program, we determine that the maximum value of $\frac{1}{\delta}\log(\frac{q^2c}{|\vartheta_2|X_0})$ is 328 for all $r-s \in [1, 321]$. Finally, searching with *Maple* in the ranges $2 \leq k \leq 340$, $k+2 \leq r \leq 328$ and $2 \leq s \leq r$ yields the set of solutions in Theorem 1.1. Thus, Theorem 1.1 is proved. $\square$

3.3 Main proof of Theorem 1.2

To complete the proof of the theorem, we present the following cases.

3.4 Case $r \leq k+1$, where $k \geq 2$

In this case, we obtain the following lemma.

Lemma 3.4. *The only non-negative integer solutions of the Diophantine equation* (2) *where* $0 \leq r \leq k+1$ *with* $k \geq 2$ *are given by* $(r, d, y) \in \{(0,0,3), (1,0,2), (2,0,2), (3,0,1), (4,1,2)\}$.

Proof. We consider the following cases.

Case 1: If $r \in \{0, 1\}$, then Eq. (2) becomes $3 \cdot 2^d = y$ and $3 \cdot 2^d = y + 1$. It is easy to verify that the only possible solutions of both equations are $(r, d, y) \in \{(0, 0, 3), (1, 0, 2)\}$.

Case 2: If $2 \leq r \leq k + 1$, from (8) with Eq. (2), it can be seen that $2^{r-2} = 3 \cdot 2^d - y$. Thus, only the possible solutions, where $y \in \{0, 1, 2, 3\}$ are $(r, d, y) \in \{(2, 0, 2), (3, 0, 1), (4, 1, 2)\}$. □

3.5 Case $r \geq k + 2$, where $k \geq 2$

Now, we consider the following results.

3.5.1 The connection between r and d

We have $r \geq k + 2$. Then combining inequality (6) with Eq. (2), we obtain

$$2^d - 1 \leq 3 \cdot 2^d - y = F_r^{(k)} \leq \varphi^{r-1} < 2^{r-1} < 2^{r+1} - 1.$$

Therefore, we obtain $d < r + 1$.

3.5.2 Finding an upper bound of r in terms of k

The aim of this case is to prove the following lemma.

Lemma 3.5. *If (r, d, y, k) is a solution in non-negative integers of Eq. (2) where $r \geq k + 2$ with $k \geq 2$, then the following inequality holds*

$$r < 4.7 \cdot 10^{14} k^4 (\log(k))^3. \tag{18}$$

Proof. Combining (4) with Eq. (2), thus we get $f_k(\varphi)\varphi^{r-1} - 3 \cdot 2^d = -y - e_k(r)$. Taking the absolute value, we deduce that $\left|f_k(\varphi)\varphi^{r-1} - 3 \cdot 2^d\right| < 3.5$. Dividing both sides by $f_k(\varphi)\varphi^{r-1}$ and using $1/2 < f_k(\varphi) < 3/4$, we get

$$|\Lambda_3| = \left|\frac{3 \cdot 2^d \varphi^{-(r-1)}}{f_k(\varphi)} - 1\right| < \frac{3.5}{f_k(\varphi)\varphi^{r-1}} < \frac{14}{\varphi^r}. \tag{19}$$

Thus, we obtain the same calculation as that in the first part of Lemma 3.2. As $d < r + 1$, we can choose $B := r + 1$. According to Theorem 2.1, we get

$$14/\varphi^r > |\Lambda_3| > \exp\{-G(1 + \log(r + 1))(5k \log k)(\log \varphi)(k \log 2)\},$$

where $G = 1.4\,(30^6)\,(3^{4.5})\,(k^2)(1 + \log(k))$. We used the fact $1 + \log k \leq 3 \log k$ for all $k \geq 2$ and $1 + \log(r + 1) \leq 2 \log r$ for all $r \geq 4$. After a simplified calculation, we obtained

$$r < 4.7 \cdot 10^{14} k^4 (\log(k))^3.$$

□

3.5.3 Absolute upper bound of r on k

The aim of this case is to demonstrate the following lemma.

Lemma 3.6. *The Eq.* (2) *has no non-negative integer solutions, where* $r \geq k+2$ *with* $k > 175$.

Proof. Assuming that $k > 175$, then we get $r < 4.7 \cdot 10^{14} k^4 (\log(k))^3 < 2^{k/2}$. From Lemma 2.4 with Eq. (2) and dividing by 2^{r-2}, it follows that

$$\left|1 - 3 \cdot 2^{d-(r-2)}\right| < \frac{2}{2^{k/2}} + \frac{3}{2^{r-2}} < \frac{5}{2^{k/2}},$$

where we used the fact that $r \geq k+2$ implies that $k \leq r-2$. Therefore, we observe that $0.25 < \left|1 - 3 \cdot 2^{d-(r-2)}\right|$. However, the right-hand side is exceedingly small for such $k > 175$, which is inconsistent with the given conditions. Hence, (2) has no solutions for $k > 175$. □

3.5.4 Reducing the bound of *r* on *k*

Let $\chi_3 := -(r-1)\log\varphi + d\log 2 + \log(3/f_k(\varphi))$. In (19), we assume that $r \geq 7$; consequently, $|\Lambda_3| < 0.82$. By Lemma 2.3, we derive $|\chi_3| < 29.3/\varphi^r$. Thus

$$0 < |-(r-1)\log\varphi + d\log 2 + \log(3/f_k(\varphi))| < 29.3\exp(-r\log\varphi).$$

Applying Lemma 2.1 for above inequality, we have $\beta := \log(3/f_k(\varphi))$ and putting

$$c := 29.3, \quad \delta := \log\varphi, \quad \psi := \frac{\log(3/f_k(\varphi))}{\log 2}, \quad \vartheta := \frac{\log\varphi}{\log 2}, \quad \vartheta_1 := -\log\varphi, \quad \vartheta_2 := \log 2.$$

In (18), we can get $X_0 := 6.1 \cdot 10^{25}$ for all $k \in [2, 175]$, which is the upper bound for r. Utilizing the *Maple* program, it was determined that 175 represents the maximum value of $\frac{1}{\delta}\log(\frac{q^2 c}{|\vartheta_2| X_0})$. Finally, we computed Eq. (2) in the ranges $2 \leq k \leq 175$, $k+2 \leq r \leq 175$, and $y \in \{0,1,2,3\}$, only possible solutions are obtained in Theorem 1.2. Thus, proved Theorem 1.2.

4 Conclusion

In this study, we examined the Diophantine equation $F_r^{(k)} + L_s^{(k)} = 3 \cdot 2^d \pm 1$, where $F_r^{(k)}$ and $L_s^{(k)}$ denote the k-generalized Fibonacci and Lucas numbers, respectively. Applying advanced results from transcendental number theory, particularly Matveev's theorem on linear forms in logarithms, alongside a continued fraction reduction method developed by de Weger, we achieved an explicit classification of all such representations as Thabit numbers of the first and second kinds.

We established two principal theorems. Theorem 1.1 identifies all integer solutions to the equation when $r > s \geq 2$. Theorem 1.2 resolves the remaining cases where $s \in \{0,1\}$ by solving the equation $F_r^{(k)} = 3 \cdot 2^d - y$, for $y \in \{0,1,2,3\}$. Key contributions include the establishment of upper bounds for r in terms of k, the demonstration of the non-existence of solutions for large k, and the computational reduction of these bounds to identify explicit solutions.

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