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Fermatian row and column sums as a family of generalized integers

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Abstract: In this paper, we introduce some feature of the Fermatian numbers. The finite sum formulas of these numbers is calculate. The exponential generating function of Fermatian numbers is found and some of its identities is calculated. Another number sequence is obtained from the partial row sums of these numbers and these numbers were examined. At the same time, another polynomial has been defined as a generalization of these numbers, depending on powers of z.
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1 Introduction

In mathematics, sequences of real numbers hold a significant place due to their inherent properties and the wide range of applications these properties enable. Among these sequences, the most

famous and important one is the Fibonacci sequence. Fibonacci numbers have found applications in numerous fields, ranging from nature and art to architecture and engineering.

Over time, numerous generalizations of these number sequences have been proposed, leading to the definition of many diverse and intriguing numbers. Fermatian numbers can be given as an example of such numbers.

An aim of this paper is to seek some further analogues for Fermatian numbers, $\underline{z}_n$, defined for integers z by

$$\underline{z}_n = \frac{\left(1-z^n\right)}{\left(1-z\right)} \tag{1.1}$$

They get their name from the French mathematician, Pierre de Fermat (1607–1665) [12]. These numbers constitute the ordered set of integer solutions of the congruence in Fermat's Little Theorem, given below as Theorem 1.1.

Theorem 1.1.

$$z^{p-1} \equiv 1 \left(\mathrm{mod}\, p\right), \ p \text{ a prime.} \tag{1.2}$$

For an integer $z > 1$, if a composite integer x divides $z^{x-1} - 1$, then x is called a Fermat pseudoprime to base z. Shanks called any integer solution of (1.2), including even numbers and other composites, a Fermatian number [5, 13]. Carlitz [3] used $\underline{z}_n = [n]$, though it is less suggestive for some of the generalized analogies. Carlitz himself used $[n]$ with other meanings [2]. Some of the results here have been established in slightly different forms in other papers [14, 16].

2 Fermatian numbers

$\underline{z}_n$ is the n-th Fermatian number of index z:

$$\underline{z}_n = \begin{cases} -z^n \underline{z}_n & (n < 0) \\ 1 + z + z^2 + \cdots + z^{n-1} & (n > 0) \\ 1 & (n = 0) \end{cases} \tag{2.1}$$

so that

$$\underline{1}_n = n, \tag{2.2}$$

and

$$\underline{1}_n ! = n!, \tag{2.3}$$

where

$$\underline{z}_n ! = \underline{z}_n \underline{z}_{n-1} \cdots \underline{1}_n \ . \tag{2.4}$$

For example, if we consider the Fermatian numbers of index 2, we have $\underline{2}_2 = 3$, and $\underline{2}_3 = 7$, so that $\underline{2}_2$ and $\underline{2}_3$ are generalized Fermatian primes, and $\underline{2}_6 = (\underline{2}_2)^2 \underline{2}_3$, but $\underline{2}_8$ cannot be represented as a product of Fermatian numbers of index 2. Some properties of these numbers may be found in [15] and Carlitz and Moser [4]. Carlitz has also used Fermatian numbers in the development of q-Bernoulli numbers and polynomials [2]. The first ten Fermatian numbers of the first ten indices are displayed in Table 1.

Table 1. First 10 Fermatian numbers of the first 10 indices

$n\rightarrow$ / $z\downarrow$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	1	3	7	15	31	63	127	255	511	1023
3	1	4	13	40	121	364	1093	3280	9841	29524
4	1	5	21	85	341	1365	5461	21845	87381	349525
5	1	6	31	156	781	3906	19531	97656	488281	2441406
6	1	7	43	259	1555	9331	55987	335923	2015539	12093235
7	1	8	57	400	2801	19608	137257	960800	6725601	47079208
8	1	9	73	585	4681	37449	299593	2396745	19173961	153391689
9	1	10	91	820	7381	66430	597871	5380840	48427561	435848050
10	1	11	111	1111	11111	111111	1111111	11111111	111111111	1111111111

The corresponding row and column sequences, $\{\underline{z}_n\}_{n=1}^{\infty}, \{\underline{z}_n\}_{z=1}^{\infty}$ are obvious from their construction, but the sequence,

$$\left\{\sum_{n=1}^{z-1}\underbrace{(z-n)}_{n}\right\} \equiv \{1,3,7,16,39,105,315,1048,\ldots\},$$

formed from adding along the forward diagonals, seems to be the sequence A103439 of [17] defined by the row and column combination double sum

$$a_n = \sum_{i=0}^{n-1}\sum_{j=0}^{i}(i-j+1)^j.$$

Row and column recurrence relations can be formed from the basic definitions and checked against the values in this table.

Variations of the identities of Simson, Cassini, Catalan and Vajda have provided opportunities for extensions and generalizations [9, 10, 11]. In its simplest Fibonacci form, the Simson identity can be expressed as

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}, \tag{2.4}$$

and it is trivial to show that the Fermatian analogue is

$$\underline{z}_n^2 - \underline{z}_{n-1}\underline{z}_{n+1} = z^{n-1}. \tag{2.5}$$

Other analogues of classical identities have been developed, but Tables 2, 3, 4 further emphasize how these Fermatians constitute a family of generalized integers.

Cassini's identity is a particularly important and interesting identity for certain sequences such as the Fibonacci sequence. This identity provides information not only about the magnitude of the numbers, but also about the structural properties of the sequence. Cassini's identity reveals a structural relationship between consecutive terms of the Fibonacci sequence, highlighting how "regular" and mathematically profound the sequence is. This identity is also connected to the

determinants of certain matrices. Cassini's identity can assist in proving other theorems related to the Fibonacci sequence. It is particularly useful in contexts where the method of mathematical induction is applied. Cassini's identity is also related to certain number theory problems or combinatorial structures involving Fibonacci numbers. In this sense, it finds applications in applied mathematics as well. Similar statements can be made for other identities (Catalan, d'Ocagne, Vajda, …) as well.

Theorem 2.1 (Cassini Identity). For $n \in \mathrm{Z}^+$ we have,

$$\underline{z}_{n-1}\underline{z}_{n+1} - \underline{z}_n^2 = z^{n-1}.$$

Proof. For the proof, we use the Equality (1.1).

$$\begin{aligned}
\underline{z}_{n-1}\underline{z}_{n+1} - \underline{z}_n^2 &= \frac{\left(1-z^{n-1}\right)}{\left(1-z\right)}\frac{\left(1-z^{n+1}\right)}{\left(1-z\right)} - \frac{\left(1-z^{n}\right)}{\left(1-z\right)}\frac{\left(1-z^{n}\right)}{\left(1-z\right)} \\
&= \frac{1-z^{n+1}-z^{n-1}+z^{2n}-1+2z^n-z^{2n}}{\left(1-z\right)^2} \\
&= \frac{-z^{n+1}-z^{n-1}+2z^n}{\left(1-z\right)^2} \\
&= \frac{-z^{n-1}\left(z-1\right)^2}{\left(1-z\right)^2} \\
&= z^{n-1}.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.2 (Catalan Identity). For $n,t \in \mathrm{Z}^+$ we have,

$$\underline{z}_{n-t}\underline{z}_{n+t} - \underline{z}_n^2 = z^{n-1}.$$

Proof. For the proof, we use the equality (1.1).

$$\begin{aligned}
\underline{z}_{n-t}\underline{z}_{n+t} - \underline{z}_n^2 &= \frac{\left(1-z^{n-t}\right)}{\left(1-z\right)}\frac{\left(1-z^{n+t}\right)}{\left(1-z\right)} - \frac{\left(1-z^{n}\right)}{\left(1-z\right)}\frac{\left(1-z^{n}\right)}{\left(1-z\right)} \\
&= \frac{1-z^{n+t}-z^{n-t}+z^{2n}-1+2z^n-z^{2n}}{\left(1-z\right)^2} \\
&= \frac{-z^{n+t}-z^{n-t}+2z^n}{\left(1-z\right)^2} \\
&= \frac{z^n\left(2-z^t-\dfrac{1}{z^t}\right)}{\left(1-z\right)^2} \\
&= \frac{-z^{n-t}\left(z^t-1\right)^2}{\left(1-z\right)^2}
\end{aligned}$$

$$= -z^{n-t}\left(\frac{1-z^t}{1-z}\right)^2$$
$$= -z^{n-t}\underline{z}_t^2.$$

Note that, if $t=1$ is taken then Cassini identitiy is obtained.

Thus, the proof is completed. □

Theorem 2.3 (d' Ocagne Identity). For $n,m \in \mathrm{Z}$, we have

$$\underline{z}_m\underline{z}_{n+1} - \underline{z}_{m+1}\underline{z}_n = \frac{z^n + z^m}{1-z}..$$

Proof.

$$\begin{aligned}\underline{z}_m\underline{z}_{n+1} - \underline{z}_{m+1}\underline{z}_n &= \frac{\left(1-z^m\right)}{\left(1-z\right)}\frac{\left(1-z^{n+1}\right)}{\left(1-z\right)} - \frac{\left(1-z^{m+1}\right)}{\left(1-z\right)}\frac{\left(1-z^n\right)}{\left(1-z\right)} \\ &= \frac{1-z^{n+1}-z^m+z^{m+n+1}-1+z^n+z^{m+1}-z^{m+n+1}}{\left(1-z\right)^2} \\ &= \frac{z^n\left(1-z\right)+z^m\left(1-z\right)}{\left(1-z\right)^2} \\ &= \frac{\left(1-z\right)\left(z^n+z^m\right)}{\left(1-z\right)^2} \\ &= \frac{z^n+z^m}{1-z}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.4 (Vajda Identity). For $n,m,r \in \mathrm{Z}$, we have

$$\underline{z}_{n+m}\underline{z}_{n+r} - \underline{z}_n\underline{z}_{n+m+r} = z^n\underline{z}_r\underline{z}_m.$$

Proof.

$$\begin{aligned}\underline{z}_{n+m}\underline{z}_{n+r} - \underline{z}_n\underline{z}_{n+m+r} &= \frac{\left(1-z^{n+m}\right)}{\left(1-z\right)}\frac{\left(1-z^{n+r}\right)}{\left(1-z\right)} - \frac{\left(1-z^n\right)}{\left(1-z\right)}\frac{\left(1-z^{n+m+r}\right)}{\left(1-z\right)} \\ &= \frac{1-z^{n+r}-z^{n+m}+z^{m+2n+r}-1+z^{n+m+r}+z^n-z^{m+2n+r}}{\left(1-z\right)^2} \\ &= \frac{\left(1-z^r\right)\left(z^n-z^{n+m}\right)}{\left(1-z\right)^2} \\ &= \frac{\left(1-z^r\right)z^n\left(1-z^m\right)}{\left(1-z\right)^2} \\ &= z^n\underline{z}_r\underline{z}_m.\end{aligned}$$

Thus, the proof is obtained. □

Theorem 2.5. (Gelin–Cesaro Identity) For $n \geq 2$, we have

$$\underline{z}_{n-1}\underline{z}_{n-2}\underline{z}_{n+1}\underline{z}_{n+2} - \underline{z}_n^4 = -z^{2n-3}(1+z)^2 - \underline{z}_n^4.$$

Proof. We will use Cassini and Catalan identities to prove the theorem. Thus,

$$\begin{aligned}\underline{z}_{n-1}\underline{z}_{n-2}\underline{z}_{n+1}\underline{z}_{n+2} - \underline{z}_n^4 &= \underline{z}_{n-1}\underline{z}_{n+1}\underline{z}_{n-2}\underline{z}_{n+2} - \underline{z}_n^4 \\ &= -z^{n-1}z^{n-2}\underline{z}_2^2 - \underline{z}_n^4 \\ &= -z^{2n-3}\,\underline{z}_2^2 - \underline{z}_n^4 \\ &= -z^{2n-3}(1+z)^2 - \underline{z}_n^4.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.6. Exponential generating function is as follows:

$$\sum_{n=0}^{\infty}\underline{z}_n\frac{x^n}{n!} = \frac{e^x - e^{zx}}{1-z}.$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\underline{z}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{1-z^n}{1-z}\right)\frac{x^n}{n!} \\ &= \frac{1}{1-z}\sum_{n=0}^{\infty}\frac{x^n - (zx)^n}{n!} \\ &= \frac{1}{1-z}\left[\sum_{n=0}^{\infty}x^n - \sum_{n=0}^{\infty}(zx)^n\right] \\ &= \frac{e^x - e^{zx}}{1-z}.\end{aligned}$$

Thus, the proof is completed. □

3 Fermatian row sums

It is obvious that from Table 1 that $\underline{z}_n = z\underline{z}_{n-1} + 1$ and from the partial row sums in Table 2 that $u_n = zu_{n-1} + n.$ Elements of these two recurrence relations are displayed in Table 2.

Thus, we see that $\underline{z}_n = \underline{z}_{n-1} + z^{n-1}$ as in $\underline{3}_5 = 121 = 40 + 81 = \underline{3}_4 + 3^4$, and the Fermatian numbers and their corresponding row sum numbers follow a neat and consistent, but not unsurprising, pattern of the form

$$\frac{z^n - 1}{z-1} \sim zu_{n-1} + n, \tag{3.1}$$

or

$$n \sim zu_{n-1} - \underline{z}_n. \tag{3.2}$$

Table 2. Partial sums $\{u_n\}$ and Fermatians $\{z_n\}$

$n\rightarrow$ / $z\downarrow$	1	2	3	4	5	6	7	Sloane #	Sloane #	Sloane: u_n =
1	1	2	3	4	5	6	7	A000027		$u_{n-1}+1$
u_n	1	3	6	10	15	21	28		A000317	$1\,u_{n-1}+n$
2	1	3	7	15	31	63	127	A000225		$\left(2^n-1\right)/1$
u_n	1	4	11	26	57	120	247		A000295	$2\,u_{n-1}+n$
3	1	4	13	40	121	364	1093	A003462		$\left(3^n-1\right)/2$
u_n	1	5	18	58	179	543	1636		A000340	$3\,u_{n-1}+n$
4	1	5	21	85	341	1365	5461	A002450		$\left(4^n-1\right)/3$
u_n	1	6	27	112	453	1818	7279		A014825	$4\,u_{n-1}+n$
5	1	6	31	156	781	3906	19531	A003463		$\left(5^n-1\right)/4$
u_n	1	7	38	194	975	4881	24412		A014827	$5\,u_{n-1}+n$
6	1	7	43	259	1555	9331	55987	A003464		$\left(6^n-1\right)/5$
u_n	1	8	51	310	1865	11196	67183		A014829	$6u_{n-1}+n$
7	1	8	57	400	2801	19608	137257	A023000		$\left(7^n-1\right)/6$
u_n	1	9	66	466	3267	22875	160132		A4830	$7u_{n-1}+n$
8	1	9	73	585	4681	37449	299593	A023001		$\left(8^n-1\right)/7$
u_n	1	10	83	668	5349	42798	342391		A048440	$8u_{n-1}+n$
9	1	10	91	820	7381	66430	597871	A002452		$\left(9^n-1\right)/8$
u_n	1	11	102	922	8303	74733	672604		A048441	$9u_{n-1}+n$

The generalizability of these Fermatians is further emphasized when we consider the conjectured series of column entries from Table 1 in Table 3 below. It opens up new connections among known sequences by means of these Fermatians, with almost Pascal triangle coefficients to generate the numbers.

Table 3. Conjectured column sequences from Table 1

z	1	2	3	4	5	6	7	8	9	v_n =
1	1	1	1	1	1	1	1	1	1	The conjectured equations $\frac{1}{z}\left\{(z+1)^n-1\right\}$ are set out in Table 4
2	3	4	5	6	7	8	9	10	11	
3	7	13	21	31	43	57	73	91	111	
4	15	40	85	156	259	400	585	120	1111	
5	31	121	341	781	1555	2801	4681	7381	11111	
6	63	364	1365	3906	9333	19608	37449	66430	111111	
7	127	1093	5461	19531	55987	137257	299593	597871	1111111	
8	255	3280	21845	97656	335923	960800	2396745	5380840	11111111	
9	511	9841	87381	488281	2015539	6725601	19173961	48427561	111111111	
10	1023	29524	349525	2441406	12093235	47079208	153391689	435848050	1111111111	

Table 4. Some z polynomials

n	1	2	3	4	5
v_n	1	$z+2$	z^2+3z+3	z^3+4z^2+6z+4	$z^4+5z^3+10z^2+10z+5$
	$\underline{(z+1)^1-1}$	$\underline{(z+1)^2-1}$	$\underline{(z+1)^3-1}$	$\underline{(z+1)^4-1}$	$\underline{(z+1)^5-1}$
	z	z	z	z	z
z	**6**			**7**	
n	$z^5+6z^4+15z^3+20z^2+15z+6$			$z^6+7z^5+21z^4+35z^3+35z^2+21z+7$	
	$\underline{(z+1)^6-1}$			$\underline{(z+1)^7-1}$	
	z			z	

Theorem 3.1. The following equality is satisfied.

$$(1-z)\sum_{i=1}^{n-1}u_i+u_n=\frac{n(n+1)}{2}.$$

Proof. The proof is shown by induction over n. □

Corollary 3.1. There is the following relation between Fermatian numbers and polynomials v_n:

$$-\underline{(z+1)}_n=v_n.$$

Proof.

$$\begin{aligned}\underline{(z+1)}_n&=\frac{\left(1-(z+1)^n\right)}{\left(1-(z+1)\right)}\\&=\frac{\left(1-(z+1)^n\right)}{-z}\\&=-v_n.\end{aligned}$$

This completes the proof. □

Theorem 3.2. For $n\geq 1$, the finite sum given below exists:

$$\underline{z}_k=\sum_{i=0}^{k-1}z^i.$$

Proof. We use the equality $\underline{z}_n=\underline{z}_{n-1}+z^{n-1}$ for the proof. Thus,

For $n=1$, $\underline{z}_1=\underline{z}_0+z^0$,

For $n=2$, $\underline{z}_2=\underline{z}_1+z^1$,

For $n=3$, $\underline{z}_3=\underline{z}_2+z^2$,

For $n=4$, $\underline{z}_4 = \underline{z}_3 + z^3$,

$\vdots$

For $n=k$, $\underline{z}_k = \underline{z}_{k-1} + z^{k-1}$.

If the necessary operations are performed, then

$$\underline{z}_k = \sum_{i=0}^{k-1} z^i$$

is obtained. □

Theorem 3.3. The limit of $\underline{z}_n$ is as follows:

$$\lim_{n\to\infty} \frac{\underline{z}_n}{\underline{z}_{n+1}} = \frac{1}{z}.$$

Proof.

$$\lim_{n\to\infty} \frac{\underline{z}_n}{\underline{z}_{n+1}} = \lim_{n\to\infty} \frac{\left(1-z^n\right)}{\left(1-z^{n+1}\right)},$$

since $z>1$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{\underline{z}_n}{\underline{z}_{n+1}} &= \lim_{n\to\infty} \frac{z^n\left(1-\frac{1}{z^n}\right)}{z^{n+1}\left(1-\frac{1}{z^{n+1}}\right)} \\ &= \lim_{n\to\infty} \frac{z^n\left(1-\frac{1}{z^n}\right)}{z^n z\left(1-\frac{1}{z^{n+1}}\right)} \\ &= \frac{1}{z}.\end{aligned}$$

This completes the proof. □

4 Conclusion

This leads into the work of Barry [1] and the connections with Fibonacci and Jacobsthal sequences as well as *q*-binomial coefficients. These also further strengthen the concept of families of generalized integers. These are also exemplified by Fontené's generalized integers [6] where he used an arbitrary sequence of real or complex numbers instead of the natural numbers. Morgan Ward [18] had independently rediscovered these and so Gould [7] developed some striking theorems for what he termed the Fontené–Ward generalized binomial coefficients, though as indicated earlier there has not been much consistency of notation, as also noted by Hoggatt and Bicknell [8].

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