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A new symmetric weighing matrix $SW(22, 16)$

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Abstract: The existence of symmetric weighing matrix $SW(22, 16)$ is settled in this note through a theorem and exhaustive search.
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1 Introduction

A *weighing matrix* $W(n, k)$ of order n and weight k is a $(0, 1, -1)$-matrix such that $WW^\top = kI_n$, $k \leq n$. For an introduction to the fundamentals of weighing matrices, the authors refer readers to the seminal paper by Chan *et al.* [2]. Henceforth, a symmetric $W(n, k)$ will be denoted as $SW(n, k)$. Recently, construction of $W(23, 16)$, $W(25, 16)$, $W(27, 16)$ and $W(29, 16)$ have been settled by Ben-Av *et al.* [1]. Georgiou *et al.* [4] have proposed a method which resulted in the very first example of $SW(30, 25)$. The smallest open case for symmetric weighing matrix with weight 16 is $SW(22, 16)$ (see [3, Table 2.88]), which is settled in this note.

2 Preliminary

Suppose $GF(q)$ be a finite field of q (a prime power) elements. Observing the properties of a finite field Paley [5] defined a matrix $Q = [q_{ij}]$ of order q which is now termed as the *Paley core*. Let $a_1, a_2, \ldots, a_q$ be the elements of $GF(q)$ arranged in some random but fixed order, then Q is defined as

$$q_{ij} = \begin{cases} 0, & \text{if } a_j - a_i = 0 \\ 1, & \text{if } a_j - a_i \text{ is a square element of } GF(q) \\ -1, & \text{if } a_j - a_i \text{ is a non-square element of } GF(q). \end{cases}$$

It has been established that $QQ^\top = qI_q$ and Q is a symmetric or skew according as $q \equiv 1 \pmod 4$ or $q \equiv -1 \pmod 4$ [5].

Let G be an additive Abelian group with elements g_i. Let $X \subseteq G$. For an arbitrary but fixed ordering of the elements of G, the matrix $M = [m_{ij}]$ defined by

$$m_{ij} = \chi(g_j - g_i),$$
$$\chi(g_j - g_i) = \begin{cases} 1, & g_j - g_i \in X \\ 0, & \text{otherwise} \end{cases},$$

is called the *type 1 incidence matrix* and the matrix $N = [n_{ij}]$, where

$$n_{ij} = \chi(g_j + g_i),$$
$$\chi(g_j + g_i) = \begin{cases} 1, & g_j + g_i \in X \\ 0, & \text{otherwise} \end{cases}$$

is called the *type 2 incidence matrix* of X in G. A matrix $A = [a_{ij}]$ of order n is called a *circulant* or *back circulant* matrix according as $a_{ij} = a_{1,j-i+1}$ or $a_{ij} = a_{1,i+j-1}$ where, $j-i+1$ & $i+j-1$ are reduced modulo n. In the definition of type 1 incidence matrix if the group G is cyclic, then the matrix in circulant.

Notation: An identity matrix and an all-1 matrix will be denoted as I and J, respectively. Their orders will be decided by the context. $N^\top$ will denote the transpose of the matrix N. A circulant matrix B with first row $(abc \ldots k)$ will be denoted as $B = \mathrm{circ}(abc \ldots k)$.

The following properties of type 1 and type 2 matrices due to Seberry [6] will be applied in the proof of the construction in Section 3.

Lemma 2.1. *If X is a type 1 matrix and Z is a type 2 matrix defined on the same Abelian group with a fixed ordering, then*

(i) $X^\top$ is a type 1 matrix and $Z^\top$ is a type 2 matrix,

(ii) $Z^\top = Z$, $XZ = Z^\top X^\top$, $ZX = X^\top Z^\top$ and $XZ^\top = ZX^\top$.

3 The construction

The construction relies on the following theorem, and the main result is a corollary of it.

Theorem 3.1. *Let* $v \equiv -1 \pmod 4$ *be a prime power and* $t < v$ *be a positive integer such that* $v = 2t + 1$ *and* $t + v$ *is a perfect square. Further, if there exists a type 1* $(0, 1, -1)$*-matrix* P *of order* v *such that* $PP^\top = tI + J$*, then there exists a symmetric weighing matrix* $SW(2v, v + t)$*.*

Proof. Consider the Paley core Q over $GF(v)$. Then it will be a type 1 matrix with following properties:

$$\begin{aligned} QQ^\top &= vI - J \quad \text{and} \\ Q^\top &= -Q. \end{aligned} \tag{1}$$

Define the matrix $R = [r_{ij}]_{v \times v}$ by

$$r_{ij} = \begin{cases} 1, & \text{if } j = v - i + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then, R is a type 2 matrix such that

$$\begin{aligned} R &= R^\top \quad \text{and} \\ RR^\top &= RR = I. \end{aligned} \tag{2}$$

Suppose the matrix P in the Theorem exists, then

$$P \text{ is type 1 matrix and } PP^\top = tI + J. \tag{3}$$

Therefore, by virtue of Lemma 2.1

$$\begin{aligned} PRQ^\top &= (PR)Q^\top \\ &= Q(PR)^\top \\ &= QRP^\top \text{ and} \\ (PR)^\top &= PR. \end{aligned} \tag{4}$$

We define a matrix

$$W = \begin{bmatrix} PR & -Q \\ Q & PR \end{bmatrix}$$

Then by virtue of equations (1), (2), (3) and (4), $WW^\top = (v + t)I_{2v}$ and $W^\top = W$. Thus W is a symmetric weighing matrix of order $2v$ and weight $v + t$. □

Corollary 3.1. *There exists a* $SW(22, 16)$*.*

Proof. Here $v = 11$. The only possible value of $t = 5$.
Consider the Paley core $Q = \text{circ}(0 - + - - - + + + - +)$.
The type 1 matrix $P = \text{circ}(000 - 0 + -0 - --)$ satisfies $PP^\top = 5I + J$.

Hence, the desired symmetric weighing matrix $SW(22,16) =$

$$
\left[\begin{array}{ccccccccccc|ccccccccccc}
0 & 0 & 0 & - & 0 & + & - & 0 & - & - & - & 0 & - & + & - & - & - & + & + & + & - & + \\
0 & 0 & - & 0 & + & - & 0 & - & - & - & 0 & + & 0 & - & + & - & - & - & + & + & + & - \\
0 & - & 0 & + & - & 0 & - & - & - & 0 & 0 & - & + & 0 & - & + & - & - & - & + & + & + \\
- & 0 & + & - & 0 & - & - & - & 0 & 0 & 0 & + & - & + & 0 & - & + & - & - & - & + & + \\
0 & + & - & 0 & - & - & - & 0 & 0 & 0 & - & + & + & - & + & 0 & - & + & - & - & - & + \\
+ & - & 0 & - & - & - & 0 & 0 & 0 & - & 0 & + & + & + & - & + & 0 & - & + & - & - & - \\
- & 0 & - & - & - & 0 & 0 & 0 & - & 0 & + & - & + & + & + & - & + & 0 & - & + & - & - \\
0 & - & - & - & 0 & 0 & 0 & - & 0 & + & - & - & - & + & + & + & - & + & 0 & - & + & - \\
- & - & - & 0 & 0 & 0 & - & 0 & + & - & 0 & - & - & - & + & + & + & - & + & 0 & - & + \\
- & - & 0 & 0 & 0 & - & 0 & + & - & 0 & - & + & - & - & - & + & + & + & - & + & 0 & - \\
- & 0 & 0 & 0 & - & 0 & + & - & 0 & - & - & - & + & - & - & - & + & + & + & - & + & 0 \\
\hline
0 & + & - & + & + & + & - & - & - & + & - & 0 & 0 & 0 & - & 0 & + & - & 0 & - & - & - \\
- & 0 & + & - & + & + & + & - & - & - & + & 0 & 0 & - & 0 & + & - & 0 & - & - & - & 0 \\
+ & - & 0 & + & - & + & + & + & - & - & - & 0 & - & 0 & + & - & 0 & - & - & - & 0 & 0 \\
- & + & - & 0 & + & - & + & + & + & - & - & - & 0 & + & - & 0 & - & - & - & 0 & 0 & 0 \\
- & - & + & - & 0 & + & - & + & + & + & - & 0 & + & - & 0 & - & - & - & 0 & 0 & 0 & - \\
- & - & - & + & - & 0 & + & - & + & + & + & + & - & 0 & - & - & - & 0 & 0 & 0 & - & 0 \\
+ & - & - & - & + & - & 0 & + & - & + & + & - & 0 & - & - & - & 0 & 0 & 0 & - & 0 & + \\
+ & + & - & - & - & + & - & 0 & + & - & + & 0 & - & - & - & 0 & 0 & 0 & - & 0 & + & - \\
+ & + & + & - & - & - & + & - & 0 & + & - & - & - & - & 0 & 0 & 0 & - & 0 & + & - & 0 \\
- & + & + & + & - & - & - & + & - & 0 & + & - & - & 0 & 0 & 0 & - & 0 & + & - & 0 & - \\
+ & - & + & + & + & - & - & - & + & - & 0 & - & 0 & 0 & 0 & - & 0 & + & - & 0 & - & -
\end{array}\right]
$$

As usual, $+$ and $-$ represent 1 and -1, respectively. $\square$

Remark 1. *The matrix P in the above corollary is the result of exhaustive computer search. The search resulted in only one inequivalent matrix P corresponding to the Paley core Q in the corollary.*

4 Conclusion

Theorem 3.1 can generate a family of symmetric weighing matrices for suitable values of v and t. However, the existence of $SW(2v, t+v)$ relies upon the existence of the matrix P. Table 1 shows the coverage of the Theorem 3.1 for $v \leq 200$.

For $v \geq 43$, the exhaustive search for the existence of the matrix P demands a high volume of memory and time. Further studies could be made to theorize the existence of matrix P or to reduce the search-space for search algorithms which would result in symmetric weighing matrices of higher orders.

Table 1. Existence of symmetric weighing matrix by Theorem 3.1

v	t	$SW(2v, t+v)$	**Existence of P**
3	1	$SW(6,4)$	Exists
11	5	$SW(22,16)$	Exists
43	21	$SW(86,64)$	Unresolved
67	33	$SW(134,100)$	Unresolved
131	65	$SW(262,196)$	Unresolved
171	85	$SW(342,256)$	Unresolved

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