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Almost repdigits in balancing and Lucas-balancing sequences

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Abstract: In this paper, we define the notion of almost repdigit as a positive integer whose digits are all equal except for at most one digit, and we search all terms of the balancing and Lucas-balancing sequences which are almost repdigits. In particular, the only almost repdigits in balancing sequence are $0, 1, 6$, and the only almost repdigits in Lucas-balancing sequence are $1, 3, 17, 99, 577, 3363$.
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1 Introduction

The balancing number B corresponding the balancer R, is a natural number that satisfies the Diophantine equation

$$1+2+\cdots+(B-1)=(B+1)+\cdots+(B+R),$$

for some natural number R and the balancing numbers are represented as $\{B_n\}_{n\geq 0}$. When B is a balancing number, $8B^2+1$ is a perfect square, and the Lucas-balancing number is the positive square root of it [3, 13] and these numbers are represented as $\{C_n\}_{n\geq 0}$. The balancing numbers satisfy the binary recurrence $B_0=0$,$B_1=1$, and $B_{n+1}=6B_n-B_{n-1}$, While the Lucas-balancing numbers, satisfies the same recurrence relation with different initial terms $C_0=1, C_1=3$. Their Binet equations are given by

$$B_n=\frac{\alpha^n-\beta^n}{4\sqrt{2}}, C_n=\frac{\alpha^n+\beta^n}{2}, n=0,1,\ldots,$$

where $\alpha=3+\sqrt{8}$ and $\beta=3-\sqrt{8}$.

A repdigit is a positive integer with a decimal expansion that contains only one distinct digit, which have the form $d(10^m-1)/9$ for some $m\geq 1$ and $1\leq d\leq 9$. As a generalization, for every given positive integer $g\geq 2$, the number N of the form $a(\frac{g^m-1}{g-1})$, where $m\geq 1$ and $a\in\{1,2,\ldots,g-1\}$ is termed the base g repdigit. Numerous studies have been carried out on the search of repdigits, sum of repdigits, concatenation of repdigits in different binary sequences, sum, product, and arithmetic functions of binary sequence terms. The search of repdigits in Fibonacci, Lucas, Pell and Pell–Lucas sequences, in the product of consecutive Fibonacci numbers and Lucas numbers has been explored in [7, 9, 11]. Further, Rayaguru and Panda [14,16,17] investigated the existence of repdigits, sum of two repdigits and cocatenation of two repdigits in balancing and Lucas-balancing sequence [15]. As an extension of these works, Sahukar and Panda investigated the repdigits in the Euler functions of Pell and associated Pell numbers [18, 19]. Considering the numbers similar to the repdigits, the Fibonacci sequence consist of three consecutive numbers $F_{12}=144, F_{13}=233, F_{14}=377$, where all digits are equal except only one digit, which is called as almost repdigits. These are the numbers of the form

$$a\left(\frac{10^m-1}{9}\right)+(b-a)10^l,\ 0\leq l<m,\ 0\leq a,b\leq 9. \tag{1}$$

The square and perfect power in almost repdigits were examined in [8,10], without being attributed a specific name. All almost repdigits are also searched in k-Fibonacci numbers and k-Lucas numbers for all $k\geq 2$ [1].

The objective of this paper is to extend this study by exploring the balancing and Lucas-balancing numbers expressible as almost repdigits. In particular, we prove the following results.

Theorem 1.1. *The only balancing numbers which are almost repdigits are* $B_0=0, B_1=1, B_2=6$.

Theorem 1.2. *The only Lucas-balancing numbers* C_n *which are in the form of almost repdigits are* $C_0=1, C_1=3, C_2=17, C_3=99, C_4=577$ *and* $C_5=3363$.

The proofs of the above theorems come from two effective methods for Diophantine equations. One of them is linear forms in logarithms of algebraic numbers due to Matveev [12], whereas the other one is a version of the reduction algorithm due to Dujella and Pethő [6], which was in fact originally introduced by Baker and Davenport in [2]. In the application of these methods, we frequently need some calculations and computations, for which we use the software *Mathematica*. To eliminate the trivial cases, the proof of the above theorems are carried out for numbers with at least three digits, since all integers having one or two digits are trivially almost repdigits. Thus, we also consider $a \geq 3$ and $n > 5$.

2 Preliminaries

Diophantine equations involving repdigits and variant binary recurrence sequences have solved by using different methods such as Baker's theory which is useful to reduce lower bounds concerning linear forms in logarithms of algebraic numbers. The prominent tools for the proof of our main results are the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. An extended version of Matveev's theorem [12] have stated in ([5], Theorem 9.4). Let $\mathbb{L}$ be an algebraic number field of degree d_L. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, d_2, \ldots, d_l$ be nonzero integers. Let

$$D = \max\{|d_1|, \cdots, |d_l|\} \text{ and } \Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, A_2, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) = \max\{d_L h(\eta_j), |\log n_j|, 0.16\}, \quad j = 1, \ldots, l,$$

where η is an algebraic number having the minimal polynomial over $\mathbb{Z}$, with degree k

$$f(X) = a_0 X^k + a_1 X^{k-1} + \cdots + a_k = a_0(X - \eta^{(1)})(X - \eta^{(2)}) \cdots (X - \eta^{(k)})$$

over the integer $a_0 > 0$ and the a_i's are relatively prime integers. The logarithmic Weil height of η is given by

$$h(n) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right)$$

with the following properties

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s|\, h(\eta), \quad s \in \mathbb{Z}. \end{aligned}$$

If η is a rational number of the form r/s, where r and $s > 0$ are relatively prime integers, then $h(\eta) = h(r/s) = \log\max\{|r|, s\}$.

We now present a theorem derived from [12, Corollary 2.3] by E. M. Matveev, resulting in a significant upper bound for the subscript n in (2) (also see [5, Theorem 9.4]).

Theorem 2.1. *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$*, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1, \ldots, A_l.$$

Another main tool for the proof of our main results is a variant of the Baker and Davenport reduction method due to de Weger [20].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Gamma = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Gamma| < c \cdot \exp(-\delta \cdot Y), \tag{3}$$

$$Y \leq X \leq X_0. \tag{4}$$

When $\beta = 0$ in Equation (2), we get

$$\Gamma = x_1\vartheta_1 + x_2\vartheta_2.$$

Let $\vartheta = -\vartheta_1/\vartheta_2$, where $\vartheta_2 \neq 0$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by $[a_0, a_1, a_2, \ldots]$, and let the k-th convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2.2. (See Lemma 3.2 in [20]) *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If Equation (3) *and Equation* (4) *hold for* x_1, x_2 *and* $\beta = 0$*, then*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\beta \neq 0$ in Equation (2), let $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$, where $\vartheta_2 \neq 0$. Then, we have

$$\frac{\Gamma}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, let $||x|| = \min\{|x - n|, n \in \mathbb{Z}\}$, the distance from x to the nearest integer.

Lemma 2.3. (See Lemma 3.3 in [20]) *Suppose that*

$$||q\psi|| > \frac{2X_0}{q}.$$

Then, the solutions of Equation (3) *and Equation* (4) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

Now, the following lemma is provided which is useful for reducing some upper bounds on the variables.

Lemma 2.4. (Lemma 1 in [4]). *Let M be a positive integer, and let p/q be a convergent of the continued fraction of the irrational number κ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\kappa q||$, where $||\cdot||$ denote the distance to the nearest integer, that is, $||x|| = \min\{|x-n| : n \in \mathbb{Z}\}$ for any real number x. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < |m\kappa - n + \mu| < AB^{-j},$$

in positive integers $m,\ n$ and j with

$$m \leq M \quad \text{and} \quad \mathrm{j} \geq \frac{\log(\mathrm{Aq}/\epsilon)}{\log \mathrm{B}}.$$

3 Main results

3.1 Proof of Theorem 1.1

Assume that

$$B_n = a\left(\frac{10^m - 1}{9}\right) + (b-a)10^l,\ 0 \leq l < m,\ 0 \leq a, b \leq 9. \tag{5}$$

A quick computer search reveals that there is no solution in the interval $n \in [4, 55]$. So from now on, we assume that $n > 55$.

Lemma 3.1. *All solutions of Equation* (5) *satisfy*

$$m\mathrm{log}10 - 1 < n\mathrm{log}\alpha < m\mathrm{log}10 + 1.77.$$

Proof. Since the balancing numbers satisfy $\alpha^{n-1} < B_n < \alpha^n$ for $n > 1$ and from Equation (5) we have $10^{m-1} < B_n < 2 \cdot 10^m$, it follows that

$$\alpha^{n-1} < B_n < 10^m \quad \text{and} \quad 10^{m-1} < B_n < \alpha^n.$$

Application of the logarithm both sides of the above inequality, gives

$$(n-1)\mathrm{log}\alpha < \log B_n < m \log 10 \quad \text{and} \quad (m-1)\mathrm{log}10 < \log B_n < n\mathrm{log}\alpha,$$

yielding

$$n\mathrm{log}\alpha < m\mathrm{log}10 + 1.77 \quad \text{and} \quad m\mathrm{log}10 - 1 < n\mathrm{log}\alpha.$$ □

Implementing the Binet formulae in Equation (5), we get

$$\frac{\alpha^n - \beta^n}{4\sqrt{2}} = a\left(\frac{10^m - 1}{9}\right) + (b-a)10^l, \tag{6}$$

i.e.,

$$9\alpha^n - 9\beta^n = a(\alpha - \beta)10^m + (\alpha - \beta)\{9(b-a)10^l - a\}. \tag{7}$$

Equation (7) is studied in two different steps.

Step 1: Rewriting Equation (7) as

$$\begin{aligned}|9\alpha^n - a(\alpha-\beta)10^m| &= |9\beta^n - (\alpha-\beta)\{9(a-b)10^l + a\}| \\ &\leq 9\alpha^{-n} + 4\sqrt{2}(8\cdot 9\cdot 10^l + 9) \\ &\leq 9\alpha^{-n} + 4\sqrt{2}(72.9\cdot 10^l) \\ &\leq 412.4\cdot 10^l,\end{aligned}$$

which yields

$$\left|\frac{9}{4\sqrt{2}}\alpha^n 10^{-m} - 1\right| \leq \frac{412.4\cdot 10^l}{4\sqrt{2}a10^m} = \frac{72.9}{10^{m-l}}. \tag{8}$$

Let

$$\Gamma = \frac{9}{4\sqrt{2}}\alpha^n 10^{-m} - 1,$$

which gives

$$|\log\Gamma| \leq \log 72.9 - (m-l)\log 10. \tag{9}$$

If $\Gamma = 0$, then $\sqrt{2} = \frac{9}{4\cdot 10^m}\alpha^n$ and hence, $\alpha^{2n} = 2(\frac{9}{4\cdot 10^m})^{-2} \in \mathbb{Q}$, which is not possible for any $n > 0$ and consequently, $\Gamma \neq 0$.

Consider $\eta_1 = \frac{9}{4\sqrt{2}}$, $\eta_2 = \alpha$, $\eta_3 = 10$ with exponents $b_1 = 1$, $b_2 = n$, $b_3 = -m$ where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $d_1, d_2, d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\sqrt{2})$ is $d_{\mathbb{L}} = 2$. Use of the properties of the logarithmic Weil height will provide

$$\begin{aligned}h(\eta_1) = h(\frac{9}{4\sqrt{2}a}) &= h(\frac{9}{a}) + h(4\sqrt{2}) \\ &\leq h(9) + h(4\sqrt{2}) \\ &\leq \log 9 + \frac{1}{2}\log 32 \leq 3.94,\end{aligned}$$

and $h(\eta_2) = h(\alpha) = \frac{\log\alpha}{2} \leq 1.77,\ \ h(\eta_3) \leq \log 10 \leq 2.31$ and thus,

$$A_1 := 7.88, A_2 = 3.54, A_3 = 4.62$$

Considering Theorem 2.1 and Equation (9), we have

$$\begin{aligned}\log\Gamma &\geq -1.4\cdot 30^{l+3} l^{4.5} d_L^2 (1+\log d_L)(1+\log D)A_1A_2\cdots A_l \\ \log\Gamma &\geq -1.25\cdot 10^1 4(1+\log n) \\ \log 72.9 - (m-l)\log 10 &\geq -1.25\cdot 10^1 4(1+\log n) \\ (m-l)\log 10 &\leq 1.25\cdot 10^1 4(1+\log n) + \log 72.9.\end{aligned}$$

Step 2: Rewriting Equation (6) differently as

$$\begin{aligned}\frac{\alpha^n - \beta^n}{4\sqrt{2}} &= \frac{1}{9}\{a10^m + 9(a-b)10^l - a\} \\ \left|\alpha^n - \frac{4\sqrt{2}(a10^m + 9(a-b)10^l)}{9}\right| &= |\beta^n - 4\sqrt{2}a| < |\alpha^{-n} + 4\sqrt{2}| < 6.\end{aligned}$$

Dividing both sides of the above inequality by $\alpha^n/4\sqrt{2}$, we get

$$\left|\frac{4\sqrt{2}(a10^{m-l}+9(a-b))}{9}\alpha^{-n}10^l-1\right|<\frac{12}{\alpha^n}.$$

Let

$$\Gamma':=\frac{4\sqrt{2}(a10^{m-l}+9(a-b))}{9}\alpha^{-n}10^l-1. \tag{10}$$

If $\Gamma'=0$, then

$$\alpha^n=4\sqrt{2}\left(\frac{a10^m}{9}+\frac{9(b-a)10^l}{9}\right),$$

and consequently, the conjugate of α^n in $\mathbb{Q}(\sqrt{2})$ will provide

$$\frac{40\sqrt{2}10^m}{9}\leq\left|4\sqrt{2}\left(\frac{a10^m}{9}+\frac{9(b-a)10^l}{9}\right)\right|=|\beta^n|<1,$$

which is not possible for any natural number m. Therefore, $\Gamma'\neq 0$.

Let $\eta_1=\frac{4\sqrt{2}(a10^{m-l}-9(a-b))}{9}$, $\eta_2=\alpha$, $\eta_3=10$, $d_1=1$, $d_2=-n$, $d_3=l$, where $\eta_1,\eta_2,\eta_3\in\mathbb{Q}(\sqrt{2})$ and $d_1,d_2,d_3\in\mathbb{Z}$. The degree of $\mathbb{L}:=\mathbb{Q}(\sqrt{2})$ is $d_{\mathbb{L}}=2$. Hence, the logarithmic Weil height will be

$$h(\eta_1)=h(\frac{4\sqrt{2}}{9})+h(a10^{m-l}-9(a-b))\leq 1.26\cdot 10^{14}(1+\log n)$$

and

$$h(\eta_2)=\frac{1}{2}\log\alpha,\quad h(\eta_3)=\log 10,$$

which implies that

$$A_1=2.52\cdot(1+\log n),A_2=3.54,A_3=4.62.$$

Since $1\leq m\leq l$ and $l<n+1$, we take $D=n+1$. Now, In view of Theorem 2.1 and Equation (10),

$$\log(\Gamma')>-3.997\cdot 10^{27}(1+\log n)^2$$

and hence

$$n\log\alpha\leq 3.997\cdot 10^{27}(1+\log n)^2+\log 12,$$

or

$$n\leq 2.14\cdot 10^{31}$$

and by Lemma 3.1, $m\leq 1.64\cdot 10^{31}$.

This is mentioned in the following lemma.

Lemma 3.2. *All solutions of Equation* (5) *satisfy*

$$m\leq l<1.64\cdot 10^{31}\quad\text{and}\quad n<2.14\cdot 10^{31}.$$

To lower the above bounds, let

$$\Lambda = \log\left(\frac{9}{4\sqrt{2}a}\alpha^n 10^{-m}\right) = \log\left(\frac{9}{4\sqrt{2}}\right) + n\log\alpha - m\log 10,$$

and this implies that

$$\Gamma = |e^{\Lambda} - 1| < \frac{72.9}{10^{m-l}}$$

The inequality $|e^z - 1| < y$ for real values of z and y, implies that $z < 2y$ and thus,

$$|\Lambda| < \frac{145.8}{10^{m-l}}$$

or

$$\left|\log\left(\frac{9}{4\sqrt{2}a}\right) + n\log\alpha - m\log 10\right| < \frac{145.8}{10^{m-l}},$$

or

$$\left|m\left(\frac{\log 10}{\log\alpha}\right) - n - \frac{\log\left(\frac{9}{4\sqrt{2}a}\right)}{\log\alpha}\right| < \frac{82.7}{10^{m-l}} < \frac{82.7}{10^{m}}.$$

In accordance to Lemma 2.4 by considering

$$\kappa = \frac{\log 10}{\log\alpha},\quad \mu = \frac{\log\left(\frac{9}{4\sqrt{2}a}\right)}{\log\alpha},\quad A = 82.7 \text{ and } B = 10.$$

Since $\|q\mu\| - M\|q\kappa\| < 0.06 := \epsilon$, and considering the continued fraction expansion of κ, the denominator q which satisfies $q > 1.9\cdot 10^{33} > 6M$ where M is defined as $m < 1.64\cdot 10^{31} := M$, is $q_{64} = 193515224029707700321265026524859$, we get

$$m \leq \frac{\log(Aq/\epsilon)}{\log B} \leq 35.50$$

and hence, $m \leq 35$.

Considering

$$\Lambda' = \log\left(\frac{4\sqrt{2}(a10^{m-l} - 9(a-b))}{9}\alpha^{-n}10^{l}\right)$$

or

$$\Lambda' = -n\log\alpha + l\log 10 + \log\left(\frac{4\sqrt{2}(a10^{m-l} - 9(a-b))}{9}\right).$$

Inequality (10) implies that

$$\Gamma' = |e^{\Lambda'} - 1| \leq \frac{12}{\alpha^n}.$$

and thus, $|\Lambda'| \leq \frac{24}{\alpha^n}$. This leads to

$$\left|l\log 10 - n\log\alpha + \log\left(\frac{4\sqrt{2}(a10^{m-l} - 9(a-b))}{9}\right)\right| \leq \frac{24}{\alpha^n}$$

or

$$\left| l\frac{\log 10}{\log \alpha} - n + \frac{\log\left(\frac{4\sqrt{2}(a10^{m-l}-9(a-b))}{9}\right)}{\log \alpha} \right| \leq \frac{24}{\log(\alpha)\alpha^n} < \frac{14}{\alpha^n}$$

By applying Lemma 2.4, consider

$$\kappa = \frac{\log 10}{\log \alpha}, \;\; \mu = \frac{\log\left(\frac{4\sqrt{2}(a10^{m-l}-9(a-b))}{9}\right)}{\log \alpha}, \;\; A = 14, \;\; B = \alpha,$$

$M\|q_\kappa\|$ has the upper bound 0.000009 as $l < M := 1.64 \cdot 10^{31}$ and the denominator of the 71-st convergent of the continued fraction expansion of κ is

$$q_\kappa = 1367697939567763980130066854527854 03.$$

The smallest value of $||Q||$ over all the values of a, b, $m > 0.000122$. Thus, $\epsilon = 0.0001 < ||q|| - M||q\kappa||$ is considered. Therefore,

$$n \leq \frac{\log(Aq/\epsilon)}{\log B} < 52.62,$$

which contradicts the fact that $n > 55$. This completes the proof of Theorem 1.1. □

3.2 Proof of Theorem 1.2

Assume that

$$C_n = a\left(\frac{10^m - 1}{9}\right) + (b-a)10^l, \; 0 \leq l < m, \; 0 \leq a, b \leq 9. \tag{11}$$

A quick computer search reveals that the only solutions existing in the interval $n \in [4, 55]$ are C_1, C_2, C_3, C_4. So from now on, the bound for the value of n is assumed as $n > 55$.

Lemma 3.3. *All solutions of Equation* (11) *satisfy*

$$m\text{log}10 + 0.693 < n\text{log}\alpha < m\text{log}10 - 3.37.$$

Proof. Since the Lucas-balancing numbers satisfy $\alpha^n \leq 2C_n \leq \alpha^{n+1}$, this gives

$$\alpha^n \leq 2C_n \leq 2 \cdot 10^m \;\text{ and }\; 2 \cdot 10^{m-1} \leq 2C_n \leq \alpha^{n+1}$$

or

$$n \log \alpha \leq \log 2 + m \log 10 \;\text{ and }\; \log 2 + (m-1)\log 10 \leq (n+1)\log \alpha$$

or

$$n \log \alpha \leq 0.693 + m \log 10 \;\text{ and }\; m \log 10 - 3.37 \leq n \log \alpha, \tag{12}$$

which proves the lemma. □

By implementing the Binet's formula of Lucas-balancing numbers, Equation (11) can be written as

$$\frac{\alpha^n+\beta^n}{2}=a\left(\frac{10^m-1}{9}\right)+(b-a)10^l, \tag{13}$$

which is studied in two different steps.

<u>Step 1:</u> Rewriting Equation (13) as

$$\frac{9}{2}(\alpha^n+\beta^n)=a10^m+9(b-a)10^l-a$$

$$\text{or}\quad \left|\frac{9}{2}\alpha^n-a10^m\right|=\left|9(b-a)10^l-\frac{9}{2}\beta^n-a\right|=81\cdot 10^l$$

$$\text{or}\quad \left|\frac{9}{2a}10^{-m}\alpha^n-1\right|\le\frac{81\cdot 10^l}{a10^m} \tag{14}$$

Let

$$\Gamma=\left|\frac{9}{2a}10^{-m}\alpha^n-1\right|,$$

which gives

$$|\log\Gamma|\le\log(81)-(m-l)\log(10)10^{m-l}. \tag{15}$$

If $\Gamma=0$, then $\sqrt{2}=q\alpha^n$, where $q\in\mathbb{Q}$ and hence, $\alpha^{2n}=2q^{-2}\in\mathbb{Q}$, which is not possible for any $n>0$. Therefore, $\Gamma\neq 0$. Let

$$\eta_1=\frac{9}{2a},\eta_2=\alpha,\eta_3=10,d_1=1,d_2=n,d_3=-m,$$

where η_1,η_2,η_3 belong to the field $\mathbb{Q}(\alpha)$ and d_1,d_2,d_3 are integers. Since the degree of the field $d_L=2$, and $10^{m-1}<C_n<\frac{\alpha^{n+1}}{2}$, we consider $D=n+2$. Application of the properties of logarithmic Weil height will give

$$h(\eta_1)=h\left(\frac{9}{2a}\right)\le h(9)+h(2a)\le 2\log 9+\log 2,$$

which implies that $2h(\eta_1)<10.18:=A_1$ and $2h(\eta_2)<1.8:=A_2$, $2h(\eta_3)<4.7:=A_3$ after proceeding as in the previous section. In view of Theorem 2.1 and Equation (15), we have

$$(m-l)\log(10)<\log(81)+8.36\cdot 10^{13}(1+\log(n+2))$$

or

$$m-l<3.645\cdot 10^{13}(1+\log(n+2)). \tag{16}$$

<u>Step 2:</u> Rewriting Equation (11) as

$$\left|\frac{\alpha^n}{2}-\left(\frac{a10^m+9(b-a)10^l}{9}\right)\right|=\left|\frac{-\beta^n}{2}-\frac{a}{9}\right|<2.5.$$

Dividing $\alpha^n/2$, we get

$$1 - \frac{2\alpha^{-n}(a10^m + 9(b-a)10^l)}{9} \le \frac{4.5}{\alpha^n} < \frac{1}{\alpha^{n-0.9}}. \tag{17}$$

Considering

$$\Gamma' = 1 - \frac{2\alpha^{-n}10^l(a10^{m-l} + 9(b-a))}{9},$$

and applying natural logarithm to both sides in (17), it gives $\log \Gamma' < (n-0.9)\log\alpha$. If $\Gamma' = 0$, then

$$\alpha^n = \frac{2(a10^m + 9(b-a)10^l)}{9}$$

and the corresponding conjugate of α^n in $\mathbb{Q}(\sqrt{2})$ is

$$\frac{2\cdot 10^{m+1}}{9} \le \left|\frac{2(a10^m + 9(b-a)10^l)}{9}\right| = |\beta^n| < 1,$$

which is not possible for any value of m and hence $\Gamma' \neq 0$. Let

$$\eta_1 = \frac{2(a10^{m-l} + 9(b-a))}{9}, \eta_2 = \alpha, \eta_3 = 10, d_1 = 1, d_2 = -n, d_3 = l,$$

where η_1, η_2, η_3 belong to the field $\mathbb{Q}(\alpha)$ and d_1, d_2, d_3 are integers with $d_L = 2$ and $D = n+2$. Using the properties of logarithmic Weil height,

$$h(\eta_1) = h\left(\frac{2(a10^{m-l} + 9(b-a))}{9}\right) \le 10.17 + (m-l)2.31$$
$$2h(\eta_1) \le 20.34 + 4.62(m-l) = A_1,$$

and continuing the same process for η_2 and η_3, we get $A_2 = 1.8$ and $A_3 = 4.7$. Application of Theorem 2.1 and Equation (17) will provide

$$\log(\Gamma') > -8.21\cdot 10^{12}(20.34 + 4.62(3.64\cdot 10^{13}(1+\log(n+2))))(1+\log(n+2)),$$

or

$$(n-0.9)\log\alpha < 8.21\cdot 10^{12}(20.34 + 4.62(3.64\cdot 10^{13}(1+\log(n+2))))(1+\log(n+2)),$$

and hence $n < 4.00009\cdot 10^{30}$ and therefore $m \le l < 3.06986\cdot 10^{30}$.

To lower the bounds, Equation (11) is revised as

$$\frac{\alpha^n + \beta^n}{2} = C_n = \frac{a10^m}{9} - \frac{a}{9} + (b-a)10^l,$$
$$\text{or} \quad \frac{\alpha^n}{2} - \frac{a10^m}{9} = \left((b-a)10^l - \frac{a}{9}\right) - \frac{\beta^n}{2},$$
$$\text{or} \quad \frac{a10^m}{9}\left(\frac{9}{2a}10^{-m}\alpha^n - 1\right) = \left\{(b-a)10^l - \frac{a}{9}\right\} - \frac{\beta^n}{2}.$$

Consider

$$\Gamma = \frac{9}{2a}10^{-m}\alpha^n - 1,$$

with

$$\log(\Gamma+1) = \Lambda_1 = \log\left(\frac{9}{2a}10^{-m}\alpha^n\right) = \log(9/2a) + n\log\alpha - m\log 10.$$

Then, we obtain that $\frac{a10^m}{9}\left|\frac{9}{2a}10^{-m}\alpha^n - 1\right| = |e^{\Lambda_1} - 1|\frac{a10^m}{9} > 0$ and hence

$$\Lambda_1 < e^{\Lambda_1} - 1 = \Gamma_1 < \frac{81}{10^{m-l}}.$$

This implies that

$$\begin{aligned}
&\log\left(\frac{9}{2a}10^{-m}\alpha^n\right) = \log(9/2a) + n\log\alpha - m\log 10 < \frac{81}{10^{m-l}}\\
\text{or}\quad &\log\left(\frac{9}{2a}10^{-m}\alpha^n\right) \le \exp\{\log 81 - (\log 10)(m-l)\}\\
\text{or}\quad &\Lambda_1 \le 10^{1.91}\exp(-2.3(m-l)),
\end{aligned}$$

which holds when $Y = m - l = 4.01 \cdot 10^{30}$. We also have

$$\frac{\Lambda_1}{\log 10} = \frac{\log(9/2a)}{\log 10} + n\frac{\log\alpha}{\log 10} - m.$$

Thus, considering $c = 10^{1.91}, \delta = 2.3, X_0 = \frac{\log(9/2a)}{\log 10}, \vartheta = -\frac{\log\alpha}{\log 10}$, $\vartheta_1 = \log\alpha$, $\vartheta_2 = \log 10$, $\beta = \log(9/2a)$, the value of q from the continued fractions p/q of ϑ which satisfies the hypothesis of Lemma 2.3 for $1 \le a \le 9$ and $q > X_0$, is $q_{61} = 34316950683475914479089643709189$, and which concludes that $m - l = Y < 34.047$, i.e., $Y \le 34$. Now, $0 \le m - l \le 34$ is considered.

Rewriting Equation (11) as

$$\frac{\alpha^n}{2} - \frac{a10^m + 9(b-a)10^l}{9} = \frac{-\beta^n}{2} - \frac{a}{9},$$

or

$$\frac{\alpha^n}{2}\left\{1 - 2\frac{a10^{m-l} + 9(b-a)}{9}10^l\alpha^{-n}\right\} = -\left\{\frac{a}{9} + \frac{1}{2\alpha^n}\right\}$$

or

$$\frac{\alpha^n}{2}\left\{1 - e^{\Lambda_1'}\right\} = -\left\{\frac{a}{9} + \frac{1}{2\alpha^n}\right\},$$

if we assume that

$$\Lambda_1' = \log\left(\frac{a10^{m-l} + 9(b-a)}{9}10^l\alpha^{-n}\right).$$

Since $\frac{a}{9} + \frac{1}{2\alpha^n} > \frac{1}{9} + \frac{1}{2\alpha^n} > 0$, then $1 - e^{\Lambda_1'} > 0$ and henceforth $\Lambda_1' > 0$. In view of Equation (17), we have

$$0 < \Lambda_1' < e^{\Lambda_1'} - 1 = |\Gamma_1'| < \frac{1}{\alpha^{n-0.9}},$$

which implies that

$$\log\left(2\frac{a10^{m-l} + 9(b-a)}{9}\right) + l\log 10 - n\log\alpha < \frac{1}{\alpha^{n-0.9}} < \alpha^{0.9}\exp(-1.76 \cdot n).$$

Consider

$$\psi' = \frac{\log(2(a10^{m-l} + 9(b-a))/9)}{\log 10}, \ c = \alpha^{0.9}, \ \delta = 1.76,$$

$$\vartheta = \frac{\log\alpha}{\log 10}, \ \vartheta_1 = -\log\alpha, \ \vartheta_2 = \log 10, \ \beta = \log(2(a10^{m-l} + 9(b-a))/9).$$

Clearly, $\beta \neq 0$ and evidently $\psi' \neq 0$ except when $a = 5, \ b = 4, \ m - l = 1$. Thus, for $\psi' \neq 0$, we find that

$$q_{70} = 16582967789052824792691327834284630 > X_0$$

satisfies the hypothesis of Lemma 2.3 and hence the application of Lemma 2.3 gives $n < 43.68$, i.e., $n \leq 43$, which is a contradiction to our assumption that $n > 55$. $\square$

4 Conclusion

We have investigated the occurrence of almost repdigits, defined as positive integers whose decimal representations consist of all identical digits except for at most one, within the balancing sequence $\{B_n\}$ and the Lucas-balancing sequence $\{C_n\}$ through a combination of theoretical arguments using Baker's theory for linear forms in logarithms of algebraic numbers and the Baker–Davenport reduction procedure and the exhaustive search using *Mathematica* and *Maple* software. We established that the only terms of the balancing sequence that are almost repdigits are $0, 1, 6$ and the only terms of the Lucas-balancing sequence that are almost repdigits are $1, 3, 17, 99, 577, 3363$. The interested readers can also extend the search for almost repdigits of base 10 to almost repdigits of base g in different sequences and sums or products of those sequences.

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