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Non-Fisherian generalized Fibonacci numbers

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Abstract: Using biology as inspiration, this paper explores a generalization of the Fibonacci sequence that involves gender biased sexual reproduction. The female, male, and total population numbers along with their associated recurrence relations are considered. We demonstrate that the generalized Fibonacci numbers being investigated are generalized third order Pell–Lucas numbers. Sequence properties, generating functions, and closed-form solutions for these new generalized Fibonacci numbers, as well as several identities involving Jacobsthal, Leonardo, and generalized Leonardo numbers are presented. The generalized Fibonacci number framework developed gives rise to many previously uncataloged sequences, and develops new connections between known sequences.

Keywords: Generalized Fibonacci numbers, Generalized third order Pell–Lucas numbers, Jacobsthal numbers, Leonardo numbers, Fisher's principle, Extraordinary sex ratios.
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1 Introduction

In 1202 the Italian mathematician Leonardo Pisano, also known as Fibonacci, wrote a book of calculation called Liber Abaci that introduced the Hindu-Arabic number system and methods of algebra to Europe. In this book he introduced and solved many problems including one involving the calculation of a sequence of integers that later would become synonymous with his name. Translated from Latin, the statement of this problem is as follows [12]:

A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from one pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

Inherent to Fibonacci's problem formulation was the assumption that the reproductive female-to-male sex ratio for rabbits is $1 : 1$. Although the population growth problem he posed is unrealistic from a biological perspective, the balanced reproductive ratio he used is not altogether unreasonable. An evolutionary model called Fisher's principle explains why most species have a sex ratio that is approximately $1 : 1$ [6, 8]. However, this is not always the case. In humans, the female-to-male sex ratio can become biased due to war, environmental factors, or sex-selective abortions. Elsewhere in the animal kingdom, examples abound of species violating Fisher's principle. Many reptile species have temperature dependent sex determination. American alligator populations, for example, typically have five females for every male [7]. Other examples of imbalanced reproductive ratios exist among birds, fish, and arthropods. In this paper we use the non-Fisherian paradigm as inspiration to explore a new generalization of Fibonacci numbers.

Remark 1.1. *There is evidence to suggest that the 10th century French mathematician Gerbert d'Aurillac, who was educated in Moorish-occupied Spain, and later became Pope Silvester II, may have been the first person to introduce the Hindu-Arabic number system to Europe* [4, 9].

2 Deriving the governing recurrence relations

Fibonacci assumed a balanced ratio of female and male offspring, and therefore could count pairs of rabbits. In order to capture population growth for a biased reproductive ratio, we split the Fibonacci sequence into two separate sequences, one for each of the sexes. We retain Fibonacci's original notion of a delay in progeny reaching sexual maturity, which in turn leads to a second order recurrence relation. We also assume that although differences in the numbers of females and males exist, each female is able to get impregnated. Let a and b be positive integers and let $a : b$ be the reproductive female-to-male sex ratio. When it is clear from the context what reproductive ratio is being consider, we denote the female population sequences as $\{W_n\}$, the male population sequence as $\{M_n\}$, and their respective n-th sequence entries as W_n and M_n. Otherwise, we use superscripts to avoid confusion, and write $\{W_n^a\}$, W_n^a, $\{M_n^{a:b}\}$, and $M_n^{a:b}$. Female and male population growth are both governed by the number of fertile females within a given generation of the population. A fertile female will give birth to "a" females and "b" males during each pregnancy. We therefore obtain the following two recurrence relations:

$$W_n = W_{n-1} + aW_{n-2}, \tag{1}$$

$$M_n = M_{n-1} + bW_{n-2}. \tag{2}$$

The family of recurrence relations in (1) produce Lucas sequences of the form:

$$U_n(P, Q) = P{\cdot}U_{n-1} - Q{\cdot}U_{n-2},$$

where $P = 1$ and $Q = -a$. To obtain recurrence relations for the male sequences, we express W_{n-2} in terms of M_{n-2} by recognizing that for each generation, n:

$$M_n = \frac{bW_n + (a-b)}{a}. \tag{3}$$

Consequently, $bW_{n-2} = aM_{n-2} - (a-b)$, and we rewrite Equation (2) thus obtaining the male recurrence relation:

$$M_n = M_{n-1} + aM_{n-2} - (a-b). \tag{4}$$

Let $\{F_n^{a:b}\}$ represent the sequence of female and male offspring in population with a reproductive ratio of $a : b$. We obtain the recurrence relations for the n-th entry of this sequence by adding recurrence relations (1) and (4) as follows:

$$\begin{aligned} F_n^{a:b} &= W_n + M_n \\ &= W_{n-1} + M_{n-1} + a(W_{n-2} + M_{n-2}) - (a-b) \\ &= F_{n-1}^{a:b} + aF_{n-2}^{a:b} - (a-b). \end{aligned} \tag{5}$$

This is a generalization of the Fibonacci recurrence relation. When $a = b = 1$, Equation (5) simplifies to $F_n^{1:1} = F_{n-1}^{1:1} + F_{n-2}^{1:1}$. Viewed in a slightly different way, when the reproductive ratio is $1\!:\!1$, recurrence relations (1) and (4) each give rise to Fibonacci recurrences in their own right, albeit with numbers representing individual females or males rather than pairs. Given the fact that the reproductive ratio is $1 : 1$ and $F_n^{a:b}$ is the sum of the female and male numbers, each fertile female will produce two offspring, i.e., a pair consisting of one female and one male. In other words, letting F_n be the n-th entry of the standard Fibonacci sequence, $\{F_n\}$, we have:

$$F_n = (W_{n-1} + M_{n-1}) + (W_{n-2} + M_{n-2}) = F_{n-1} + F_{n-2}.$$

Remark 2.1. *Situations in which $a = b$ are straight-forward and of less interest than those involving biased reproductive ratios. In cases with parity between the number of female and male offspring, the generalized Fibonacci sequence, $\{F_n^{a:b}\}$, simply becomes a Lucas sequence, $U_n(P, Q)$, where $P = 1$ and $Q = -2a$. A few sequences that involve Fisherian growth are listed in Appendix C.*

We can further transform the second-order nonhomogeneous recurrence relations obtained in (4) and (5) into a homogeneous third order recurrence relations. For the male recurrence relation we consider the difference

$$M_n - M_{n-1} = \big(M_{n-1} + aM_{n-2} - (a-b)\big) - \big(M_{n-2} + aM_{n-3} - (a-b)\big).$$

By rearranging terms and simplifying the expression we obtain:

$$M_n = 2M_{n-1} + (a-1)M_{n-2} - aM_{n-3}. \tag{6}$$

Proceeding similarly for the generalized Fibonacci recurrence relation, we obtain:

$$F_n^{a:b} = 2F_{n-1}^{a:b} + (a-1)F_{n-2}^{a:b} - aF_{n-3}^{a:b}. \tag{7}$$

Remark 2.2. *Notice that when $a = 1$, equations* (6) *and* (7) *become $M_n = 2M_{n-1} + M_{n-3}$ and $F_n^{a:b} = 2F_{n-1}^{a:b} - F_{n-3}^{a:b}$. These recurrence relations represent a well-known Fibonacci identity and are equivalent to $M_n = M_{n-1} + M_{n-2}$ and $F_n^{a:b} = F_{n-1}^{a:b} + F_{n-2}^{a:b}$ respectively.*

The recurrence relation in (6) and (7) produces generalized Tribonacci numbers. More specifically, they give rise to generalized third order Pell sequences defined by:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, \text{where } r = 2,\ s = a - 1,\ t = -a \text{ and } a \in \mathbb{N}^*. \tag{8}$$

A vast amount of research has been devoted to the generalizion of Tribonacci and Pell numbers. Catarino [1] investigated k-Pell numbers of the form $P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}$ for $n \geq 2$, $P_{k,0} = 0$ and $P_{k,1} = 1$. Trojnar-Spelina and Włoch [15] introduced a generalization of Pell and Pell–Lucas numbers defined by $Q_{k,n} = kQ_{k,n-1} + (k-1)Q_{k,n-2}$ for $k \geq 2$, $n \geq 2$ and respective initial values $Q_0 = 0$ and $Q_1 = 1$ or $Q_0 = Q_1 = 2$. Soykan [13] considered generalized third-order Pell numbers defined by the same recurrence relation as (8) and $r = 2$, albeit with different coefficients $s = t = 1$. While somewhat related to our work, none of this aforementioned research covers the generalized Pell and Pell–Lucas numbers considered in this paper. In the following section, we present generating functions and Binet's formulas associated with the new generalized third-order Pell sequences $\{M_n\}$ and $\{F_n^{a:b}\}$. Throughout this paper, we alternate between using the second-order nonhomogeneous recurrence relations (4) and (5), and their equivalent third-order homogeneous recurrence relations (6) and (7), depending upon which are more convenient for the situation at hand.

3 Closed-form solutions and generating functions

As previously mentioned, the family of female recurrence relations in (1) are associated with well-known Lucas sequences [5]. They have characteristic equations: $x^2 - x - a = 0$, with a discriminant $D = 1 - 4a$ and roots: $(1 \pm \sqrt{D})/2$. Using Binet's formula, a closed-form solution for the n-th term of these sequences is:

$$W_n = \frac{1}{\sqrt{D}} \left[\left(\frac{1 + \sqrt{D}}{2} \right)^n - \left(\frac{1 - \sqrt{D}}{2} \right)^n \right], \text{ where } n \geq 1. \tag{9}$$

Remark 3.1. *Notice that since we want the initial two terms of female sequence $W_1 = W_2 = 1$, we start the sequence at $n = 1$, as opposed to $n = 0$, which is common practice.*

We now turn our attention to deriving a closed-form solution for the n-th entry of male sequences. The characteristic equation of the third order homogeneous recurrence relations for $\{M_n\}$ in (6) is $x^4 - 2x^2 - (a-1)x + a = (x^2 - x - a)(1 - x)$. The roots of this equation are: $(1 \pm \sqrt{D})/2$ and 1, where, D once again equals $1 - 4a$. Using Spickerman's generalized of Binet's formula for Tribonacci sequences in [14], the n-th solution of the males sequence can be written as:

$$c_1 1^n + c_2\Big(\frac{1+\sqrt{D}}{2}\Big)^n + c_3\Big(\frac{1-\sqrt{D}}{2}\Big)^n = c_1 + c_2\Big(\frac{1+\sqrt{D}}{2}\Big)^n + c_3\Big(\frac{1-\sqrt{D}}{2}\Big)^n,$$

where c_1, c_2, and c_3 are constants and $n \geq 1$.

We use solution (9) for the female sequences along with the established relationship between male and female sequences with a reproductive ratio of $a : b$ from Equation (3), and obtain the following:

$$\begin{aligned} M_n &= \frac{bW_n - (a-b)}{a} \\ &= \frac{\frac{b}{\sqrt{D}}\Big[\Big(\frac{1+\sqrt{D}}{2}\Big)^n - \Big(\frac{1-\sqrt{D}}{2}\Big)^n\Big] - (a-b)}{a} \\ &= \frac{b}{a\sqrt{D}}\Big[\Big(\frac{1+\sqrt{D}}{2}\Big)^n - \Big(\frac{1-\sqrt{D}}{2}\Big)^n\Big] - \frac{(a-b)}{a}, \text{where } n \geq 1. \end{aligned} \tag{10}$$

With solutions (9) and (10) in hand, we are now in a position to write down the closed-form solution for the generalized Fibonacci sequence with a reproductive ratio of $a : b$:

$$\begin{aligned} F_n^{a:b} &= W_n + M_n \\ &= \frac{1}{\sqrt{D}}\Big[\Big(\frac{1+\sqrt{D}}{2}\Big)^n - \Big(\frac{1-\sqrt{D}}{2}\Big)^n\Big] + \frac{b}{a\sqrt{D}}\Big[\Big(\frac{1+\sqrt{D}}{2}\Big)^n - \Big(\frac{1-\sqrt{D}}{2}\Big)^n\Big] - \frac{(a-b)}{a} \\ &= \frac{a+b}{a\sqrt{D}}\Big[\Big(\frac{1+\sqrt{D}}{2}\Big)^n - \Big(\frac{1-\sqrt{D}}{2}\Big)^n\Big] - \frac{(a-b)}{a}, \text{where } n \geq 1. \end{aligned} \tag{11}$$

In addition to the closed-form solutions we derived, combinatorial series solutions also exist for the female, male, and generalized Fibonacci sequences. Using a result from Croot [3], we know that a solution for the n-th term of a Lucas sequence $U_n(P, Q) = PU_{n-1} - QU_{n-2}$, where $U_0 = 0, U_1 = 1$ and $n \geq 3$ is:

$$U_n = \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} \binom{n-1-k}{k} (-Q)^k (P)^{n-1-2k}.$$

Letting $P = 1$ and $Q = -a$, for $n \geq 3$ we can express the n-th term of the female sequence as:

$$W_n = \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^k \binom{n-1-k}{k}. \tag{12}$$

As we previously did when deriving Equation (10), we use the male and female sequence relationship $M_n = (bW_n + a - b)/a$ from Equation (3), and obtain the following expression for the male sequence for $n \geq 3$:

$$M_n = \frac{b\Big(\sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^k \binom{n-1-k}{k}\Big) + (a-b)}{a} = b \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^{k-1} \binom{n-1-k}{k} + \frac{a-b}{a}. \tag{13}$$

Combining Equations (12) and (13) yields the following expression for the generalized Fibonacci sequence with a female-to-male reproductive ratio of $a:b$ for $n \geq 3$:

$$\begin{aligned}
F_n^{a:b} &= W_n + M_n \\
&= \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^k \binom{n-1-k}{k} + b \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^{k-1} \binom{n-1-k}{k} + \frac{a-b}{a} \\
&= (a+b) \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^{k-1} \binom{n-1-k}{k} + \frac{a-b}{a}.
\end{aligned} \tag{14}$$

Beyond obtaining closed-form solutions, we would also like to derive generating functions for our number sequences. We begin by considering the sequence $\{W_n\}$. The generating function for Lucas sequences is a well known result. Typically, however, the two first terms of these sequences are 0 and 1. In the case of $\{W_n\}$, we desire that the initial two terms both be 1. This alters the result slightly. In the interest of clarity, we present here this classical result. Let $G_W(x)$ be the generating function we seek. Proceeding in typical fashion, we obtain the following four equations:

$$\begin{aligned}
G_W(x) &= g_1 + g_2 x + g_3 x^2 + \dots \\
-xG_W(x) &= - g_1 x - g_2 x^2 - \dots \\
-ax^2 G_W(x) &= - a g_1 x^2 - \dots \\
(1 - x - ax^2) G_W(x) &= g_1 + (g_2 - g_1)x + (g_3 - g_2 - a g_1)x^2 + \dots .
\end{aligned}$$

Solving the last equation for $G_W(x)$ yields:

$$G_W(x) = \frac{g_1 + (g_2 - g_1)x + (g_3 - g_2 - a g_1)x^2}{1 - x - ax^2}.$$

Provided $g_1 = g_2 = 1$, and using the fact that $g_3 = g_2 + a g_1$, we obtain the following generating function for the female sequence, W_n:

$$G_W(x) = \frac{1}{1 - x - ax^2} \; . \tag{15}$$

Proceeding in a similar manner, we determine the generating functions for the generalized third order Pell and Pell–Lucas sequences. Let $G(x)$ be the generic generating function we seek. We then obtain the following:

$$\begin{aligned}
G(x) &= g_1 + g_2 x + g_3 x^2 + g_4 x^3 \cdots \\
-2xG(x) &= - 2g_1 x - 2g_2 x^2 - 2g_3 x^3 \cdots \\
-(a-1)x^2 G(x) &= - (a-1)g_1 x^2 - (a-1)g_2 x^3 \cdots \\
+ax^3 G(x) &= + a g_1 x^3 \cdots \\
(1 - x - (a-1)x^2 + ax^3)G(x) &= g_1 + (g_2 - 2g_1)x + (g_3 - 2g_2 - (a-1)g_1)x^2 + (g_4 - 2g_3 - (a-1)g_2 + a g_1)x^3 \dots .
\end{aligned}$$

Solving the last equation for $G(x)$ yields:

$$G(x) = \frac{g_1 + (g_2 - 2g_1)x + (g_3 - 2g_2 - (a-1)g_1)x^2 + (g_4 - 2g_3 - (a-1)g_2 + a g_1)x^3}{1 - x - (a-1)x^2 + ax^3}.$$

Provided $g_1 = g_2$, $g_3 = g_2 + ag_1 - (a - b)$, and using the fact that $g_4 = 2g_3 + (a - 1)g_2 - ag_1$, we obtain the following generating function:

$$G(x) = \frac{g_1 - g_1 x + (-(a - b))x^2}{1 - 2x - (a - 1)x^2 + ax^3} \ . \tag{16}$$

In the case of the male sequence, $\{M_n\}$, where $g_1 = g_2 = 1$, the generating function $G_M(x)$ is:

$$G_M(x) = \frac{1 - x - (a - b)x^2}{1 - 2x - (a - 1)x^2 + ax^3} \ . \tag{17}$$

In the case of the non-Fisherian generalized Fibonacci sequence, $\{F_n^{a:b}\}$, where $g_1 = g_2 = 2$, the generating function $G_F(x)$ is:

$$G_F(x) = \frac{2 - 2x - (a - b)x^2}{1 - 2x - (a - 1)x^2 + ax^3} \ . \tag{18}$$

Remark 3.2. *Observe that when* $a = b = 1$.

$$G_F(x) = \frac{2 - 2x}{1 - 2x + x^3} = \frac{2(1 - x)}{(1 - x - x^2)(1 - x)} = 2\left(\frac{1}{1 - x - x^2}\right) = 2G_W(x).$$

In other words, when the female and male reproductive ratio is 1:1, *the generating function for the generalized Fibonacci sequence, or the generalized third order Pell–Lucas sequence, is twice that of the generating function for the female sequence from* (15).

4 Sequence differences and relationships

Given the biased nature of the reproductive ratios at work in the sequences under consideration, it is interesting to examine differences, both with respect to successive terms of a single sequence, as well as differences between corresponding terms of different sequences.

We begin by pointing out a formula for computing the reproductive ratio $a : b$ given two corresponding elements of related female and male sequences, $\{W_n\}$ and $\{M_n\}$. Observe that by rewriting Equation (3), we obtain $a(M_n - 1) = b(W_n - 1)$. Which means that

$$a/b = (W_n - 1)/(M_n - 1).$$

Restricting our consideration to a single sequence, we know that as n grows large, the ratio W_{n+1}/W_n of successive terms of the female sequence $\{W_n\}$ with a reproductive ratio of $a : b$ tends to $\varphi_a = (1 + \sqrt{D})/2$, where $D = 1 + 4a$. This turns out to also be the case for the male sequence $\{M_n\}$ and the generalized Fibonacci sequence $\{F_n^{a:b}\}$. In spite of the fact that these sequences are nonhomogeneous and both contain the term $-(a - b)$, as n grows large, this term becomes negligible and the ratio between successive terms in each of these sequences approaches φ_a. In other words:

$$\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = \lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{F_{n+1}^{a:b}}{F_n^{a:b}} = \frac{1 + \sqrt{D}}{2} = \varphi_a.$$

Given the fact that the growth of a population is governed by the number of fertile females, and keeping in mind the form of the homogeneous third order recurrence relations (6) and (7), this is perhaps not all that surprising.

Turning our attention to differences in the number of female and male offspring produced with a biased sex ratio of $a:b$, let $\{\Delta_n\}$ represent the sequence of differences between corresponding entries in the female and male sequences $\{W_n\}$ and $\{M_n\}$. Using Equations (9) and (10), we obtain the following expression for the n-th difference:

$$\Delta_n = \mid W_n - M_n \mid = \frac{\mid a-b \mid}{a\sqrt{D}} \left[\left(\frac{1+\sqrt{D}}{2} \right)^n - \left(\frac{1-\sqrt{D}}{2} \right)^n \right] - \frac{a-b}{a}. \tag{19}$$

Using Equations (12) and (13), we can also express the n-th difference, Δ_n, as follows:

$$\Delta_n = \mid W_n - M_n \mid = \mid a-b \mid \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a^{k-1} \binom{n-1-k}{k} - \frac{a-b}{a}. \tag{20}$$

It is also interesting to consider differences between populations with different reproductive ratios. Let $\{F_n^{a_1:b_1}\}$ and $\{F_n^{a_2:b_2}\}$ be two generalized Fibonacci sequences with respective reproductive ratios $a_1 : b_1$ and $a_2 : b_2$, and let $\{M_n^{a_1:b_1}\}$ and $\{M_n^{a_2:b_2}\}$ represent their associated male sequences. Let D_1 and D_2 be the respective discriminants for the closed-form solutions of the sequences involving the female reproductive numbers a_1 and a_2 such that $D_1 = 1 + 4a_1$ and $D_2 = 1 + 4a_2$. Then using Equation (10), we can represent the difference between n-th entries of the two male sequences as:

$$\begin{aligned} \left(\Delta M_{a_2:b_2}^{a_1:b_1}\right)_n &= M_n^{a_1:b_1} - M_n^{a_2:b_2} \\ &= \frac{b_1}{a_1\sqrt{D_1}} \Big[\Big(\frac{1+\sqrt{D_1}}{2} \Big)^n - \Big(\frac{1-\sqrt{D_1}}{2} \Big)^n \Big] \\ &- \frac{b_2}{a_2\sqrt{D_2}} \Big[\Big(\frac{1+\sqrt{D_2}}{2} \Big)^n - \Big(\frac{1-\sqrt{D_2}}{2} \Big)^n \Big] + \frac{(a_2b_1 - b_2a_1)}{a_1a_2}. \end{aligned} \tag{21}$$

Similarly, using Equation (11), the difference between the n-th entries of the two generalized Fibonacci sequences becomes:

$$\begin{aligned} \left(\Delta F_{a_2:b_2}^{a_1:b_1}\right)_n &= F_n^{a_1:b_1} - F_n^{a_2:b_2} \\ &= \frac{a_1+b_1}{a_1\sqrt{D_1}} \Big[\Big(\frac{1+\sqrt{D_1}}{2} \Big)^n - \Big(\frac{1-\sqrt{D_1}}{2} \Big)^n \Big] \\ &- \frac{a_2-b_2}{a_2\sqrt{D_2}} \Big[\Big(\frac{1+\sqrt{D_2}}{2} \Big)^n - \Big(\frac{1-\sqrt{D_2}}{2} \Big)^n \Big] + \frac{(a_2b_1 - b_2a_1)}{a_1a_2}. \end{aligned} \tag{22}$$

Using the series solutions from Equations (13) and (14) we obtain the corresponding expressions:

$$
\begin{aligned}
\left(\Delta M^{a_1:b_1}_{a_2:b_2}\right)_n &= M^{a_1:b_1}_n - M^{a_2:b_2}_n \\
&= b_1 \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a_1^{k-1} \binom{n-1-k}{k} + \frac{a_1-b_1}{a_1} - b_2 \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a_2^{k-1} \binom{n-1-k}{k} - \frac{a_2-b_2}{a_2} \\
&= \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} \left[b_1 a_1^{k-1} - b_2 a_2^{k-1} \right] \binom{n-1-k}{k} + \frac{a_1 b_2 - a_2 b_1}{a_1 a_2},
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\left(\Delta F^{a_1:b_1}_{a_2:b_2}\right)_n &= F^{a_1:b_1}_n - F^{a_2:b_2}_n \\
&= (a_1+b_1) \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a_1^{k-1} \binom{n-1-k}{k} - (a_2+b_2) \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a_2^{k-1} \binom{n-1-k}{k} - \frac{a_2-b_2}{a_2} \\
&= \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} \left[(a_1+b_1) a_1^{k-1} - (a_2+b_2) a_2^{k-1} \right] \binom{n-1-k}{k} + \frac{a_1 b_2 - a_2 b_1}{a_1 a_2}.
\end{aligned} \tag{24}
$$

Remark 4.1. *Notice that the sum in Equation* (24) *involves the difference between the products of the total offspring for each generation and a power of the female reproductive rate for each of the respective populations. Notice also that in the case where* $a_1 = a_2$, *Equations* (22) *and* (24) *simplify to:*

$$
\left(\Delta F^{a_1:b_1}_{a_2:b_2}\right)_n = \frac{b_1-b_2}{a_1\sqrt{D_1}} \left[\left(\frac{1+\sqrt{D_1}}{2}\right)^n - \left(\frac{1-\sqrt{D_1}}{2}\right)^n \right] + \frac{b_2-b_1}{a_1} \tag{25}
$$

and

$$
\left(\Delta F^{a_1:b_1}_{a_2:b_2}\right)_n = (b_1-b_2) \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} a_1^{k-1} \binom{n-1-k}{k} + \frac{b_2-b_1}{a_1}. \tag{26}
$$

Given the fact that

$$
\frac{b_2-b_1}{a_1} = -\frac{a_1-b_1}{a_1} - \left(-\frac{a_1-b_2}{a_1}\right),
$$

in both cases

$$
\left(\Delta F^{a_1:b_1}_{a_2:b_2}\right)_n = M^{a:b_1}_n - M^{a:b_2}_n.
$$

In other words, as expected, if the female reproductive rates agree between two populations, the n-th difference in their generalized Fibonacci sequences is simply the difference between the n-th entries of the male sequences.

Growth of the male population is a function of the number of fertile females and the male reproduction number, b. Consequently, corresponding terms of two different male sequences that share the same female reproductive number, a, but have different male reproductive numbers are linearly related, and one can easily compute one sequence's entry given the corresponding entry in the other sequence, or calculate the difference between two corresponding entries.

Theorem 4.1. *Given two male sequences $\{M_n^{a:b}\}$ and $\{M_n^{a:b+k}\}$, with the same female reproductive number a, and respective male reproductive numbers b and $b+k$, where $k \geq 0$, the difference between their n-th entries is $k\left(\frac{M_n^{a:b}-1}{b}\right)$.*

Proof. Let $\{M_n^{a:b}\}$ and $\{M_n^{a:b+k}\}$ be two male sequences with female-to-male reproductive ratios $a : b$ and $a : b+k$, where $k \geq 0$. Using the relationship between male and female sequence entries from Equation (3), the difference in the corresponding n-th entries of the two sequences can be rewritten as:

$$M_n^{a:b+k} - M_n^{a:b} = \frac{(b+k)W_n + (a-(b+k))}{a} - \frac{bW_n + (a-b)}{a} = \frac{k(W_n-1)}{a}.$$

As previously mentioned, Equation (3) can be written as $a(M_n - 1) = b(W_n - 1)$, which implies that $W_n - 1 = a(M_n - 1)/b$. Substituting this expression in place of $W_n - 1$ and adding the applicable superscript of $a : b$ we obtain:

$$\frac{k(W_n-1)}{a} = \frac{k(M_n-1)}{b} = \frac{k(M_n^{a:b}-1)}{b}. \qquad \square$$

Corollary 4.1. *Let $\{M_n^{a:b}\}$ represent the sequence of male offspring for a population with a female-to-male reproductive ratio of $a : b$. Then for a corresponding n and $k \geq 0$, the male sequence is:*

$$M_n^{a:b+k} = \frac{(b+k)M_n^{a:b} - k}{b}. \tag{27}$$

Proof. Let $\{M_n^{a:b}\}$ and $\{M_n^{a:b+k}\}$ be two male sequences with respective female-to-male reproductive ratios $a : b$ and $a : b+k$, where $k \geq 0$. Using Theorem 4.1, we have:

$$M_n^{a:b+k} = M_n^{a:b} + \frac{k(M_n^{a:b}-1)}{b} = \frac{(b+k)M_n^{a:b} - k}{b}. \qquad \square$$

Corollary 4.2. *Let $\{F_n^{a:b}\}$ and $\{F_n^{a:b+k}\}$ be two generalized Fibonacci sequences with the same female reproductive number a, and respective male reproductive numbers b and $b+k$, where $k \geq 0$. Let $\{M_n^{a:b}\}$ be the male sequence associated with $\{F_n^{a:b}\}$. Then for a fixed n:*

$$\left(\Delta F_{a_2:b_2}^{a_1:b_1}\right)_n = F_n^{a:b+k} - F_n^{a:b} = k\left(\frac{M_n^{a:b}-1}{b}\right). \tag{28}$$

Proof. Suppose that $\{F_n^{a:b}\}$ and $\{F_n^{a:b+k}\}$ are two generalized Fibonacci sequences both with the female reproductive number a, and respective male reproductive numbers b and $b+k$, where $k \geq 0$. Suppose further that $\{M_n^{a:b}\}$ and $\{M_n^{a:b+k}\}$ are their corresponding male sequences. From Equation (26) we know that for fixed a and n, $\left(\Delta F_{a_2:b_2}^{a_1:b_1}\right)_n = M_n^{a:b+k} - M_n^{a:b}$. Applying Theorem 4.1, the result immediately follows. $\square$

5 Some even-numbered non-Fisherian Fibonacci sequences

We can express the n-th generalized Fibonacci sequence number exclusively in terms of either its corresponding female or its corresponding male sequence number.

Theorem 5.1. *Given a generalized Fibonacci sequence* $\{F_n^{a:b}\}$ *with a reproductive ratio* $a : b$*, and corresponding female and male sequences* $\{W_n^a\}$ *and* $\{M_n^{a:b}\}$*, then*

$$F_n^{a:b} = \frac{(a+b)W_n^a + (a-b)}{a} = \frac{(a+b)M_n^{a:b} - (a-b)}{b}. \tag{29}$$

Proof. Let $\{F_n^{a:b}\}$ be a generalized Fibonacci sequence with reproductive ratio $a:b$, and let $\{W_n^a\}$ and $\{M_n^{a:b}\}$ be the corresponding female and male sequences. Since $F_n^{a:b} = W_n^a + M_n^{a:b}$, by using Equation (3) and expressing $M_n^{a;b}$ in terms of W_n^a or vice versa, we obtain the above result. □

Remark 5.1. *Observe that in the case where* $a = b$*, Equation* (29) *not surprisingly becomes* $F_n^{a:b} = 2W_n^a = 2M_n^{a:b}$*. Fisherian population growth produces equal numbers (pairs) of male and female offspring, and each entry in these generalized Fibonacci sequences is therefore divisible by* 2*. A similar relationship can sometimes occur between generalized Fibonacci numbers with imbalanced reproductive ratios and male sequences numbers that share the same female reproductive number* a*, but differ in terms of their male reproductive numbers. A few such examples are* $F_n^{1:3} = 2M_n^{1:2}$*,* $F_n^{3:1} = 2M_n^{3:2}$*, and* $F_n^{5:3} = 2M_n^{5:4}$*. To discover under what circumstances this occurs, we begin by using Equation* (5) *in conjunction with Equation* (27).

Theorem 5.2. *Let* $\{F_n^{a:b}\}$ *be a generalized Fibonacci sequence with reproductive ratio* $a:b$ *and let* $\{M_n^{a:b+k}\}$ *be a male sequence with a male reproductive number* $b+k$*, where* $k \geq 0$*. Then,*

$$F_n^{a:b} = \frac{(a+b)M_n^{a:b+k} + 2k + b - a}{b+k}. \tag{30}$$

Proof. Let $\{F_n^{a:b}\}$ be a generalized Fibonacci sequence with reproductive ratio $a:b$ and let $\{W_n^a\}$ and $\{M_n^{a:b}\}$ be its corresponding female and male sequences. Using Equation (5), we know that $F_n^{a:b} = W_n^a + M_n^{a:b}$. Enlisting the help of Equations (3) and (27), we rewrite W_n^a and $M_n^{a:b}$ in terms of $M_n^{a:b+k}$ and obtain the following:

$$F_n^{a:b} = W_n^a + M_n^{a:b} = \frac{aM_n^{a:b+k} - (a-(b+k))}{b+k} + \frac{bM_n^{a:b+k} + k}{b+k} = \frac{(a+b)M_n^{a:b+k} + 2k + b - a}{b+k}.$$

□

Using a similar approach, we derive an equivalent expression for a generalized Fibonacci sequence with a reproductive ratio $a : b+k$, where $k \geq 0$, and a male sequence with a reproductive ratio of $a : b$.

Theorem 5.3. *Given a male sequence* $\{M_n^{a:b}\}$ *with a reproductive ratio of* $a : b$ *and a generalized Fibonacci sequence* $\{F_n^{a:b+k}\}$ *with a reproductive ratio* $a : b+k$*, where* $k \geq 0$*,*

$$F_n^{a:b+k} = \frac{(a+b+k)M_n^{a:b} - k - (a-b)}{b}. \tag{31}$$

Proof. Suppose $\{W_n^a\}$ is a female sequence with reproductive number a. Let $\{M_n^{a:b+k}\}$ and $\{F_n^{a:b+k}\}$ represent the male and generalized Fibonacci sequences with a reproductive ratio $a : b+k$, where $k \geq 0$. Using Equations (3) and (27) to rewrite the female and male sequences in terms of a male sequence with a reproductive ratio of $a : b$, we obtain the following:

$$\begin{aligned}
F_n^{a:b+k} &= W_n^a + M_n^{a:b+k} \\
&= \frac{aM_n^{a:b} - (a-b)}{b} + \frac{(b+k)M_n^{a:b} - k}{b} \\
&= \frac{(a+b+k)M_n^{a:b} - k - (a-b)}{b}.
\end{aligned}$$

□

Remark 5.2. *Considering the case where $k = 0$, we see that the equations in Theorems 5.2 and 5.3 represent generalizations of Equation* (29) *in Theorem 5.1.*

Using Theorems 5.2 and 5.3, we can now establish criteria for which the generalized Fibonacci sequences in the equations of Theorems 5.2 and 5.3 are twice that of the male sequence involved in these equations.

- For Theorem 5.2, if a, b, and k are such that $a = 2k + b$, then Equation (30) becomes:
$$F_n^{a:b} = \frac{(a+b)M_n^{a:b+k} + 2k + b - a}{b+k} = \frac{2(b+k)M_n^{a:b+k} + 2k + b - (2k+b)}{b+k} = 2M_n^{a:b+k}.$$
- For Theorem 5.3, if a, b, and k are such that $a = b - k$, then Equation (31) becomes:
$$F_n^{a:b+k} = \frac{(a+b+k)M_n^{a:b} - k - (a-b)}{b} = \frac{(2b)M_n^{a:b} - k - ((b-k)-b)}{b} = 2M_n^{a:b}.$$

In both instances, all of the generalized Fibonacci numbers under consideration are even.

6 A few non-Fisherian Fibonacci sequence identities

There are a great many identities involving non-Fisherian generalized Fibonacci sequences. We present here a few involving Jacobsthal and Leonardo numbers that the reader may find interesting.

6.1 Sequences involving Jacobsthal numbers

Recall that Jacobsthal numbers, J_n are the terms of a Lucas sequence $U_n(P, Q)$, where $P = 1$ and $Q = -2$.

Theorem 6.1. *Let $\{W_n\}$ represent the female sequence in a population with a female reproductive number $a = 2$. Then the following holds:*

$$W_n = \begin{cases} 2W_{n-1} + 1 & \text{if } n \text{ is even;} \\ 2W_{n-1} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{32}$$

Proof. Let $\{W_n\}$ be the female sequence with a reproductive number $a = 2$. Substituting this value into Equations (1) and (9), we obtain $W_n = W_{n-1} + 2W_{n-2}$ and $W_n = (2^n - (-1)^n)/3$. Combining these expressions, we obtain:

$$\begin{aligned}
W_n &= \frac{1}{3}\Big(2^{n-1} - (-1)^{n-1}\Big) + \frac{2}{3}\Big(2^{n-2} - (-1)^{n-2}\Big) \\
&= \frac{1}{3}\Big(2^{n-1} - (-1)^{n-1}\Big) + \frac{1}{3}\Big(2^{n-1} - 2(-1)^{n-1}\Big) \\
&= \frac{2}{3}\Big(2^{n-1} - (-1)^{n-1}\Big) + (-1)^{n-1} = 2W_{n-1} - (-1)^{n-1}.
\end{aligned}$$

Therefore, if n is even, then $W_n = 2W_{n-1} + 1$, and if n is odd, then $W_n = 2W_{n-1} - 1$. □

Theorem 6.2. *Let $\{W_n\}$ and $\{M_n\}$ be the respective female and male sequences of a population with a female-to-male reproductive ratio of $2:1$. Then,*

$$M_n = \begin{cases} W_{n-1} & \text{if } n \text{ is even;} \\ W_{n-1} + 1 & \text{if } n \text{ is odd.} \end{cases} \tag{33}$$

Proof. Let $\{W_n\}$ and $\{M_n\}$ be the respective female and male sequences of a population with a reproductive ratio of $2:1$. Using these values in Equations (3) and (9) yields:

$$M_n = \frac{W_n + 1}{2} = \frac{\frac{1}{3}\Big(2^n - (-1)^n\Big) + 1}{2} = \frac{1}{3}(2^{n-1}) - \frac{1}{6}(-1)^n + \frac{1}{2}.$$

Considering the even and odd cases for n, we obtain the following:

- If n is even:
$$M_n = \frac{1}{3}(2^{n-1}) + \frac{1}{3} = \frac{1}{3}(2^{n-1} - (-1)^{n-1}) = W_{n-1}.$$
- If n is odd:
$$M_n = \frac{1}{3}(2^{n-1}) + \frac{2}{3} = \frac{1}{3}(2^{n-1}) - \frac{1}{3}(-1) + \frac{3}{3} = \frac{1}{3}(2^{n-1} - (-1)^{n-1}) + 1 = W_{n-1} + 1.$$ □

Corollary 6.1. *Let $\{F_n^{2:1}\}$ represent the generalized Fibonacci sequence with a female-to-male reproductive ratio of $2:1$. Then,*

$$F_n^{2:1} = \begin{cases} 2^{n-1} & \text{if } n \text{ is even;} \\ 2^{n-1} + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\{F_n^{2:1}\}$ represent the generalized Fibonacci sequence with a female-to-male reproductive ratio of $2:1$. Let $\{W_n\}$ and $\{M_n\}$ represent the corresponding female and male sequences. Then, using the fact that $F_n^{2:1} = M_n + W_n$ along with Theorem 6.2, and considering the even and odd cases for n, we obtain the following:

- If n is even:
$$F_n^{2:1} = W_{n-1} + W_n = \frac{1}{3}\Big(2^{n-1} - (-1)^{n-1}\Big) + \frac{1}{3}\Big(2^n - (-1)^n\Big) = \frac{1}{3}\Big(3 \cdot 2^{n-1}\Big) = 2^{n-1}.$$
- If n is odd:
$$\begin{aligned}
F_n^{2:1} &= W_{n-1} + 1 + W_n = \frac{1}{3}\Big(2^{n-1} - (-1)^{n-1}\Big) + 1 + \frac{1}{3}\Big(2^n - (-1)^n\Big) \\
&= \frac{1}{3}\Big(3 \cdot 2^{n-1}\Big) + 1 = 2^{n-1} + 1.
\end{aligned}$$ □

Theorem 6.3. *Let $\{W_n\}$ and $\{M_n\}$ be the respective female and male sequences of a population. If the female-to-male reproductive ratio is $2:4$, then $M_n = 2W_n - 1$.*

Proof. Let $\{W_n\}$ and $\{M_n\}$ represent the female and male sequences with a female-to-male reproductive ratio of $2:4$. Then using Equation (3) to express M_n in terms of W_n yields:

$$M_n = \frac{4W_n + (2-4)}{2} = 2W_n - 1. \qquad \square$$

Corollary 6.2. *Let $\{F_n^{2:4}\}$ represent the generalized Fibonacci sequence of a population with a female-to-male reproductive ratio of $2:4$, and let $\{W_n\}$ represent the corresponding female sequence. Then the n-th entry of the generalized Fibonacci sequence, $F_n^{2:4} = 3W_n - 1$.*

Proof. Suppose $\{F_n^{2:4}\}$ is the generalized Fibonacci sequence with a reproductive ratio of $2:4$, and let $\{W_n\}$ and $\{M_n\}$ represent the corresponding female and male sequences. Then, given the fact that $F_n^{2:4} = W_n + M_n$ and applying Theorem 6.3, the result immediate follows. $\square$

Theorem 6.4. *Let $\{F_n^{2:4}\}$ be the generalized Fibonacci sequence with a reproductive ratio of $2:4$. Then,*

$$F_n^{2:4} = \begin{cases} 2^n - 2 & \text{if } n \text{ is even;} \\ 2^n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\{W_n\}$ and $\{F_n^{2:4}\}$ represent the female and generalized Fibonacci sequences of a population with a reproductive ratio of $2:4$. Then using Corollary 6.2 and the closed-form solution for W_n from Equation (9), we obtain:

$$F_n^{2:4} = 3W_n - 1 = \frac{3}{3}\Big(2^n - (-1)^n\Big) - 1 = 2^n - (-1)^n - 1.$$

Therefore, if n is even, $2^n - (-1)^n - 1 = 2^n - 2$, and if n is odd, $2^n - (-1)^n - 1 = 2^n$. $\square$

Theorem 6.5. *Let $\{W_n\}$ and $\{M_n\}$ be the female and male sequences of a population with a female-to-male reproductive ratio of $2:3$, then*

$$M_n = \begin{cases} 2^{n-1} - 1 & \text{if } n \text{ is even;} \\ 2^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\{W_n\}$ and $\{M_n\}$ be the female and male sequences with a reproductive ratio of $2:3$. Then, applying Equations (3) and (9) with $a = 2$ and $b = 3$, yields the following:

$$M_n = \frac{\frac{3}{3}\Big(2^n - (-1)^n\Big) - 1}{2} = 2^{n-1} - \frac{(-1)^n}{2} - \frac{1}{2}.$$

Consequently, $M_n = 2^{n-1} - 1$ if n is even, and $M_n = 2^{n-1}$ if n is odd. $\square$

Theorem 6.6. *Let $\{F_n^{2:3}\}$ represent the generalized Fibonacci sequence with a female-to-male reproductive ratio of $2:3$ and let $\{W_n\}$ and $\{M_n\}$ represent the corresponding female and male sequences, then*

$$M_n = \begin{cases} W_n + W_{n-1} - 1 & \text{if } n \text{ is even;} \\ W_n + W_{n-1} & \text{if } n \text{ is odd.} \end{cases} \qquad F_n^{2:3} = \begin{cases} 2W_n + W_{n-1} - 1 & \text{if } n \text{ is even;} \\ 2W_n + W_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose that $\{F_n^{2:3}\}$ is the generalized Fibonacci sequence with a reproductive ratio of $2:3$ and $\{W_n\}$ and $\{M_n\}$ are the corresponding female and male sequences. Since $a = 2$, the female sequence is a Jacobsthal sequence, J_n. Using the Jacobsthal identity $J_n + J_{n-1} = 2^{n-1}$ along with Theorem 6.5, the male sequence identity is established. Secondly, given the fact that $F_n^{2:3} = M_n + W_n$, the generalized Fibonacci sequence result immediately follows. $\square$

Corollary 6.3. *Let $\{F_n^{2:3}\}$ represent the generalized Fibonacci sequence of a population with a female-to-male reproductive ratio of $2:3$, then*

$$F_n^{2:3} = \begin{cases} \frac{1}{3}(5 \cdot 2^{n-1} - 4) & \text{if } n \text{ is even;} \\ \frac{1}{3}(5 \cdot 2^{n-1} + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose that $\{F_n^{2:3}\}$ is the generalized Fibonacci sequence with a reproductive ratio of $2:3$ and let $\{W_n\}$ be the corresponding female sequence. Using Theorem 6.6 and Equation (9), we obtain the following:

- If n is even:
$$F_n^{2:3} = 2W_n + W_{n-1} - 1 = \frac{1}{3}\Big(2 \cdot 2^n - 2(-1)^n + 2^{n-1} - (-1)^{n-1}\Big) - 1 = \frac{1}{3}\Big(5 \cdot 2^{n-1} - 4\Big).$$
- If n is odd:
$$F_n^{2:3} = 2W_n + W_{n-1} = \frac{1}{3}\Big(2 \cdot 2^n - 2(-1)^n + 2^{n-1} - (-1)^{n-1}\Big) = \frac{1}{3}\Big(5 \cdot 2^{n-1} + 1\Big). \quad \square$$

6.2 Sequences involving Leonardo and generalized Leonardo numbers

We remind the reader that Leonardo numbers, L_n, are defined by the nonhomogeneous second order recurrence relation $L_n = L_{n+1} + L_{n+2} + 1$, where $L_1 = L_2 = 1$, and that they are related to Fibonacci numbers, F_n, according to the following: $L_n = 2F_{n+1} - 1$ [1,2]. Kuhapatanakul and Chobsorn generalized Leonardo numbers by using a fixed positive integer k in place of 1 in the above recurrence relation for L_n [10].

Theorem 6.7. *Let $\{M_n\}$ be the male sequence associated with the generalized Fibonacci sequence, $\{F_n^{1:b}\}$, with a reproductive female-to-male ratio of $1:b$, where $b \geq 1$. Let F_n represent the standard Fibonacci sequence. Then, $M_n = bF_n - (b-1)$.*

Proof. Let $\{W_n\}$ and $\{M_n\}$ represent the female and male sequences and suppose the population reproductive ratio is $1 : b$, where $b \geq 1$. Given the relationship between male and female sequences from Equation (3) along with the fact that $a = 1$, which means that W_n is the standard Fibonacci sequence, F_n, the result immediately follows. $\square$

Remark 6.1. *Observe that when the reproductive ratio is $1:2$, Theorem 6.7 says $M_n = 2F_n - 1$. Alternatively, using Equation (4), we have that $M_n = M_{n-1} + M_{n-2} + 1$. In both cases, we recognize that the male sequence is the Leonardo sequence. In general, for reproductive ratios $1 : b$ with $b \geq 2$, the male sequences are generalized Leonardo sequences with recurrence relations: $M_n = M_{n-1} + M_{n-2} + (b-1)$.*

Theorem 6.8. *Let* $\{F_n^{1:b}\}$ *represent the generalized Fibonacci sequence with a female-to-male reproductive ratio of* $1 : b$*, where* $b \geq 1$*. Then* $F_n^{1:b} = (b+1)F_n - (b-1)$*, where* F_n *is the* n*-th entry in the standard Fibonacci sequence.*

Proof. Suppose $\{F_n^{1:b}\}$ is the generalized Fibonacci sequence with a female-to-male reproductive ratio of $1 : b$, where $b \geq 1$, and let $\{W_n\}$ and $\{M_n\}$ be the associated female and male sequences. Since $a = 1$, W_n is the standard Fibonacci sequence, $\{F_n\}$. Given Theorem 6.7 and the fact that $F_n^{1:b} = M_n + W_n$, the result is established. $\square$

Remark 6.2. *Once again, we see that if the reproductive ratio is* $1 : b$ *with* $b \geq 2$*,* $F_n^{1:b}$ *also is a generalized Leonardo sequence with recurrence relation* $F_n^{1:b} = F_{n-1}^{1:b} + F_{n-2}^{1:b} + (b-1)$.

7 Conclusion

The female, male, and generalized Fibonacci recurrence relations presented in Section 2 give rise to many fascinating integer sequences, some of which are cataloged in Sloane's *Online Encyclopedia of Integer Sequences* (OEIS) and others that are new [11]. In particular, many of the generalized third order Pell and Pell–Lucas sequences of the respective male and generalized Fibonacci numbers, have yet to be cataloged in the OEIS.

Beyond the creation of new integer sequences, the generalized Fibonacci number framework we have developed serves to associate sequences that up to this point did not appear to be connected. For example, given a female-to-male reproductive ratio of 2 : 1, the male sequence (A005578) is related to the female sequence (A001045), and the sum of these two sequences produces the generalized Fibonacci sequence (A052531). To highlight some of these connections, we group female, male, and generalized Fibonacci sequences associated with specific reproductive ratios in the following appendices. The data is organized according to whether the sequences represent polygynous, polyandrous, or Fisherian population growth. In cases where sequences have previously been cataloged, their OEIS sequence numbers are included.

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Appendix A Examples of polygynous population growth

2:1													
W_n :	1,	1,	3,	5,	11,	21,	43,	85,	171,	341,	683,	1365,...	A001045
M_n :	1,	1,	2,	3,	6,	11,	22,	43,	86,	171,	342,	683,...	A005578
$F_n^{2:1}$:	2,	2,	5,	8,	17,	32,	65,	128,	257,	512,	1025,	2048,...	A052531
3:1													
W_n :	1,	1,	4,	7,	19,	40,	97,	217,	508,	1159,	2683,	6160,...	A006130
M_n :	1,	1,	2,	3,	7,	14,	33,	73,	170,	387,	895,	2054,...	
$F_n^{3:1}$:	2,	2,	6,	10,	26,	54,	130,	290,	678,	1546,	3578,	8214,...	
3:2													
W_n :	1,	1,	4,	7,	19,	40,	97,	217,	508,	1159,	2683,	6160,...	A006130
M_n :	1,	1,	3,	5,	13,	27,	65,	145,	339,	773,	1789,	4107,...	
$F_n^{3:2}$:	2,	2,	7,	12,	32,	67,	162,	362,	847,	1932,	4472,	10267,...	
4:1													
W_n :	1,	1,	5,	9,	29,	65,	181,	441,	1165,	2929,	7589,	19305,...	A006131
M_n :	1,	1,	2,	3,	8,	17,	46,	111,	292,	733,	1898,	4827,...	
$F_n^{4:1}$:	2,	2,	7,	12,	37,	82,	227,	552,	1457,	3662,	9487,	24132,...	
4:2													
W_n :	1,	1,	5,	9,	29,	65,	181,	441,	1165,	2929,	7589,	19305,...	A006131
M_n :	1,	1,	3,	5,	15,	33,	91,	221,	583,	1465,	3795,	9653,...	
$F_n^{4:2}$:	2,	2,	8,	14,	44,	98,	272,	662,	1748,	4394,	11384,	28958,...	
4:3													
W_n :	1,	1,	5,	9,	29,	65,	181,	441,	1165,	2929,	7589,	19305,...	A006131
M_n :	1,	1,	4,	7,	22,	49,	136,	331,	874,	2197,	5692,	14479,...	
$F_n^{4:3}$:	2,	2,	9,	16,	51,	114,	317,	772,	2039,	5126,	13281,	33784,...	
5:1													
W_n :	1,	1,	6,	11,	41,	96,	301,	781,	2286,	6191,	17621,	48576,...	A015440
M_n :	1,	1,	2,	3,	9,	20,	61,	157,	458,	1239,	3525,	9716,...	
$F_n^{5:1}$:	2,	2,	8,	14,	50,	116,	362,	938,	2744,	7430,	21146,	58292,...	
5:2													
W_n :	1,	1,	6,	11,	41,	96,	301,	781,	2286,	6191,	17621,	48576,...	A015440
M_n :	1,	1,	3,	5,	17,	39,	121,	313,	915,	2477,	7049,	19431,...	
$F_n^{5:2}$:	2,	2,	9,	16,	58,	135,	422,	1094,	3201,	8668,	24670,	68007,...	
5:3													
W_n :	1,	1,	6,	11,	41,	96,	301,	781,	2286,	6191,	17621,	48576,...	A015440
M_n :	1,	1,	4,	7,	25,	58,	181,	469,	1372,	3715,	10573,	29146,...	
$F_n^{5:3}$:	2,	2,	10,	18,	66,	154,	482,	1250,	3658,	9906,	28194,	77722,...	
5:4													
W_n :	1,	1,	6,	11,	41,	96,	301,	781,	2286,	6191,	17621,	48576,...	A015440
M_n :	1,	1,	5,	9,	33,	77,	241,	625,	1829,	4953,	14097,	38861,...	
$F_n^{5:4}$:	2,	2,	11,	20,	74,	173,	542,	1406,	4115,	11144,	31718,	87437,...	

Appendix B Examples of polyandrous population growth

							1:2						
$W_n:$	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,...	A000045
$M_n:$	1,	1,	3,	5,	9,	15,	25,	41,	67,	109,	177,	287,...	A001595
$F_n^{1:2}$:	2,	2,	5,	8,	14,	23,	38,	62,	101,	164,	266,	431,...	
							1:3						
$W_n:$	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,...	A000045
$M_n:$	1,	1,	4,	7,	13,	22,	37,	61,	100,	163,	265,	430,...	A111314
$F_n^{1:3}$:	2,	2,	6,	10,	18,	30,	50,	82,	134,	218,	354,	577,...	
							1:4						
$W_n:$	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,...	A000045
$M_n:$	1,	1,	5,	9,	17,	29,	49,	81,	133,	217,	353,	573,...	
$F_n^{1:4}$:	2,	2,	7,	12,	22,	37,	62,	102,	167,	272,	442,	717,...	
							1:5						
$W_n:$	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,...	A000045
$M_n:$	1,	1,	6,	11,	21,	36,	61,	101,	166,	271,	441,	716,...	
$F_n^{1:5}$:	2,	2,	8,	14,	26,	44,	74,	122,	200,	326,	530,	860,...	
							2:3						
$W_n:$	1,	1,	3,	5,	11,	21,	43,	85,	171,	341,	683,	1365,...	A001045
$M_n:$	1,	1,	4,	7,	16,	31,	64,	127,	256,	511,	1024,	2047,...	
$F_n^{2:3}$:	2,	2,	7,	12,	27,	52,	107,	202,	427,	852,	1707,	3412,...	
							2:4						
$W_n:$	1,	1,	3,	5,	11,	21,	43,	85,	171,	341,	683,	1365,...	A001045
$M_n:$	1,	1,	5,	9,	21,	41,	85,	169,	341,	681,	1365,	2729,...	
$F_n^{2:4}$:	2,	2,	8,	14,	32,	62,	128,	254,	512,	1022,	2048,	3754,...	
							2:5						
$W_n:$	1,	1,	3,	5,	11,	21,	43,	85,	171,	341,	683,	1365,...	A001045
$M_n:$	1,	1,	6,	11,	26,	51,	106,	211,	426,	851,	1706,	3411,...	
$F_n^{2:5}$:	2,	2,	9,	16,	37,	72,	149,	296,	597,	1192,	2389,	4776,...	
							3:4						
$W_n:$	1,	1,	4,	7,	19,	40,	97,	217,	508,	1159,	2683,	6160,...	A006130
$M_n:$	1,	1,	5,	9,	25,	60,	130,	290,	678,	1146,	3178,	5496,...	
$F_n^{3:4}$:	2,	2,	9,	16,	44,	100,	227,	507,	1186,	2305,	5861,	11656,...	
							3:5						
$W_n:$	1,	1,	4,	7,	19,	40,	97,	217,	508,	1159,	2683,	6160...	A006130
$M_n:$	1,	1,	6,	11,	31,	66,	161,	361,	846,	1931,	4971,	10766,...	
$F_n^{3:5}$:	2,	2,	10,	18,	50,	106,	258,	578,	1354,	3090,	7654,	16926,...	

Appendix C Examples of Fisherian population growth

							1:1						
W_n :	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,...	A000045
M_n :	1,	1,	2,	3,	5,	8,	13,	21,	34,	55,	89,	144,...	A000045
$F_n^{1:1}$:	2,	2,	4,	6,	10,	16,	26,	42,	68,	110,	178,	288,...	A006355
							2:2						
W_n :	1,	1,	3,	5,	11,	21,	43,	85,	171,	341,	683,	1365,...	A001045
M_n :	1,	1,	3,	5,	11,	21,	43,	85,	171,	341,	683,	1365,...	A001045
$F_n^{2:2}$:	2,	2,	6,	10,	22,	41,	86,	170,	342,	682,	1366,	2730,...	A078008
							3:3						
W_n :	1,	1,	4,	7,	19,	40,	97,	217,	508,	1159,	2683,	6160,...	A006130
M_n :	1,	1,	4,	7,	19,	40,	97,	217,	508,	1159,	2683,	6160,...	A006130
$F_n^{3:3}$:	2,	2,	8,	14,	38,	80,	194,	434,	1016,	2318,	5366,	12320,...	A228661
							4:4						
W_n :	1,	1,	5,	9,	29,	65,	181,	441,	1165,	2929,	7589,	19305,...	A006131
M_n :	1,	1,	5,	9,	29,	65,	181,	441,	1165,	2929,	7589,	19305,...	A006131
$F_n^{4:4}$:	2,	2,	10,	18,	58,	130,	362,	882,	2330,	5858,	15178,	38610,...	A102446
							5:5						
W_n :	1,	1,	6,	11,	41,	96,	301,	781,	2286,	6191,	17621,	48576,...	A015440
M_n :	1,	1,	6,	11,	41,	96,	301,	781,	2286,	6191,	17621,	48576,...	A015440
$F_n^{5:5}$:	2,	2,	12,	22,	82,	192,	602,	1562,	4572,	12382,	35242,	97152,...	
							6:6						
W_n :	1,	1,	7,	13,	55,	133,	463,	1261,	4039,	11605,	35839,	105469,...	A015441
M_n :	1,	1,	7,	13,	55,	133,	463,	1261,	4039,	11605,	35839,	105469,...	A015441
$F_n^{5:5}$:	2,	2,	14,	26,	110,	266,	926,	2522,	8078,	23210,	71678,	210938,...	