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On certain inequalities for φ, ψ, σ and related functions, III

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Abstract: We obtain generalizations of certain results from [2] and [4]. The unitary variants are also considered. Some new arithmetic functions and their inequalities are also considered.
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1 Introduction

Let $\varphi(n)$, $\psi(n)$ and $\sigma(n)$ denote the classical arithmetic functions, representing Euler's totient, Dedekind's function, and the sum of divisors functions, respectively. Let $\varphi^*(n)$, $\sigma^*(n)$ denote the unitary analogues of the functions φ and σ. It is well-known that $\varphi(1) = \psi(1) = \sigma(1) = \varphi^*(1) = \sigma^*(1) = 1$ and that these functions are multiplicative, and for prime powers $n = p^a$ (p prime, $a \geq 1$ integer) one has

$$\varphi(p^a) = p^a \cdot \left(1 - \frac{1}{p}\right),\ \psi(p^a) = p^a \cdot \left(1 + \frac{1}{p}\right),\ \sigma(p^a) = \frac{p^{a+1} - 1}{p - 1} \tag{1}$$

and

$$\varphi^*(p^a) = p^a - 1, \ \ \sigma^*(p^a) = p^a + 1 \tag{2}$$

(see [5] and [3]).

In part II of this series [4], by refining earlier results by S. Dimitrov [1], the following inequalities have been proved (in what follows, $n \geq 2$):

Theorem 1.

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq \varphi^2(n) + 2\psi^2(n) \geq 3n^2 + 2n + 3. \tag{3}$$

Theorem 2.

$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq \psi^2(n) + 2\varphi(n)\psi(n) \geq 3n^2 + 2n - 1, \tag{4}$$

Theorem 3.

$$\big(\varphi^*(n)\big)^2 + \big(\psi(n)\big)^2 + \big(\sigma^*(n)\big)^2 \geq \big(\psi^*(n)\big)^2 + 2\big(\sigma^*(n)\big)^2 \geq 3n^2 + 2n + 3, \tag{5}$$

Theorem 4.

$$\varphi^*(n)\psi(n) + \varphi^*(n)\sigma^*(n) + \psi(n)\sigma^*(n) \geq 2\varphi^*(n)\sigma^*(n) + \big(\sigma^*(n)\big)^2 \geq 3n^2 + 2n - 1. \tag{6}$$

In the paper [2], S. Dimitrov proved the following inequalities:

Theorem 5. (Theorem 1 of [2]).

$$\varphi^3(n) + \psi^3(n) + \sigma^3(n) \geq 3n^3 + 3n^2 + 9n + 1. \tag{7}$$

Theorem 6. (Theorem 2 of [2]).

$$\varphi^4(n) + \psi^4(n) + \sigma^4(n) \geq 3n^4 + 4n^3 + 18n^2 + 4n + 3. \tag{8}$$

Theorem 7. (Theorem 5 of [2]).

$$\begin{aligned} &\varphi^3(n)\big(\psi(n) + \sigma(n)\big) + \psi^3(n)\big(\varphi(n) + \sigma(n)\big) + \sigma^3(n)\big(\varphi(n) + \psi(n)\big) \\ &\geq 6n^4 + 8n^3 + 12n^2 + 8n - 2. \end{aligned} \tag{9}$$

In what follows, we will generalize all the above inequalities, and among others, these generalizations will provide also refinements of inequalities (7)–(9).

Inspired by the auxiliary results proved in this process, certain new arithmetic functions will be considered, too.

2 Main results

Let $k \geq 2$, $n \geq 2$ be integers. Then one has

Theorem 8.

$$\varphi^k(n) + \psi^k(n) + \sigma^k(n) \geq \varphi^k(n) + 2\psi^k(n) \geq (n-1)^k + 2(n+1)^k. \tag{10}$$

Theorem 9.

$$\begin{aligned}\varphi^k(n)\psi^k(n)+\varphi^k(n)\sigma^k(n)+\sigma^k(n)\psi^k(n) &\geq \psi^{2k}(n)+2\varphi^k(n)\psi^k(n)\\ &\geq 2(n^2-1)^k+(n+1)^{2k}.\end{aligned} \tag{11}$$

Theorem 10.

$$\big(\varphi^*(n)\big)^k+\big(\sigma^*(n)\big)^k+\big(\psi(n)\big)^k \geq \big(\varphi^*(n)\big)^k+2\big(\sigma^*(n)\big)^k \geq (n-1)^k+2(n+1)^k. \tag{12}$$

Theorem 11.

$$\begin{aligned}\big(\varphi^*(n)\big)^k\psi^k(n)+\big(\varphi^*(n)\big)^k\big(\sigma^*(n)\big)^k+\psi^k(n)\big(\sigma^*(n)\big)^k &\geq \big(\sigma^*(n)\big)^k+2\big(\varphi^*(n)\big)^k\\ &\geq (n+1)^k+2(n^k-1).\end{aligned} \tag{13}$$

Theorem 12.

$$\begin{aligned}&\varphi^k(n)\big(\psi(n)+\sigma(n)\big)+\psi^k(n)\big(\varphi(n)+\sigma(n)\big)+\sigma^k(n)\big(\varphi(n)+\psi(n)\big)\\ &\geq 2\psi(n)\big(\psi^k(n)+\psi^{k-1}(n)\varphi(n)+\psi^k(n)\big)\\ &\geq 2(n+1)[(n-1)^k+2n(n+1)^{k-1}].\end{aligned} \tag{14}$$

Theorem 13.

$$\begin{aligned}&\big(\varphi^*(n)\big)^k\big(\psi(n)+\sigma^*(n)\big)+\big(\psi(n)\big)^k\cdot\big(\varphi^*(n)+\sigma^*(n)\big)\\ &\quad+\big(\sigma^*(n)\big)^k\cdot\big(\varphi^*(n)+\psi(n)\big)\\ &\geq 2\sigma^*(n)\big[\big(\varphi^*(n)\big)^k+\varphi^*(n)\big(\sigma^*(n)\big)^{k-1}+\big(\sigma^*(n)\big)^k\big]\\ &\geq 2(n+1)\cdot\big[(n-1)^k+2n(n+1)^{k-1}\big].\end{aligned} \tag{15}$$

We need the following auxiliary results:

Lemma 1. *Let* $x_i > i$ $(i=1,2,\ldots,k)$, $r\geq 1$. *Then*

$$\prod_{i=1}^{r}(x_i-1)^k+2\prod_{i=1}^{r}(x_i+1)^k \geq \left(\prod_{i=1}^{r}x_i-1\right)^k+2\left(\prod_{i=1}^{r}x_i+1\right). \tag{16}$$

Proof. We will use induction upon r. For $r=1$, there is equality. Assume that (16) holds true for r, we will prove it for $r+1$. We have

$$\begin{aligned}&\prod_{i=1}^{r}(x_i-1)^k\cdot(x_{i+1}-1)^k+2\prod_{i=1}^{r}(x_i+1)^k\cdot(x_{r+1}+1)^k\\ &=\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j\cdot x_{r+1}^{k-j}\right)\prod_{i=1}^{r}(x_i-1)^k+2\left(\sum_{j=0}^{k}\binom{k}{j}x_{r+1}^{k-j}\right)\cdot\prod_{i=1}^{r}(x_i+1)^k\\ &=x_{r+1}^k\left(\prod_{i=1}^{r}(x_i-1)^k+2\prod_{i=1}^{r}(x_i+1)^k\right)\\ &\quad+\sum_{j=1}^{k}\binom{k}{j}x_{r+1}^{k-j}\cdot\left((-1)^j\prod_{i=1}^{r}(x_i-1)^k+\prod_{i=1}^{r}(x_i+1)^k\right).\end{aligned}$$

As $x_i > 1$ and $x_{r+1} > 1$, we get

$$(-1)^j \cdot \prod_{i=1}^{r}(x_i - 1)^k + \prod_{i=1}^{r}(x_i + 1)^k > 0$$

and thus above is, by induction

$$> x_{r+1}^k \left(\prod_{i=1}^{r}(x_i - 1)^k + 2\prod_{i=1}^{r}(x_i + 1)^k \right) \geq x_{r+1}^k \left[\left(\prod_{i=1}^{r} x_i - 1 \right)^k + 2\left(\prod_{i=1}^{r} x_i + 1 \right)^k \right].$$

Then, we have

$$\begin{aligned}
x_{r+i}^k \left[\left(\prod_{i=1}^{r} x_i - 1 \right)^k + 2\left(\prod_{i=1}^{r} x_i + 1 \right)^k \right] &= x_{r+i}^k \left[\sum_{j=1}^{k} \binom{k}{j} \left(2\prod_{i=1}^{r} x_i^j + (-1)^{k-j} \prod_{i=1}^{r} x_i^j \right) \right] \\
&= \sum_{j=0}^{k} \binom{k}{j} \left(2x_{r+i}^k \prod_{i=1}^{r} x_i^j + (1)^{k-j} x_{r+1}^k \prod_{i=1}^{r} x_i^j \right) \\
&> \sum_{j=0}^{k} \binom{k}{j} \left(2\prod_{i=1}^{r+1} x_i^j + (1)^{k-j} \prod_{i=1}^{r+1} x_i^j \right) \\
&= \left(\prod_{i=1}^{r+1} x_i - 1 \right)^k + 2\left(\prod_{i=1}^{r+1} x_i + 1 \right)^k.
\end{aligned}$$

We have used that the coefficients of the polynomial in variables $x_1, \ldots, x_{r+1}$ are positive numbers, $x_r > 1$ and trivially $x_r^{i+1} > x_r^i$. Thus the proof of (16) is completed. □

The proof of the following lemma is similar:

Lemma 2. *Let* $x_i > 1$, $r \geq 1$. *Then*

$$\begin{aligned}
&\prod_{i=1}^{r}(x_i - 1)^k + \prod_{i=1}^{r}(x_i - 1)\prod_{i=1}^{r}(x_i + 1)^{k-1} + \prod_{i=1}^{r}(x_i + 1)^k \\
&\geq \left(\prod_{i=1}^{r} x_i - 1 \right)^k + 2\prod_{i=1}^{r} x_i \cdot \left(\prod_{i=1}^{r} x_i + 1 \right)^{k-1}.
\end{aligned} \tag{17}$$

Lemma 3. *Let* $x_i > 1$, $r \geq 1$. *Then*

$$\prod_{i=1}^{r}(x_i + 1)^k + 2\prod_{i=1}^{r}(x_i^k - 1) \geq \prod_{i=1}^{r}(x_i + 1)^k + 2\left(\prod_{i=1}^{r} x_i^k - 1 \right). \tag{18}$$

Lemma 4. *Let* $x_i > 1$, $r \geq 1$ $(i = 1, 2, \ldots, r)$. *Then*

$$\prod_{i=1}^{r}(x_i + 1)^{2k} + 2\prod_{i=1}^{r}(x_i^2 - 1)^k \geq \left(\prod_{i=1}^{r} x_i + 1 \right)^{2k} + 2\left(\prod_{i=1}^{r} x_i^2 - 1 \right)^k. \tag{19}$$

Proof. Inequality (18) is proved in [4] for $k = 1$ and $k = 2$. Thus, we will assume in what follows that $k \geq 3$. Let thus assume (18) is true for r and we will prove it for $r + 1$. One has:

$$\begin{aligned}
&\prod_{i=1}^{r+1}(x_i+1)^k + 2\prod_{i=1}^{r+1}(x_i^k-1) \\
&= \prod_{i=1}^{r+1}(x_i+1)^k \cdot (x_{r+1}+1)^k + 2\prod_{i=1}^{r+1}({x_i}^k-1)(x_{r+1}^k-1) \\
&= \left(\sum_{j=0}^{k}\binom{k}{j}x_{r+1}^j\right)\prod_{i=1}^{r}(x_i+1)^k + 2x_{r+1}^k\cdot\prod_{i=1}^{r}(x_i^k-1) - 2\prod_{i=1}^{r}(x_i^k-1) \\
&= x_{r+1}^k\left(\prod_{i=1}^{r}(x_i+1)^k + 2\prod_{i=1}^{r}(x_i^k-1)\right) + \left(\sum_{j=0}^{k-1}\binom{k}{j}x_{i+1}^j\right)\cdot\prod_{i=1}^{r}(x_i+1)^k - 2\prod_{i=1}^{r}(x_i-1).
\end{aligned}$$

The final sum contains two expressions, the first being the product of x_{r+1}^k with the sum in parentheses, the second being the remaining two terms. We now prove that the second expression is non-negative, then we prove that the first expression is sufficient to derive the desired bound. Notice that

$$\sum_{j=0}^{k-1}\binom{k}{j}x_{r+1}^j \geq 1 + kx_{r+1} > 2$$

(we use $k \geq 2$ and $x_{r+1} > 1$), therefore

$$\begin{aligned}
\left(\sum_{j=0}^{k-1}\binom{k}{j}x_{r+1}^j\right)\prod_{i=1}^{r}(x_i+1)^k - 2\prod_{i=1}^{r}(x_i^k-1) &> 2\prod_{i=1}^{r}(x_i+1)^k - 2\prod_{i=1}^{r}(x_i^k+1) \\
&= 2\prod_{i=1}^{k}\left(x_i^j+1+\sum_{j=1}^{k=1}\binom{k}{j}x_i^k\right) - 2\prod_{i=1}^{r}(x_i^k+1) \geq 0
\end{aligned}$$

and thus the second expression is non-negative. Moreover, using the induction hypothesis we can now write the first expression as

$$\begin{aligned}
x_{r+1}^k\cdot\left(\prod_{i=1}^{r}(x_i+1)^k + 2\prod_{I=1}^{r}(x_i^k-1)\right) &\geq x_{r+1}^k\cdot\left[\left(\prod_{i=1}^{r}x_i+1\right)^k + 2\left(\prod_{i=1}^{r}x_i^k-1\right)\right] \\
&= \left(\prod_{i=1}^{r+1}x_i + x_{r+1}\right)^k + 2\prod_{i=1}^{r+1}x_i^k - 2x_{r+1}^k
\end{aligned} \tag{20}$$

we now put $a = \prod_{i=1}^{r} x_i$, $b = x_{r+1}$, and write (20) to $(ax+x)^k + 2a^kx^k - 2x^k$. To complete the proof, it is now sufficient to show that

$$(ax+x)^k + 2a^kx^k - 2x^k \geq (ax+1)^k + 2(a^kx^k-1),$$

which is equivalent to

$$(a+1)^k - 2 \geq \left(a+\frac{1}{x}\right)^k - \frac{2}{x^k}. \tag{21}$$

By letting

$$f(x) = \left(a + \frac{1}{x}\right)^k - \frac{2}{x^k},$$

it is immediate that

$$f'(x) = \frac{[k(2 - (ax+1)^{k-1})]}{x^{k+1}} < 0$$

for $x > 1$. Thus f is decreasing, giving $f(1) > f(a)$ for $x > 1$ and (21) follows. The proof of Lemma 4 is similar. $\square$

Proof of Theorems 8–13

The middle inequalities are consequences of the well-known inequalities $\sigma(n) \geq \psi(n)$ and $\psi(n) \geq \sigma^*(n)$, so we have to prove the last inequalities.

Applying Lemma 1 to $x_i = p_i^{a_i}$, where $n = p_1^{a_1} \cdots p_r^{a_r} =$ prime factorization of n, first remark that Theorem 10 follows immediately. Also Theorem 11 follows by Lemma 3.

To prove Theorem 8, by application of Lemma 1, we proceed as in [4], by remarking that if $n = \prod_{i=1}^{r} p_i^{a_i}$ is the prime factorization of n, then

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{r} \frac{\varphi(p_i)}{p_i} \text{ and } \frac{\psi(n)}{n} = \prod_{i=1}^{r} \frac{\psi(p_i)}{p_i},$$

thus it is sufficient to apply Lemma 1 for $x_i = p_i$.

To complete the proof, remark that $\frac{1}{p_i^{a_i}} \leq \frac{1}{p_i}$, so dividing both sides of (10) by $n^k = p_1^{ka_1} \cdots p_r^{ka_r}$, the inequalities follows by Lemma 1 for $x_i = p_i$.

The proof of Theorem 9 follows by Lemma (4). Applying Lemma 4 to $x_i = p_i$, and to finish the proof of Theorem 9, we have to show that

$$\left(1 + \frac{1}{p_1 \cdots p_r}\right)^{2k} + 2\left(1 - \frac{1}{p_1^2 \cdots p_r^2}\right)^k \geq \left(1 + \frac{1}{p_1^{a_1} \cdots p_r^{a_r}}\right)^{2k} + 2\left(1 - \frac{1}{p_i^{2a_1} \cdots p_i^{2a_r}}\right)^k \tag{22}$$

Let $f(a_1) = 1 + \left(x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}\right)^{2k} + 2(1 - x_1^{2a_1} \cdots x_r^{2a_r})^k$, where $x_i = \frac{1}{p_i}$ $(i = 1, \ldots, r)$. Then an easy computation gives

$$f'(a_1) = 2kt \cdot (1+t)^{k-1} \cdot \ln x_1 \cdot [(1+t)^k - 2t(1-t)^{k-1}],$$

where $t = x_1^{a_1} \cdots x_r^{a_r}$. Now $g(t) = (1+t)^k - 2t(1-t)^{k-1} > 0$ as $2t < 1+t$, $0 < 1-t < 1+t$. Thus, we get $f'(a_1) < 0$, implying $f(a_1) \leq f(1)$, as $\ln x_1 < 0$. Thus, for the function

$$F(a_1, a_2, \ldots, a_r) = (1 + x_1^{a_1} \cdots x_r^{a_r})^{2k} + 2(1 - x_1^{2a_1} \cdots x_r^{2a_r})^k$$

of r variables, one has $F(a_1, a_2, \ldots, a_r) \leq F(1, a_2, \ldots, a_r)$.

Let now define $f_1(a_2) = F(1, a_2, \ldots, a_r)$. An easy computation gives

$$f_1'(a_2) = 2kp(1+p)^{k-1} \cdot \ln x_2 \cdot [(1+p)^k - 2p(1-p)^{k-1}],$$

where $p = x_1 x_2^{a_2} \cdots x_r^{a_r}$. As $2p < 1+p$, $1-p < 1+p$, we get $(1+p)^k - 2p(1-p)^{k-1} > 0$. As $\ln x_2 < 0$, we get $f_1'(a_2) < 0$, implying $F(1, a_2, \ldots, a_r) \leq F(1, 1, \ldots, a_r)$. By repeating the argument r times, we get $F(a_1, \ldots, a_r) \leq f(1, 1, \ldots, 1)$, so relation (22) follows.

For the second inequality of Theorem 12, remark that since $\psi(n) \geq n+1$, it will be sufficient to prove that

$$\varphi^k(n) + \psi^{k-1}(n)\varphi(n) + \psi^k(n) \geq (n-1)^k + 2n(n+1)^{k-1}. \tag{23}$$

Dividing both sides of (23) by n^k, and by remarking again that $\varphi(n)/n$ and $\psi(n)/n$ depend only on the prime factors $p_1, \ldots, p_r$ of n, it will be sufficient to prove the inequality for $n = p_1 \cdots p_r$. This follows by Lemma 2, by selecting $x_i = p_i$ $(i = 1, 2, \ldots, r)$.

For the proof of Theorem 13, remark that we have to prove the second inequality of relation (15). Since $\sigma^*(n) \geq n+1$, it will be sufficient to prove that

$$(\varphi^*(n))^k + \varphi^*(n)(\sigma^*(n))^{k-1} + (\sigma^*(n))^k \geq (n-1)^k + 2n(n+1)^{k-1}. \tag{24}$$

This follows by Lemma 2 for $x_i = p_i^{a_i}$ $(i = 1, 2, \ldots, r)$, for $n = p_1^{a_1} \cdots p_r^{a_r}$ being the prime factorization of n.

3 Notes on new arithmetic functions and inequalities

First remark that as Lemmas 1, 2, 3, 4 are valid for any $x_i > 1$, we can apply them in a variety of other situations.

Let $s > 0$ be a real number, and apply Lemma 1 for $x_i = p_i^{sa_i}$ $(i = 1, 2, \ldots, r)$. Then $x_i \geq 2^{sa_i} \geq 2^s > 1$ as $s > 0$.

Define now the following extensions of $\varphi^*(n)$ and $\sigma^*(n)$:

$$\varphi_s^*(n) = \prod_{i=1}^r (p_i^{sa_i} - 1), \ \sigma_s^*(n) = \prod_{i=1}^r (p_i^{sa_i} + 1). \tag{25}$$

Then by Lemma 1 we get:

Theorem 14.

$$\big(\varphi_s^*(n)\big)^k + 2\big(\sigma_s^*(n)\big)^k \geq (n^s - 1)^k + 2(n^s + 1)^k. \tag{26}$$

By applying Lemma 3 for the same x_i, we get

Theorem 15.

$$\big(\sigma_s^*(n)\big)^k + 2\varphi_{sk}^*(n) \geq (n^s + 1)^k + 2(n^{sk} - 1) \tag{27}$$

for any integer $h \geq 1$, and $s > 0$ a real number.

By Lemma 2, we get

$$\big(\varphi_s^*(n)\big)^k + \varphi_s^*(n)\big(\sigma_s^*(n)\big)^{k-1} + \big(\sigma_s^*(n)\big)^k \geq (n^s - 1)^l + 2n^s \cdot (n^s + 1)^{k-1}. \tag{28}$$

Remark 1. *We can remark that, when s is a positive integer, then $\varphi_s^*(n) = \varphi^*(n^s)$ and $\sigma_s^*(n) = \sigma^*(n^s)$, but this is not true if s is not a positive integer.*

By example , when $s = 1/2$,

$$\varphi_{1/2}^*(n) = \prod_{i=1}^{r} \left(\sqrt{p_i}^{a_i} - 1\right) = \frac{\varphi^*(n)}{\sigma_{1/2}^*(n)},$$

so from (26)*, after elementary transformations, we get*

$$\left(\varphi^*(n)\right)^2 + 2\left(\sigma_{1/2}^*(n)\right)^4 \geq (3n + 2\sqrt{n} + 3)\left(\sigma_{1/2}^*(n)\right)^2. \quad (29)$$

Let now $x_i = 1\frac{p_i}{1}$ $(i = 1, 2, \ldots, n)$ in Lemma 1, with p_i $(i = 1, 2, \ldots, n)$ being the prime factors of n. Let us introduce the new arithmetic function:

$$\widetilde{\psi} = n \cdot \prod_{p|n} \left(2 + \frac{1}{p}\right), \quad (30)$$

which is an analogue of Dedekind's arithmetical function. Let $\gamma(n) = p_1 \cdots p_r$ be the "core" function of n (see [5]*). Then we get from Lemma 1:*

Theorem 16.

$$\frac{1}{(\gamma(n))^k} + 2\left(\frac{\widetilde{\psi}(n)}{n}\right)^k \geq \left(\frac{\psi(n)}{n} - 1\right)^k + 2\left(\frac{\psi(n)}{n} + 1\right)^k. \quad (31)$$

Let us now introduce the following extension of the core function $\gamma(n)$:

$$\frac{1}{\gamma_k(n)} = \prod_{i=1}^{r} \left[\left(1 + \frac{1}{p_i}\right)^k - 1\right]. \quad (32)$$

For $k = 1$ we get $\gamma_k(n) = p_1 \cdots p_r = \gamma(n)$. Now, by Lemma 3, applied to $x_i = 1 + 1/p_i$ $(i = 1, 2, \ldots, r)$, we get:

Theorem 17.

$$\left(\frac{\widetilde{\psi}(n)}{n}\right)^k + \frac{2}{\gamma_k(n)} \geq \left(\frac{\psi(n)}{n} + 1\right)^k + 2\left[\left(\frac{\psi(n)}{n}\right)^k - 1\right]. \quad (33)$$

Applying Lemma 2 for the same values of x_i, we get

Theorem 18.

$$\frac{1}{(\gamma(n))^k} + \frac{1}{\gamma(n)} \cdot \left(\frac{\widetilde{\psi}(n)}{n}\right)^{k-1} + \left(\frac{\widetilde{\psi}(n)}{n}\right)^k \geq \left(\frac{\psi(n)}{n} - 1\right)^k + 2\frac{\psi(n)}{n} \cdot \left[\frac{\psi(n)}{n} + 1\right]^{k-1}. \quad (34)$$

Applying Lemma 4 for the same values of x_i, we get

Theorem 19.

$$\left(\frac{\widetilde{\psi}(n)}{n}\right)^{2k} + \frac{2}{(\gamma_2(n))^k} \geq \left(\frac{\psi(n)}{n} + 1\right)^{2k} + 2\left[\left(\frac{\psi(n)}{n}\right)^2 - 1\right]^k. \quad (35)$$

Finally, applying Lemma 4 to $x_i = p_i^{sa_i}$ $(i = 1, \ldots, r)$, $s > 0$, we get the inequalities

Theorem 20.

$$\left(\sigma_s^*(n)\right)^{2k} + \left(\varphi_s^*(n)\right)^k \left(\sigma_s^*(n)\right)^k \geq (n^s + 1)^{2k} + 2(n^{2s} - 1)^k. \quad (36)$$

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