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A note on newly introduced arithmetic functions φ^+ and σ^+

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Abstract: In a recent paper [7], the authors introduced new arithmetic functions φ^+, σ^+ related to the classical functions φ, and σ, respectively. In this note, we study the behavior of

$$\sum_{\substack{n\leq x\\ \omega(n)=2}} (\varphi^+ - \varphi)(n), \quad \text{and} \quad \sum_{\substack{n\leq x\\ \omega(n)=2}} (\sigma^+ - \sigma)(n),$$

for any real number $x \geq 6$.

Keywords: Arithmetic functions, Inequalities for arithmetic functions, Euler's totient function, Sum of positive divisors function.
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1 Introduction

An arithmetic function or a number-theoretic function ([1], chap. 2) is a function defined on the positive integers. There are many well-known arithmetic functions like $\mu(n), \varphi(n), \sigma(n), \Lambda(n)$ etc. In recent times, many authors [2–4, 6] have worked enormously on arithmetic functions. In [7] the authors introduced six new arithmetic functions $\varphi^+, \varphi^-, \psi^+, \psi^-, \sigma^+, \sigma^-$. In this note, we prove the following theorems considering the functions $(\varphi^+ - \varphi)(n)$ and $(\sigma^+ - \sigma)(n)$.

Theorem 1.1. *For any real number $x \geq 6$ we have*

$$\sum_{\substack{n\leq x\\ \omega(n)=2}} \big(\varphi^+ - \varphi\big)(n) < x + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p\leq x}\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)}{p^k},$$

where $\omega(n)$ is the number of distinct prime divisors of n.

Theorem 1.2. *For any real number $x \geq 6$ we have*

$$\sum_{\substack{n\leq x\\ \omega(n)=2}} \big(\sigma^+ - \sigma\big)(n) < x + \frac{1}{6}\sum_{k=0}^{\infty}\sum_{p\leq x}(-1)^k\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)(2\lceil\frac{x}{p}\rceil+1)}{p^{k-1}},$$

where $\omega(n)$ is the number of distinct prime divisors of n.

Note that, we define $\big(\varphi^+ - \varphi\big)(n) = \varphi^+(n) - \varphi(n)$ and $\big(\sigma^+ - \sigma\big)(n) = \sigma^+(n) - \sigma(n)$. Throughout this paper we consider p, q to be prime numbers.

2 Important definitions

Definition 2.1. *For any prime number p, we define the p-adic valuation of an integer n to be*

$$\nu_p(n) = \begin{cases} \max\{k \in \mathbb{Z}_{\geq 0} : p^k \mid n\}, & \text{if } n \neq 0,\\ \infty, & \text{if } n = 0,\end{cases}$$

where $\mathbb{Z}_{\geq 0} = \mathbb{Z}^+ \cup \{0\}$.

Definition 2.2. [7, p. 852] *For Euler's totient function φ, we define φ^+ by*

$$\varphi^+(n) = \prod_{p|n}(\varphi(p^{v_p(n)}) + 1),$$

for a positive integer $n > 1$ and $\varphi^+(1) = 1$.

Definition 2.3. [7, p. 852] *For sum of positive divisors function σ, we define σ^+ by*

$$\sigma^+(n) = \prod_{p|n}(\sigma(p^{v_p(n)}) + 1),$$

for a positive integer $n > 1$ and $\sigma^+(1) = 1$.

3 Proof of Theorem 1.1

Since $\varphi^+(n) = \prod_{p|n}(\varphi(p^{v_p(n)}) + 1)$, we have

$$\sum_{n\leq x}\varphi^+(n) = \sum_{n\leq x}\prod_{p|n}(\varphi(p^{v_p(n)}) + 1).$$

Since we are considering all those positive integers greater than 1 having exactly two distinct prime divisors, we have

$$\sum_{\substack{n\le x\\ \omega(n)=2}} \varphi^+(n) = \sum_{\substack{n\le x\\ \omega(n)=2}} \prod_{p|n}(\varphi(p^{v_p(n)})+1) \tag{1}$$

$$= \sum_{\substack{n\le x\\ \omega(n)=2}} \big(1+\varphi(n)+\varphi(n)\sum_{p|n}\frac{1}{\varphi(p^{v_p(n)})}\big)$$

$$< x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n)\sum_{p|n}\frac{1}{\varphi(p^{v_p(n)})}.$$

Since for any prime number p and for any positive integer n such that $p \mid n$ we have $\varphi(p^{v_p(n)}) \geq \varphi(p) = (p-1)$, i.e., $\frac{1}{\varphi(p^{v_p(n)})} \leq \frac{1}{p-1}$, it follows that

$$x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n)\sum_{p|n}\frac{1}{\varphi(p^{v_p(n)})} \leq x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n)\sum_{p|n}\frac{1}{p-1}$$

$$= x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{1}{p-1}\sum_{\substack{d\le \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\,\alpha\ge 0,\,\beta\ge 1}} \varphi(pd)\,.$$

Since for any positive integer $n>1$ we have $\varphi(n)<n$, we can write

$$x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{1}{p-1}\sum_{\substack{d\le \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\,\alpha\ge 0,\,\beta\ge 1}} \varphi(pd) < x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{1}{p-1}\sum_{\substack{d\le \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\,\alpha\ge 0,\,\beta\ge 1}} pd$$

$$= x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{p}{p-1}\sum_{\substack{d\le \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\,\alpha\ge 0,\,\beta\ge 1}} d.$$

Note that,

$$\sum_{\substack{d\le \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\,\alpha\ge 0,\,\beta\ge 1}} d \leq \sum_{d\le \frac{x}{p}} d,$$

therefore we have

$$x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{p}{p-1}\sum_{\substack{d\le \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\,\alpha\ge 0,\,\beta\ge 1}} d \leq x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{p}{p-1}\sum_{d\le \frac{x}{p}} d$$

$$< x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \sum_{p\le x}\frac{p}{p-1}\cdot\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)}{2}$$

$$= x + \sum_{\substack{n\le x\\ \omega(n)=2}} \varphi(n) + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p\le x}\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)}{p^k}. \tag{2}$$

From (1) and (2) we conclude that

$$\sum_{\substack{n\leq x\\ \omega(n)=2}} \big(\varphi^+ - \varphi\big)(n) \quad < \quad x + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p\leq x} \frac{\lceil \frac{x}{p}\rceil(\lceil \frac{x}{p}\rceil + 1)}{p^k}.$$

This completes the proof. □

4 Proof of Theorem 1.2

Since $\sigma^+(n) = \prod_{p|n}(\sigma(p^{v_p(n)}) + 1)$, we have

$$\sum_{n\leq x}\sigma^+(n) \quad = \quad \sum_{n\leq x}\prod_{p|n}(\sigma(p^{v_p(n)}) + 1).$$

Since we are considering all those positive integers greater than 1 having exactly two distinct prime divisors, we have

$$\begin{aligned}\sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma^+(n) \quad &= \quad \sum_{\substack{n\leq x\\ \omega(n)=2}} \prod_{p|n}(\sigma(p^{v_p(n)}) + 1) \\ &= \quad \sum_{\substack{n\leq x\\ \omega(n)=2}} \big(1 + \sigma(n) + \sigma(n)\sum_{p|n}\frac{1}{\sigma(p^{v_p(n)})}\big) \\ &< \quad x + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n) + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n)\sum_{p|n}\frac{1}{\sigma(p^{v_p(n)})}.\end{aligned} \tag{3}$$

Since for any prime number p and for any positive integer n such that $p \mid n$ we have $\sigma(p^{v_p(n)}) \geq \sigma(p) = (p+1)$, i.e., $\frac{1}{\sigma(p^{v_p(n)})} \leq \frac{1}{p+1}$, it follows that

$$\begin{aligned}x + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n) + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n)\sum_{p|n}\frac{1}{\sigma(p^{v_p(n)})} \quad &\leq \quad x + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n) + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n)\sum_{p|n}\frac{1}{p+1} \\ &= \quad x + \sum_{\substack{n\leq x\\ \omega(n)=2}} \sigma(n) + \sum_{p\leq x}\frac{1}{p+1}\sum_{\substack{d\leq \frac{x}{p}\\ pd=n\\ d=p^\alpha q^\beta,\, \alpha\geq 0,\, \beta\geq 1}} \sigma(pd).\end{aligned}$$

Since for any positive integer $n > 1$ we have

$$\sigma(n) = \sum_{\substack{t|n\\ t>0}} t \leq \sum_{\substack{t|n\\ t>0}} n < \sum_{1\leq t\leq n} n = n^2,$$

we get

$$
\begin{aligned}
x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\sum_{p\le x}\frac{1}{p+1}\sum_{\substack{d\le\frac{x}{p}\\ pd=n\\ d=p^{\alpha}q^{\beta},\,\alpha\ge0,\,\beta\ge1}}\sigma(pd). \quad &< \quad x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\sum_{p\le x}\frac{1}{p+1}\sum_{\substack{d\le\frac{x}{p}\\ pd=n\\ d=p^{\alpha}q^{\beta},\,\alpha\ge0,\,\beta\ge1}}p^2d^2.\\
&= \quad x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\sum_{p\le x}\frac{p^2}{p+1}\sum_{\substack{d\le\frac{x}{p}\\ pd=n\\ d=p^{\alpha}q^{\beta},\,\alpha\ge0,\,\beta\ge1}}d^2.
\end{aligned}
$$

Note that,

$$
\sum_{\substack{d\le\frac{x}{p}\\ pd=n\\ d=p^{\alpha}q^{\beta},\alpha\ge0,\beta\ge1}}d^2\le\sum_{d\le\frac{x}{p}}d^2,
$$

therefore we have

$$
\begin{aligned}
x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\sum_{p\le x}\frac{p^2}{p+1}\sum_{\substack{d\le\frac{x}{p}\\ pd=n\\ d=p^{\alpha}q^{\beta},\,\alpha\ge0,\,\beta\ge1}}d^2 \;&\le\; x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\sum_{p\le x}\frac{p^2}{p+1}\sum_{d\le\frac{x}{p}}d^2\\
&<\; x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\sum_{p\le x}\frac{p^2}{p+1}\cdot\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)(2\lceil\frac{x}{p}\rceil+1)}{6}\\
&=\; x+\sum_{\substack{n\le x\\ \omega(n)=2}}\sigma(n)+\frac{1}{6}\sum_{k=0}^{\infty}\sum_{p\le x}(-1)^k\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)(2\lceil\frac{x}{p}\rceil+1)}{p^{k-1}}.
\end{aligned}
\tag{4}
$$

From (3) and (4) we conclude that

$$
\sum_{\substack{n\le x\\ \omega(n)=2}}\big(\sigma^{+}-\sigma\big)(n)<x+\frac{1}{6}\sum_{k=0}^{\infty}\sum_{p\le x}(-1)^k\frac{\lceil\frac{x}{p}\rceil(\lceil\frac{x}{p}\rceil+1)(2\lceil\frac{x}{p}\rceil+1)}{p^{k-1}}.
$$

This completes the proof. □

5 Conclusion

In this note, we only consider all positive integers having exactly two distinct prime divisors. We can of course consider positive integers having three or more prime divisors but observe that if $\mathrm{M}=\prod_i^r x_i$, then

$$
\prod_{i=1}^{r}(x_i+1)=1+\mathrm{M}\Big(1+\sum_{1\le i_1\le r}\frac{1}{x_{i_1}}+\sum_{\substack{1\le i_1,i_2\le r\\ i_1<i_2}}\frac{1}{x_{i_1}x_{i_2}}+\cdots+\sum_{\substack{1\le i_1,i_2,\ldots,i_{r-1}\le r\\ i_1<i_2<\cdots<i_{r-1}}}\frac{1}{x_{i_1}x_{i_2}\cdots x_{i_{r-1}}}\Big),
\tag{5}
$$

that is, if we consider a positive integer $n = \prod_{i=1}^{\omega(n)\geq 3} p_i^{v_{p_i}(n)}$ (p_i are prime numbers), then we can consider $\mathrm{M} = \prod_{i=1}^{\omega(n)} f(p_i^{v_{p_i}(n)}) = f(n)$, i.e., $x_i = f(p_i^{v_{p_i}(n)})$, $r = \omega(n) \geq 3$ where* f is a multiplicative arithmetic function like φ, σ, etc. Then, from (5) we can write:

$$\begin{aligned} f^+(n) &= \prod_{i=1}^{\omega(n)} \left(f(p_i^{v_{p_i}(n)}) + 1\right) \\ &= 1 + \mathrm{M}\Big(1 + \sum_{1\leq i_1\leq r} \frac{1}{x_{i_1}} + \sum_{\substack{1\leq i_1,i_2\leq r\\ i_1<i_2}} \frac{1}{x_{i_1}x_{i_2}} + \cdots + \sum_{\substack{1\leq i_1,i_2,\ldots,i_{r-1}\leq r\\ i_1<i_2<\cdots<i_{r-1}}} \frac{1}{x_{i_1}x_{i_2}\cdots x_{i_{r-1}}}\Big). \end{aligned} \quad (6)$$

Clearly, if $r = \omega(n) \geq 3$ we have to consider all such i_j where j runs up to at least 2 but the above Equation (6) shows that the calculation will be then more complicated for $\sum_{n\leq x} f^+(n)$. Although the authors [7] have introduced ψ^+ in connection with Dedekind's function [5], the present work does not yield a parallel result. This naturally leads to the question: can a similar result be established for ψ^+?

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* In [7], the authors also mentioned to consider more general functions like $f^+(n), f^-(n)$ for a multiplicative arithmetic function f.