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On robust multiplication method for higher even-dimensional rhotrices

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Abstract: Rhotrices (heart-oriented) are often multiplied either by heart-based or row-column multiplication method. The element-wise multiplication method for higher even-dimensional rhotrices has recently been introduced in [9]. However, this type of multiplication method, though simple, is less robust. Hence, we present a multiplication method called "Robust Multiplication Method" (RMM) for higher even-dimensional rhotrices (hl-rhotrices), and a number of rediscovered properties of hl-rhotrices. Analysis and examples of RMM for some hl-rhotrices are presented for demonstration purposes.
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1 Introduction

In recent years, rhotrices have found applications in several aspects of real-life problems [7]. Rhotrices, as paradigms of matrices, are concerned with representing arrays of numbers in mathematical rhomboid form, introduced by Ajibade in 2003 [1], as an extension of ideas on matrix–tertions and matrix–noitrets proposed by Atanassov and Shannon in 1998 [4], and as represented in [2, 3].

A rhotrix has rows and columns. The row of a rhotrix is an array of entries running from the top-left to the right bottom while its column is an array of entries running from the top-right to the left bottom of the rhotrix whenever it is rotated anticlockwise through angle 45 degrees, see [19].

Thus, a rhotrix R of dimension 3 is a rhomboidal array defined in [1] as:

$$\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle.$$

The vertical axis is the set of values $\{a, c, e\}$ and the horizontal axis is the set of values $\{b, c, d\}$ of the rhotrix R above. The entry 'c' at the center of R is called the heart of R denoted as $h(R)$. It is the intersection of the major vertical and the major horizontal axes, and the above rhotrix is mathematically written as:

$$\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle.$$

The vertex of the rhotrix R is an entry at any of the four corners of the rhotrix, that is, entries a, b, e and d in the rhotrix R above. Two rhotrices can be added up only if they have the same dimension.

The addition and multiplication of two rhotrices as presented in [1] are given below. Given two rhotrices R and S,

$$R+S = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(S) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(S) & d+j \\ & e+k & \end{array} \right\rangle.$$

Multiplication ($\circ$) operation of two rhotrices R and S is defined as:

$$R \circ S = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(S) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & ah(S)+fh(R) & \\ bh(S)+gh(R) & h(R)h(S) & dh(S)+jh(R) \\ & eh(S)+kh(R) & \end{array} \right\rangle.$$

In the concluding section of [1], the author was challenged by further development regarding how a rhotrix can be converted to a matrix and vice versa for its mathematical enrichment. In quest to solving this challenge, Sani [17] in 2004 proposed the first alternative rhotrix multiplication method called the row-column multiplication method. This procedure has given room for more literatures in rhotrix algebra [15]. Thus, the row-column multiplication method presented in [17, 18] is defined as:

$$R \circ S = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(S) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(S) & aj+dk \\ & bj+ek & \end{array} \right\rangle.$$

In many aspects, rhotrices of odd dimensions (heart-oriented) are well-known in literature but even-dimensional rhotrices (heartless or hl-rhotrices) are still a novelty. Isere in 2017 classified heart-oriented rhotrices as classical rhotrices and even-dimensional rhotrices (hl-rhotrices) as non-classical rhotrices [7]. For detailed studies on classical rhotrices, see [5, 6, 10–16].

In this work, we introduce the robust multiplication method for higher even-dimensional rhotrices and derive their identity and inverse operations. Section 2 discusses the preliminary background, while Section 3 presents the robust multiplication method for high hl-rhotrices and their properties. In Section 4, some numerical examples are presented for the purpose of illustration. Finally, we give the concluding remarks in Section 5.

Remark 1.1. *By high hl-rhotrices we mean even-dimensional rhotrices of high order, higher than dimension* 2*. So, we may sometime use higher even-dimensional rhotrices or high hl-rhotrices interchangeably. Even-dimensional rhotrices and hl-rhotrices may also be used interchangeably.*

2 Preliminaries

An introduction to even-dimensional rhotrices was first presented in [8] where the author showed that it was still mathematically tractable to extract the heart of a rhotrix and still obtain an algebraic rhotrix. Moreover, doing so enables one to obtain even-dimensional rhotrices (see [8, 9, 20]). Interestingly, the objects called Quaternions by A- and V-tertions in [2, 3] are even-dimensional rhotrices provided their sides are geometrically equal.

Definition 2.1 ([8])**.** *Even-dimensional rhotrix is a rhotrix with even cardinality and a special type of rhotrix where the heart has been extracted. An example is presented below.*

$$A = \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle,$$

where a, b, d and $e \in \Re$.

Definition 2.2. *The minors of a higher even-dimensional rhotrix are the matrices and rhotrices of dimension two that can be gotten from the higher rhotrix.*

Definition 2.3 ([21])**.** *Robust multiplication method (RMM) for higher even-dimensional rhotrices, is a rhotrix multiplication method that splits the high hl-rhotrix into its minors of rhotrices and matrices of dimension two, and multiplies the corresponding minors using the row-column multiplication operation for matrices and rhotrices, and then inserts the product entries into the high hl-rhotrix.*

For example, below are minors of R_4.

$$\left\langle \begin{array}{ccc} & a_{11} & \\ a_{31} & & a_{13} \\ & a_{33} & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & c_{11} & \\ c_{21} & & c_{12} \\ & c_{22} & \end{array} \right\rangle, \text{ and } \begin{bmatrix} a_{21} & a_{12} \\ a_{32} & a_{23} \end{bmatrix},$$

as obtained from

$$R_4 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle.$$

Definition 2.4. *The index ρ of an hl-rhotrix A is the number of minor rhotrices of dimension 2 (R_2) that can be obtained from A. This index is always a whole number; cf.* [6].

2.1 Multiplication of hl-rhotrices

The element-wise multiplication of higher hl-rhotrices is presented in [9]. This has been the only multiplication method for high hl-rhotrices in literature up till now. Each entry is obtained by multiplying the corresponding elements.

Consider the set of any two 4-dimensional rhotrices,

$$R_4 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle \text{ and } S_4 = \left\langle \begin{array}{ccccc} & & b_{11} & & \\ & b_{21} & d_{11} & b_{12} & \\ b_{31} & d_{21} & & d_{12} & b_{13} \\ & b_{32} & d_{22} & b_{23} & \\ & & b_{33} & & \end{array} \right\rangle.$$

By the element-wise multiplication method, we have

$$\begin{aligned} R_4 \circ S_4 &= \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & & b_{11} & & \\ & b_{21} & d_{11} & b_{12} & \\ b_{31} & d_{21} & & d_{12} & b_{13} \\ & b_{32} & d_{22} & b_{23} & \\ & & b_{33} & & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccccc} & & a_{11}b_{11} & & \\ & a_{21}b_{21} & c_{11}d_{11} & a_{12}b_{12} & \\ a_{31}b_{31} & c_{21}d_{21} & & c_{12}d_{12} & a_{13}b_{13} \\ & a_{32}b_{32} & c_{22}d_{22} & a_{23}b_{23} & \\ & & a_{33}b_{33} & & \end{array} \right\rangle. \end{aligned}$$

The multiplication above is very simple and beautiful but less robust. Hence, we present a more robust multiplication method for high hl-rhotrices in Section 3.

2.2 Identity and inverse elements of hl-rhotrices

For a 2-dimensional hl-rhotrix (R_2), the identity and inverse elements, as presented in [8], are given below:

(i) Consider an hl-rhotrix R of n-dimensional, if I is also an hl-rhotrix of n-dimensional such that: $R \circ I = R = I \circ R$. Then I is an identity element.

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} \right\rangle.$$

(ii) The concept of identity element makes the inverse of a rhotrix meaningful. If for an hl-rhotrix R we can find another hl-rhotrix S such that $R \circ S = S \circ R = I$, then S will be the inverse of R. Consider R_2 for example, let

$$R = \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle.$$

Then,

$$S = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -d & & -b \\ & a & \end{array} \right\rangle.$$

This implies that

$$R^{-1} = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -d & & -b \\ & a & \end{array} \right\rangle.$$

Remark 2.1. *For an hl-rhotrix R_2 to be invertible or non-singular, $ae \neq bd$ must hold.*

3 Main results

In this section, we develop an alternative multiplication method for higher even-dimensional rhotrices (hl-rhotrices), called the Robust Multiplication Method (RMM). For the purpose of illustration, we start with 4-dimensional rhotrices up to 12-dimensional rhotrices, and these are presented below.

3.1 Multiplication of high hl-rhotrices

(i) The multiplication of 4-dimensional rhotrices (R_4) is defined as:

$$\begin{aligned} R_4 \times S_4 &= \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle \times \left\langle \begin{array}{ccccc} & & b_{11} & & \\ & b_{21} & d_{11} & b_{12} & \\ b_{31} & d_{21} & & d_{12} & b_{13} \\ & b_{32} & d_{22} & b_{23} & \\ & & b_{33} & & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccccc} & & \alpha_{11} & & \\ & \alpha_{21} & \beta_{11} & \alpha_{12} & \\ \alpha_{31} & \beta_{21} & & \beta_{12} & \alpha_{13} \\ & \alpha_{32} & \beta_{22} & \alpha_{23} & \\ & & \alpha_{33} & & \end{array} \right\rangle, \end{aligned}$$

Solution: First, we obtain the minors of R_4 and S_4 as:

$$\left\langle \begin{array}{ccc} & a_{11} & \\ a_{31} & & a_{13} \\ & a_{33} & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & b_{11} & \\ b_{31} & & b_{13} \\ & b_{33} & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & c_{11} & \\ c_{21} & & c_{12} \\ & c_{22} & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & d_{11} & \\ d_{21} & & d_{12} \\ & d_{22} & \end{array} \right\rangle, \begin{bmatrix} a_{21} & a_{12} \\ a_{32} & a_{23} \end{bmatrix}, \begin{bmatrix} b_{21} & b_{12} \\ b_{32} & b_{23} \end{bmatrix}.$$

Then, the row-column multiplications of the systems are:

$$\left\langle \begin{array}{ccc} & a_{11} & \\ a_{31} & & a_{13} \\ & a_{33} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & b_{11} & \\ b_{31} & & b_{13} \\ & b_{33} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a_{11}b_{11}+a_{13}b_{31} & \\ a_{31}b_{11}+a_{33}b_{31} & & a_{13}b_{33}+a_{11}b_{13} \\ & a_{33}b_{33}+a_{31}b_{13} & \end{array} \right\rangle;$$

$$\left\langle \begin{array}{ccc} & c_{11} & \\ c_{21} & & c_{12} \\ & c_{22} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & d_{11} & \\ d_{21} & & d_{12} \\ & d_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & c_{11}d_{11}+c_{12}d_{21} & \\ c_{21}d_{11}+c_{22}d_{21} & & c_{12}d_{22}+c_{11}d_{12} \\ & c_{22}d_{22}+c_{21}d_{12} & \end{array} \right\rangle;$$

$$\begin{bmatrix} a_{21} & a_{12} \\ a_{32} & a_{23} \end{bmatrix} \circ \begin{bmatrix} b_{21} & b_{12} \\ b_{32} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{21}b_{21}+a_{12}b_{32} & a_{12}b_{23}+a_{21}b_{12} \\ a_{32}b_{21}+a_{23}b_{32} & a_{23}b_{23}+a_{32}b_{12} \end{bmatrix}.$$

with the following equations:

$$\begin{array}{llll} a_{11}b_{11}+a_{13}b_{31}=\alpha_{11}; & a_{31}b_{11}+a_{33}b_{31}=\alpha_{31}; & a_{33}b_{33}+a_{31}b_{13}=\alpha_{33}; & a_{13}b_{33}+a_{11}b_{13}=\alpha_{13}; \\ c_{11}d_{11}+c_{12}d_{21}=\beta_{11}; & c_{21}d_{11}+c_{22}d_{21}=\beta_{21}; & c_{22}d_{22}+c_{21}d_{12}=\beta_{22}; & c_{12}d_{22}+c_{11}d_{12}=\beta_{12}; \\ a_{21}b_{21}+a_{12}b_{32}=\alpha_{21}; & a_{32}b_{21}+a_{23}b_{32}=\alpha_{32}; & a_{23}b_{23}+a_{32}b_{12}=\alpha_{23}; & a_{12}b_{23}+a_{21}b_{12}=\alpha_{12}. \end{array}$$

Then, the RMM of R_4 is:

$$R_4 \times S_4 = \left\langle \begin{array}{ccccc} & & a_{11}b_{11}+a_{13}b_{31} & & \\ & a_{21}b_{12}+a_{12}b_{32} & c_{11}d_{11}+c_{12}d_{21} & a_{12}b_{23}+a_{21}b_{12} & \\ a_{31}b_{11}+a_{33}b_{31} & c_{21}d_{11}+c_{22}d_{21} & & c_{12}d_{22}+c_{11}d_{12} & a_{13}b_{33}+a_{11}b_{13} \\ & a_{32}b_{21}+a_{23}b_{32} & c_{22}d_{22}+c_{21}d_{12} & a_{23}b_{23}+a_{32}b_{12} & \\ & & a_{33}b_{33}+a_{31}b_{13} & & \end{array} \right\rangle$$

(ii) Similarly, the multiplication of 6-dimensional Rhotrices (R_6) is defined as:

$$R_6 \times S_6 = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ a_{41} & c_{31} & a_{32} & & a_{23} & c_{13} & a_{14} \\ & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & \\ & & a_{43} & c_{33} & a_{34} & & \\ & & & a_{44} & & & \end{array} \right\rangle \times \left\langle \begin{array}{ccccccc} & & & b_{11} & & & \\ & & b_{21} & d_{11} & b_{12} & & \\ & b_{31} & d_{21} & b_{22} & d_{12} & b_{13} & \\ b_{41} & d_{31} & b_{32} & & b_{23} & d_{13} & b_{14} \\ & b_{42} & d_{32} & b_{33} & d_{23} & b_{24} & \\ & & b_{43} & d_{33} & b_{34} & & \\ & & & b_{44} & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccc} & & & \alpha_{11} & & & \\ & & \alpha_{21} & \beta_{11} & \alpha_{12} & & \\ & \alpha_{31} & \beta_{21} & \alpha_{22} & \beta_{12} & \alpha_{13} & \\ \alpha_{41} & \beta_{31} & \alpha_{32} & & \alpha_{23} & \beta_{13} & \alpha_{14} \\ & \alpha_{42} & \beta_{32} & \alpha_{33} & \beta_{23} & \alpha_{24} & \\ & & \alpha_{43} & \beta_{33} & \alpha_{34} & & \\ & & & \alpha_{44} & & & \end{array} \right\rangle,$$

where

$$\alpha_{11} = a_{11}b_{11} + a_{14}b_{41};\quad \alpha_{12} = a_{12}b_{34} + a_{21}b_{12};\quad \alpha_{13} = a_{13}b_{24} + a_{31}b_{13};\quad \alpha_{14} = a_{14}b_{44} + a_{11}b_{14};$$
$$\beta_{11} = c_{11}d_{11} + c_{13}d_{31};\quad \beta_{12} = c_{12}d_{23} + c_{21}d_{12};\quad \beta_{13} = c_{13}d_{33} + c_{11}d_{13};\quad \alpha_{21} = a_{21}b_{21} + a_{12}b_{43};$$
$$\alpha_{22} = a_{22}b_{22} + a_{23}b_{32};\quad \alpha_{23} = a_{23}b_{33} + a_{22}b_{23};\quad \alpha_{24} = a_{24}b_{24} + a_{42}b_{13};\quad \beta_{21} = c_{21}d_{21} + c_{12}d_{32};$$
$$\beta_{23} = c_{23}d_{23} + c_{32}d_{12};\quad \alpha_{31} = a_{31}b_{31} + a_{13}b_{42};\quad \alpha_{32} = a_{32}b_{22} + a_{33}b_{32};\quad \alpha_{33} = a_{33}b_{33} + a_{32}b_{23};$$
$$\alpha_{34} = a_{34}b_{34} + a_{43}b_{12};\quad \beta_{31} = c_{31}d_{11} + c_{33}d_{31};\quad \beta_{32} = c_{32}d_{21} + c_{23}d_{32};\quad \beta_{33} = c_{33}d_{33} + c_{31}d_{31};$$
$$\alpha_{41} = a_{41}b_{11} + a_{44}b_{41};\quad \alpha_{42} = a_{42}b_{31} + a_{24}b_{42};\quad \alpha_{43} = a_{43}b_{21} + a_{34}b_{43};\quad \alpha_{44} = a_{44}b_{44} + a_{41}b_{14}.$$

(iii) Then, the multiplication of 8-dimensional rhotrices (R_8) is also defined as:

$$R_8 \times S_8 =$$

$$\left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & c_{11} & a_{12} & & & \\
 & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\
 & a_{41} & c_{31} & a_{32} & c_{22} & a_{23} & c_{13} & a_{14} & \\
a_{51} & c_{41} & a_{42} & c_{32} & & c_{23} & a_{24} & c_{14} & a_{15} \\
 & a_{52} & c_{42} & a_{43} & c_{33} & a_{34} & c_{24} & a_{25} & \\
 & & a_{53} & c_{43} & a_{44} & c_{34} & a_{35} & & \\
 & & & a_{54} & c_{44} & a_{45} & & & \\
 & & & & a_{55} & & & &
\end{array} \right\rangle \times \left\langle \begin{array}{ccccccccc}
 & & & & b_{11} & & & & \\
 & & & b_{21} & d_{11} & b_{12} & & & \\
 & & b_{31} & d_{21} & b_{22} & d_{12} & b_{13} & & \\
 & b_{41} & d_{31} & b_{32} & d_{22} & b_{23} & d_{13} & b_{14} & \\
b_{51} & d_{41} & b_{42} & d_{32} & & d_{23} & b_{24} & d_{14} & b_{15} \\
 & b_{52} & d_{42} & b_{43} & d_{33} & b_{34} & d_{24} & b_{25} & \\
 & & b_{53} & d_{43} & b_{44} & d_{34} & b_{35} & & \\
 & & & b_{54} & d_{44} & b_{45} & & & \\
 & & & & b_{55} & & & &
\end{array} \right\rangle .$$

Then, the RMM of R_8 is:

$$R_8 \times S_8 = \left\langle \begin{array}{ccccccccc}
 & & & & \alpha_{11} & & & & \\
 & & & \alpha_{21} & \beta_{11} & \alpha_{12} & & & \\
 & & \alpha_{31} & \beta_{21} & \alpha_{22} & \beta_{12} & \alpha_{13} & & \\
 & \alpha_{41} & \beta_{31} & \alpha_{32} & \beta_{22} & \alpha_{23} & \beta_{13} & \alpha_{14} & \\
\alpha_{51} & \beta_{41} & \alpha_{42} & \beta_{32} & & \beta_{23} & \alpha_{24} & \beta_{14} & \alpha_{15} \\
 & \alpha_{52} & \beta_{42} & \alpha_{43} & \beta_{33} & \alpha_{34} & \beta_{24} & \alpha_{25} & \\
 & & \alpha_{53} & \beta_{43} & \alpha_{44} & \beta_{34} & \alpha_{35} & & \\
 & & & \alpha_{54} & \beta_{44} & \alpha_{45} & & & \\
 & & & & \alpha_{55} & & & &
\end{array} \right\rangle$$

where

$$\alpha_{11} = a_{11}b_{11} + a_{15}b_{51};\quad \alpha_{12} = a_{12}b_{45} + a_{21}b_{12};\quad \alpha_{13} = a_{13}b_{35} + a_{31}b_{13};\quad \alpha_{14} = a_{14}b_{25} + a_{41}b_{14};$$
$$\alpha_{15} = a_{15}b_{55} + a_{11}b_{15};\quad \beta_{11} = c_{11}d_{11} + c_{14}d_{41};\quad \beta_{12} = c_{12}d_{34} + c_{21}d_{12};\quad \beta_{13} = c_{13}d_{24} + c_{31}d_{13};$$
$$\beta_{14} = c_{14}d_{44} + c_{11}d_{14};\quad \alpha_{21} = a_{21}b_{21} + a_{12}b_{54};\quad \alpha_{22} = a_{22}b_{22} + a_{24}b_{42};\quad \alpha_{23} = a_{23}b_{34} + a_{32}b_{23};$$
$$\alpha_{24} = a_{24}b_{44} + a_{22}b_{24};\quad \alpha_{25} = a_{25}b_{25} + a_{52}b_{14};\quad \beta_{21} = c_{21}d_{21} + c_{12}d_{43};\quad \beta_{22} = c_{22}d_{22} + c_{23}d_{32};$$
$$\beta_{23} = c_{23}d_{33} + c_{22}d_{23};\quad \beta_{24} = c_{24}d_{24} + c_{42}d_{13};\quad \alpha_{31} = a_{31}b_{31} + a_{13}b_{53};\quad \alpha_{32} = a_{32}b_{32} + a_{23}b_{43};$$
$$\alpha_{34} = a_{34}b_{34} + a_{43}b_{23};\quad \alpha_{35} = a_{35}b_{35} + a_{53}b_{13};\quad \beta_{31} = c_{31}d_{31} + c_{13}d_{42};\quad \beta_{32} = c_{32}d_{22} + c_{33}d_{32};$$
$$\beta_{33} = c_{33}d_{33} + c_{32}d_{23};\quad \beta_{34} = c_{34}d_{34} + c_{43}d_{12};\quad \alpha_{41} = a_{41}b_{41} + a_{14}b_{52};\quad \alpha_{42} = a_{42}b_{22} + a_{44}b_{42};$$
$$\alpha_{43} = a_{43}b_{32} + a_{34}b_{43};\quad \alpha_{44} = a_{44}b_{44} + a_{42}b_{24};\quad \alpha_{45} = a_{45}b_{45} + a_{54}b_{12};\quad \beta_{41} = c_{41}d_{11} + c_{44}d_{41};$$
$$\beta_{42} = c_{42}d_{31} + c_{24}d_{42};\quad \beta_{43} = c_{43}d_{21} + c_{34}d_{43};\quad \beta_{44} = c_{44}d_{44} + c_{41}d_{14};\quad \alpha_{51} = a_{51}b_{11} + a_{55}b_{51};$$
$$\alpha_{52} = a_{52}b_{41} + a_{25}b_{52};\quad \alpha_{53} = a_{53}b_{31} + a_{35}b_{53};\quad \alpha_{54} = a_{54}b_{21} + a_{45}b_{54};\quad \alpha_{55} = a_{55}b_{55} + a_{51}b_{15}.$$

(iv) Multiplication of 10-dimensional rhotrices (R_{10}) is also defined as:

$$R_{10} \times S_{10} = \left\langle \begin{array}{ccccccccccc}
 & & & & & a_{11} & & & & & \\
 & & & & a_{21} & c_{11} & a_{12} & & & & \\
 & & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & & \\
 & & a_{41} & c_{31} & a_{32} & c_{22} & a_{23} & c_{13} & a_{14} & & \\
 & a_{51} & c_{41} & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & c_{14} & a_{15} & \\
a_{61} & c_{51} & a_{52} & c_{42} & a_{43} & & a_{34} & c_{24} & a_{25} & c_{15} & a_{16} \\
 & a_{62} & c_{52} & a_{53} & c_{43} & a_{44} & c_{34} & a_{35} & c_{25} & a_{26} & \\
 & & a_{63} & c_{53} & a_{54} & c_{44} & a_{45} & c_{35} & a_{36} & & \\
 & & & a_{64} & c_{54} & a_{55} & c_{45} & a_{46} & & & \\
 & & & & a_{65} & c_{55} & a_{56} & & & & \\
 & & & & & a_{66} & & & & &
\end{array} \right\rangle$$

$$\times \left\langle \begin{array}{ccccccccccc}
 & & & & & b_{11} & & & & & \\
 & & & & b_{21} & d_{11} & b_{12} & & & & \\
 & & & b_{31} & d_{21} & b_{22} & d_{12} & b_{13} & & & \\
 & & b_{41} & d_{31} & b_{32} & d_{22} & b_{23} & d_{13} & b_{14} & & \\
 & b_{51} & d_{41} & b_{42} & d_{32} & b_{33} & d_{23} & b_{24} & d_{14} & b_{15} & \\
b_{61} & d_{51} & b_{52} & d_{42} & b_{43} & & b_{34} & d_{24} & b_{25} & d_{15} & b_{16} \\
 & b_{62} & d_{52} & b_{53} & d_{43} & b_{44} & d_{34} & b_{35} & d_{25} & b_{26} & \\
 & & b_{63} & d_{53} & b_{54} & d_{44} & b_{45} & d_{35} & b_{36} & & \\
 & & & b_{64} & d_{54} & b_{55} & d_{45} & b_{46} & & & \\
 & & & & b_{65} & d_{55} & b_{56} & & & & \\
 & & & & & b_{66} & & & & &
\end{array} \right\rangle$$

Then, the RMM of R_{10} is:

$$R_{10} \times S_{10} = \left\langle \begin{array}{ccccccccccc}
 & & & & & \alpha_{11} & & & & & \\
 & & & & \alpha_{21} & \beta_{11} & \alpha_{12} & & & & \\
 & & & \alpha_{31} & \beta_{21} & \alpha_{22} & \beta_{12} & \alpha_{13} & & & \\
 & & \alpha_{41} & \beta_{31} & \alpha_{32} & \beta_{22} & \alpha_{23} & \beta_{13} & \alpha_{14} & & \\
 & \alpha_{51} & \beta_{41} & \alpha_{42} & \beta_{32} & \alpha_{33} & \beta_{23} & \alpha_{24} & \beta_{14} & \alpha_{15} & \\
\alpha_{61} & \beta_{51} & \alpha_{52} & \beta_{42} & \alpha_{43} & & \alpha_{34} & \beta_{24} & \alpha_{25} & d_{15} & \alpha_{16} \\
 & \alpha_{62} & \beta_{52} & \alpha_{53} & \beta_{43} & \alpha_{44} & \beta_{34} & \alpha_{35} & d_{25} & \alpha_{26} & \\
 & & \alpha_{63} & \beta_{53} & \alpha_{54} & \beta_{44} & \alpha_{45} & \beta_{35} & \alpha_{36} & & \\
 & & & \alpha_{64} & \beta_{54} & \alpha_{55} & \beta_{45} & \alpha_{46} & & & \\
 & & & & \alpha_{65} & \beta_{55} & \alpha_{56} & & & & \\
 & & & & & \alpha_{66} & & & & &
\end{array} \right\rangle$$

where

$$\alpha_{11} = a_{11}b_{11} + a_{16}b_{61}, \quad \alpha_{12} = a_{21}b_{12} + a_{12}b_{56}, \quad \alpha_{13} = a_{31}b_{13} + a_{13}b_{46}, \quad \alpha_{14} = a_{41}b_{14} + a_{14}b_{36},$$
$$\alpha_{15} = a_{51}b_{15} + a_{15}b_{26}, \quad \alpha_{16} = a_{11}b_{16} + a_{16}b_{66}, \quad \beta_{11} = c_{11}d_{11} + c_{15}d_{51}, \quad \beta_{12} = c_{21}d_{12} + c_{12}d_{45},$$
$$\beta_{13} = c_{31}d_{13} + c_{13}d_{35}, \quad \beta_{14} = c_{41}d_{14} + c_{14}d_{25}, \quad \beta_{15} = c_{11}d_{15} + c_{15}d_{55}, \quad \alpha_{17} = a_{21}b_{21} + a_{12}b_{65},$$
$$\alpha_{18} = a_{22}b_{22} + a_{25}b_{52}, \quad \alpha_{21} = a_{23}b_{45} + a_{32}b_{23}, \quad \alpha_{22} = a_{42}b_{24} + a_{24}b_{35}, \quad \alpha_{23} = a_{22}b_{25} + a_{25}b_{55},$$
$$\alpha_{25} = a_{62}b_{15} + a_{26}b_{26}, \quad \beta_{21} = c_{21}d_{21} + c_{12}d_{54}, \quad \beta_{22} = c_{22}d_{22} + c_{24}d_{42}, \quad \beta_{23} = c_{32}d_{23} + c_{23}d_{34},$$

$$\begin{array}{llll}
\beta_{24}=c_{22}d_{24}+c_{24}d_{44}, & \beta_{25}=c_{52}d_{14}+c_{25}d_{25}, & \alpha_{31}=a_{31}b_{31}+a_{13}b_{64}, & \alpha_{32}=a_{32}b_{32}+a_{23}b_{54}, \\
\alpha_{33}=a_{33}b_{33}+a_{34}b_{43}, & \alpha_{34}=a_{33}b_{34}+a_{34}b_{44}, & \alpha_{35}=a_{53}b_{24}+a_{35}b_{35}, & \alpha_{36}=a_{63}b_{14}+a_{36}b_{36}, \\
\beta_{31}=c_{31}d_{31}+c_{13}d_{53}, & \beta_{32}=c_{32}d_{32}+c_{23}d_{43}, & \beta_{34}=c_{43}d_{23}+c_{34}d_{34}, & \beta_{35}=c_{53}d_{13}+c_{35}d_{35}, \\
\alpha_{41}=a_{41}b_{41}+a_{14}b_{63}, & \alpha_{42}=a_{42}b_{42}+a_{24}b_{53}, & \alpha_{43}=a_{43}b_{33}+a_{44}b_{43}, & \alpha_{44}=a_{43}b_{34}+a_{44}b_{44}, \\
\alpha_{45}=a_{45}b_{45}+a_{54}b_{23}, & \alpha_{46}=a_{64}b_{13}+a_{46}b_{46}, & \beta_{41}=c_{41}d_{41}+c_{14}d_{52}, & \beta_{42}=c_{42}d_{22}+c_{44}d_{42}, \\
\beta_{43}=c_{43}d_{32}+c_{34}d_{43}, & \beta_{44}=c_{42}d_{24}+c_{44}d_{44}, & \beta_{45}=c_{54}d_{12}+c_{45}d_{45}, & \alpha_{51}=a_{51}b_{51}+a_{15}b_{62}, \\
\alpha_{52}=a_{52}b_{22}+a_{55}b_{52}, & \alpha_{53}=a_{53}b_{42}+a_{35}b_{53}, & \alpha_{54}=a_{54}b_{32}+a_{45}b_{54}, & \alpha_{55}=a_{52}b_{25}+a_{55}b_{55}, \\
\alpha_{56}=a_{65}b_{12}+a_{56}b_{56}, & \beta_{51}=c_{51}d_{11}+c_{55}d_{51}, & \beta_{52}=c_{52}d_{41}+c_{25}d_{52}, & \beta_{53}=c_{53}d_{31}+c_{35}d_{53}, \\
\beta_{54}=c_{54}d_{21}+c_{45}d_{54}, & \beta_{55}=c_{51}d_{15}+c_{55}d_{55}, & \alpha_{61}=a_{61}b_{11}+a_{66}b_{61}, & \alpha_{62}=a_{62}b_{51}+a_{26}b_{62}, \\
\alpha_{63}=a_{63}b_{41}+a_{36}b_{63}, & \alpha_{64}=a_{64}b_{31}+a_{46}b_{64}, & \alpha_{65}=a_{65}b_{21}+a_{56}b_{65}, & \alpha_{66}=a_{61}b_{16}+a_{66}b_{66}.
\end{array}$$

(v) Finally, the multiplication of 12-dimensional rhotrices (R_{12}) is similarly defined as:

$$R_{12}\times S_{12}=\left\langle\begin{array}{ccccccccccccc}
 & & & & & & a_{11} & & & & & & \\
 & & & & & a_{21} & c_{11} & a_{12} & & & & & \\
 & & & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & & & \\
 & & & a_{41} & c_{31} & a_{32} & c_{22} & a_{23} & c_{13} & a_{14} & & & \\
 & & a_{51} & c_{41} & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & c_{14} & a_{15} & & \\
 & a_{61} & c_{51} & a_{52} & c_{42} & a_{43} & c_{33} & a_{34} & c_{24} & a_{25} & c_{15} & a_{16} & \\
a_{71} & c_{61} & a_{62} & c_{52} & a_{53} & c_{43} & & c_{34} & a_{35} & c_{25} & a_{26} & c_{16} & a_{17} \\
 & a_{72} & c_{62} & a_{63} & c_{53} & a_{54} & c_{44} & a_{45} & c_{35} & a_{36} & c_{26} & a_{27} & \\
 & & a_{73} & c_{63} & a_{64} & c_{54} & a_{55} & c_{45} & a_{46} & c_{36} & a_{37} & & \\
 & & & a_{74} & c_{64} & a_{65} & c_{55} & a_{56} & c_{46} & a_{47} & & & \\
 & & & & a_{75} & c_{65} & a_{66} & c_{56} & a_{57} & & & & \\
 & & & & & a_{76} & c_{66} & a_{67} & & & & & \\
 & & & & & & a_{77} & & & & & &
\end{array}\right\rangle$$

$$\times\left\langle\begin{array}{ccccccccccccc}
 & & & & & & b_{11} & & & & & & \\
 & & & & & b_{21} & d_{11} & b_{12} & & & & & \\
 & & & & b_{31} & d_{21} & b_{22} & d_{12} & b_{13} & & & & \\
 & & & b_{41} & d_{31} & b_{32} & d_{22} & b_{23} & d_{13} & b_{14} & & & \\
 & & b_{51} & d_{41} & b_{42} & d_{32} & b_{33} & d_{23} & b_{24} & d_{14} & b_{15} & & \\
 & b_{61} & d_{51} & b_{52} & d_{42} & b_{43} & d_{33} & b_{34} & d_{24} & b_{25} & d_{15} & b_{16} & \\
b_{71} & d_{61} & b_{62} & d_{52} & b_{53} & d_{43} & & d_{34} & b_{35} & d_{25} & b_{26} & d_{16} & b_{17} \\
 & b_{72} & d_{62} & b_{63} & d_{53} & b_{54} & d_{44} & b_{45} & d_{35} & b_{36} & d_{26} & b_{27} & \\
 & & b_{73} & d_{63} & b_{64} & d_{54} & b_{55} & d_{45} & b_{46} & d_{36} & b_{37} & & \\
 & & & b_{74} & d_{64} & b_{65} & d_{55} & b_{56} & d_{46} & b_{47} & & & \\
 & & & & b_{75} & d_{65} & b_{66} & d_{56} & b_{57} & & & & \\
 & & & & & b_{76} & d_{66} & b_{67} & & & & & \\
 & & & & & & b_{77} & & & & & &
\end{array}\right\rangle$$

Then, the RMM of R_{12} is:

$$R_{12} \times S_{12} = \left\langle \begin{array}{ccccccccccccc}
 & & & & & & \alpha_{11} & & & & & & \\
 & & & & & \alpha_{21} & \beta_{11} & \alpha_{12} & & & & & \\
 & & & & \alpha_{31} & \beta_{21} & \alpha_{22} & \beta_{12} & \alpha_{13} & & & & \\
 & & & \alpha_{41} & \beta_{31} & \alpha_{32} & \beta_{22} & \alpha_{23} & \beta_{13} & \alpha_{14} & & & \\
 & & \alpha_{51} & \beta_{41} & \alpha_{42} & \beta_{32} & \alpha_{33} & \beta_{23} & \alpha_{24} & \beta_{14} & \alpha_{15} & & \\
 & \alpha_{61} & \beta_{51} & \alpha_{52} & \beta_{42} & \alpha_{43} & \beta_{33} & \alpha_{34} & \beta_{24} & \alpha_{25} & \beta_{15} & \alpha_{16} & \\
\alpha_{71} & \beta_{61} & \alpha_{62} & \beta_{52} & \alpha_{53} & \beta_{43} & & \beta_{34} & \alpha_{35} & \beta_{25} & \alpha_{26} & \beta_{16} & \alpha_{17} \\
 & \alpha_{72} & \beta_{62} & \alpha_{63} & \beta_{53} & \alpha_{54} & \beta_{44} & \alpha_{45} & \beta_{35} & \alpha_{36} & \beta_{26} & \alpha_{27} & \\
 & & \alpha_{73} & \beta_{63} & \alpha_{64} & \beta_{54} & \alpha_{55} & \beta_{45} & \alpha_{46} & \beta_{36} & \alpha_{37} & & \\
 & & & \alpha_{74} & \beta_{64} & \alpha_{65} & \beta_{55} & \alpha_{56} & \beta_{46} & \alpha_{47} & & & \\
 & & & & \alpha_{75} & \beta_{65} & \alpha_{66} & \beta_{56} & \alpha_{57} & & & & \\
 & & & & & \alpha_{76} & \beta_{66} & \alpha_{67} & & & & & \\
 & & & & & & \alpha_{77} & & & & & &
\end{array} \right\rangle,$$

where

$$\begin{array}{llll}
\alpha_{11} = a_{11}b_{11} + a_{17}b_{71}, & \alpha_{12} = a_{21}b_{12} + a_{12}b_{67}, & \alpha_{13} = a_{31}b_{13} + a_{13}b_{57}, & \alpha_{14} = a_{41}b_{14} + a_{14}b_{47}, \\
\alpha_{15} = a_{51}b_{15} + a_{15}b_{37}, & \alpha_{16} = a_{61}b_{16} + a_{16}d_{27}, & \alpha_{17} = a_{11}b_{17} + a_{17}b_{77}, & \beta_{11} = c_{11}d_{11} + c_{16}d_{61}, \\
\beta_{12} = c_{21}d_{12} + c_{12}d_{56}, & \beta_{13} = c_{31}d_{13} + c_{13}d_{46}, & \beta_{14} = c_{41}d_{14} + c_{14}d_{36}, & \beta_{15} = c_{51}d_{15} + c_{15}d_{26}, \\
\beta_{16} = c_{11}d_{16} + c_{16}d_{66}, & \alpha_{21} = a_{21}b_{21} + a_{12}b_{76}, & \alpha_{22} = a_{22}b_{22} + a_{26}b_{62}, & \alpha_{23} = a_{32}b_{23} + a_{23}b_{56}, \\
\alpha_{24} = a_{42}b_{24} + a_{24}b_{46}, & \alpha_{25} = a_{52}b_{25} + a_{25}b_{36}, & \alpha_{26} = a_{22}d_{16} + a_{26}d_{66}, & \alpha_{27} = a_{72}b_{16} + a_{27}b_{27}, \\
\beta_{21} = c_{21}d_{21} + c_{12}d_{65}, & \beta_{22} = c_{22}d_{22} + c_{25}d_{52}, & \beta_{23} = c_{32}d_{23} + c_{23}d_{45}, & \beta_{24} = c_{42}d_{24} + c_{24}d_{35}, \\
\beta_{25} = c_{22}d_{25} + c_{25}d_{55}, & \beta_{26} = c_{62}d_{15} + c_{26}d_{26}, & \alpha_{31} = a_{31}b_{31} + a_{13}b_{75}, & \alpha_{32} = a_{32}b_{32} + a_{23}b_{65}, \\
\alpha_{33} = a_{33}b_{33} + c_{35}b_{53}, & \alpha_{34} = a_{43}b_{34} + a_{34}b_{45}, & \alpha_{35} = a_{33}b_{35} + a_{35}b_{55}, & \alpha_{36} = a_{63}b_{25} + a_{36}b_{36}, \\
\alpha_{37} = a_{73}b_{15} + a_{37}b_{37}, & \beta_{31} = c_{31}d_{31} + c_{13}d_{64}, & \beta_{32} = c_{32}d_{32} + c_{23}d_{54}, & \beta_{33} = c_{33}d_{33} + c_{34}d_{43}, \\
\beta_{34} = c_{33}d_{34} + c_{34}d_{44}, & \beta_{35} = c_{53}d_{24} + c_{35}d_{35}, & \beta_{36} = c_{63}d_{14} + c_{36}d_{36}, & \alpha_{41} = a_{41}b_{41} + a_{14}b_{74}, \\
\alpha_{42} = a_{42}b_{42} + a_{24}b_{64}, & \alpha_{43} = a_{43}b_{43} + a_{34}b_{54}, & \alpha_{45} = a_{54}b_{34} + a_{45}b_{45}, & \alpha_{46} = a_{64}b_{24} + a_{46}b_{46}, \\
\alpha_{47} = a_{74}b_{14} + a_{47}b_{47}, & \beta_{41} = c_{41}d_{41} + c_{14}d_{63}, & \beta_{42} = c_{42}d_{42} + c_{24}d_{53}, & \beta_{43} = c_{43}d_{33} + c_{44}d_{43}, \\
\beta_{44} = c_{43}d_{34} + c_{44}d_{44}, & \beta_{45} = c_{54}d_{23} + c_{45}d_{45}, & \beta_{46} = c_{64}d_{13} + c_{46}d_{46}, & \alpha_{51} = a_{51}b_{51} + a_{15}b_{73}, \\
\alpha_{52} = a_{52}b_{52} + a_{25}b_{63}, & \alpha_{53} = a_{53}b_{33} + a_{55}b_{53}, & \alpha_{54} = a_{54}b_{43} + a_{45}b_{54}, & \alpha_{55} = a_{53}b_{35} + a_{55}b_{55}, \\
\alpha_{56} = a_{65}b_{23} + a_{56}b_{56}, & \alpha_{57} = a_{75}b_{13} + a_{57}b_{57}, & \beta_{51} = c_{51}d_{51} + c_{15}d_{62}, & \beta_{52} = c_{52}d_{22} + c_{55}d_{52}, \\
\beta_{53} = c_{53}d_{42} + c_{35}d_{53}, & \beta_{54} = c_{54}d_{32} + c_{45}d_{54}, & \beta_{55} = c_{55}d_{55} + c_{52}d_{25}, & \beta_{56} = c_{65}d_{12} + c_{56}d_{56}, \\
\alpha_{61} = a_{61}b_{11} + a_{66}b_{61}, & \alpha_{62} = a_{62}b_{22} + a_{66}b_{62}, & \alpha_{63} = a_{63}b_{52} + a_{36}b_{63}, & \alpha_{64} = c_{64}d_{31} + c_{46}d_{64}, \\
\alpha_{65} = a_{65}b_{32} + a_{56}b_{65}, & \alpha_{66} = a_{62}b_{26} + a_{66}b_{66}, & \alpha_{67} = a_{76}b_{12} + a_{67}b_{67}, & \beta_{61} = c_{61}d_{11} + c_{66}d_{61}, \\
\beta_{62} = c_{62}d_{51} + c_{26}d_{62}, & \beta_{63} = c_{63}d_{41} + c_{36}d_{63}, & \beta_{64} = c_{64}d_{31} + c_{46}d_{64}, & \beta_{65} = c_{65}d_{21} + c_{56}d_{65}, \\
\beta_{66} = c_{61}d_{16} + c_{66}d_{66}, & \alpha_{71} = a_{71}b_{11} + a_{77}b_{71}, & \alpha_{72} = a_{72}b_{61} + a_{27}b_{72}, & \alpha_{73} = a_{73}b_{51} + a_{37}b_{73}, \\
\alpha_{74} = a_{74}b_{41} + a_{47}b_{74}, & \alpha_{75} = a_{75}b_{31} + a_{57}b_{75}, & \alpha_{76} = a_{76}b_{21} + a_{67}b_{76}, & \alpha_{77} = a_{71}b_{17} + a_{77}b_{77}.
\end{array}$$

Remark 3.1. *The multiplication of all high hl-rhotrices by RMM follows the same pattern as illustrated above. First, split them into their minors of R_2 and M_2, and multiply them accordingly, using the row-column method. Second, return their corresponding products into the product rhotrix. For example, each of the R_6 has three minors of R_2 and three minors of M_2 which are multiplied by the row-column method, and are returned to the product rhotrix, and so on.*

Theorem 3.1. *Let R_n be an hl-rhotrix of dimension n for all $n \in 2\mathbb{N}$. Then,*

(a) *there exist $\rho = \frac{n}{2}$ number of minor rhotrices of dimension 2 (R_2) in R_n.*

(b) *there exists*

$$m = \sum_{i=0}^{j} i,$$

$j = \dfrac{n-2}{2}$, number of 2-dimensional matrices (M_2) in R_n.

Proof. **(a)** We prove using the principle of mathematical induction. From Definition 2.4, ρ is the index corresponding to the number of R_2 (minors) that can be obtained from $R_n, \forall\, n \in 2\mathbb{N}$. Then:

$$\rho = \frac{n}{2} \Rightarrow 2\rho = n, \ \forall\, n \in 2\mathbb{N}.$$

Now, suppose $\rho = 1$, then $n = 2$ which is in $2\mathbb{N}$. This gives a rhotrix of dimension 2. So, it is true for $\rho = 1$.

When $\rho = 2$, then $n = 4 \in 2\mathbb{N}$. Then we have two minors of R_2 in R_4. It is true for $\rho = 2$. Similarly, when $\rho = 3$, then $n = 6 \in 2\mathbb{N}$. Then we have three minors of R_2 in R_6.

Suppose $\rho = k$, then $n = 2k \in 2\mathbb{N}$. Then we have k minors of R_2 in R_{2k}. It is true also for $\rho = k$.

Let $\rho = k+1$, then $n = 2(k+1) \in 2\mathbb{N}$. Then we have $k+1$ minors of R_2 in $R_{2(k+1)}$. It is also true for $\rho = k+1$.

Thus, $\rho = \frac{n}{2}$ holds for all values of $n \in 2\mathbb{N}$.

(b) Consider

$$m = \sum_{i=0}^{j} i, \quad j = \frac{n-2}{2}.$$

If $j = 0$, then $m = 0 \ \Rightarrow \ n = 2 \in 2\mathbb{N}$ (from $j = \dfrac{n-2}{2}$). Rightfully, there is no minor of M_2 in R_2.

Suppose $j = 1$, then $m = 1 \ \Rightarrow \ n = 4 \in 2\mathbb{N}$. That is, there is only one M_2 in R_4.

When $j = 2$, then $m = 3 \ \Rightarrow \ n = 6 \in 2\mathbb{N}$, meaning there are three M_2 in R_6.

Now, suppose that $j = k$, then $m = 0 + 1 + \cdots + k = \sum_{i=0}^{k} i \ \Rightarrow \ n = 2(k+1) \in 2\mathbb{N}$. That is, there are $\sum_{i=0}^{k} i$ number of M_2 in $R_{2(k+1)}$.

Finally, suppose that $j = k+1$, then $m = 0 + 1 + \cdots + 2k + 1 = \sum_{i=0}^{k+1} i \ \Rightarrow \ n = 2(k+2) \in 2\mathbb{N}$. That is, there are $\sum_{i=0}^{k+1} i$ number of M_2 in $R_{2(k+2)}$.

Thus, $m = \sum_{i=0}^{j} i, j = \dfrac{n-2}{2}$ is true for all values of j and $n \in 2\mathbb{N}$. □

Remark 3.2. *The proof of Theorem 3.1 above can be visualized from the illustrated examples in Subsection* 3.1 (i)–(v), *for all values of $n \in 2\mathbb{N}$.*

Corollary 3.1. *Let R_n be an hl-rhotrix of dimension n. Then, the index ρ of an hl-rhotrix R_n is given as*

$$\rho = j+1, \ j = \frac{n-2}{2}.$$

Proof. The proof follows from Theorem 3.1. □

Remark 3.3. *Corollary* 3.1 *above shows the relationship between* j *and* ρ*, and that there is a one-to-one correspondence between the two functions.*

Generally, the properties of these hl-rhotrices are summarized in Table 1 below:

Table 1. Rediscovered properties of hl-rhotrices

$\boldsymbol{R_n}$	$\boldsymbol{R_2}$	$\boldsymbol{M_2}$
2	1	0
4	2	1
6	3	3
8	4	6
10	5	10
$\vdots$	$\vdots$	$\vdots$
n	$\frac{n}{2}$	$\sum_{i=0}^{j} i, \quad j = \frac{n-2}{2}$

3.2 Identity of hl-rhotrices under RMM operation

Consider an hl-rhotrix R_n of dimension n, if *I* is also an hl-rhotrix of dimension n such that: $R_n \circ I_n = R_n = I_n \circ R_n$. Then, *I* is an identity element. The procedure is as follows:
(i) We look for b_{11}, b_{21}, d_{11}, b_{12}, b_{31}, d_{21}, d_{12}, b_{13}, b_{32}, d_{22}, b_{23}, b_{33}, such that:

$$R_4 \circ S_4 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & & b_{11} & & \\ & b_{21} & d_{11} & b_{12} & \\ b_{31} & d_{21} & & d_{12} & b_{13} \\ & b_{32} & d_{22} & b_{23} & \\ & & b_{33} & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle.$$

Using the multiplication result for $R_4 \circ I_4$, we obtain:

$$\begin{array}{llll} a_{11}b_{11} + a_{13}b_{31} = a_{11}; & a_{13}b_{33} + a_{11}b_{13} = a_{13}; & a_{31}b_{11} + a_{33}b_{31} = a_{31}; & a_{33}b_{33} + a_{31}b_{13} = a_{33}; \\ c_{11}d_{11} + c_{12}d_{21} = c_{11}; & c_{21}d_{11} + c_{22}d_{21} = c_{21}; & c_{22}d_{22} + c_{21}d_{12} = c_{22}; & c_{12}d_{22} + c_{11}d_{12} = c_{12}; \\ a_{21}b_{12} + a_{12}b_{32} = a_{21}; & a_{12}b_{23} + a_{21}b_{12} = a_{12}. & a_{32}b_{21} + a_{23}b_{32} = a_{32}; & a_{23}b_{23} + a_{32}b_{12} = a_{23}; \end{array}$$

and obtain the multiplication of the system as:
$b_{11} = b_{33} = 1$ and $b_{31} = b_{13} = 0$;
$b_{21} = b_{23} = 1$ and $b_{32} = b_{12} = 0$;
$d_{11} = d_{22} = 1$ and $d_{21} = d_{12} = 0$.

Note that $R_4 \circ I_4 = R_4 = I_4 \circ R_4$. Thus, the identity rhotrix in this case is

$$I_4 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & & 0 & 0 \\ & 0 & 1 & 1 & \\ & & 1 & & \end{array} \right\rangle.$$

(ii) Following the same analysis as above, gives the identity of R_6 as

$$I_6 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 1 & 0 & & \\ & 1 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & \\ & & 0 & 1 & 1 & & \\ & & & 1 & & & \end{array} \right\rangle.$$

(iii) Similarly, the identity rhotrix of R_8 is

$$I_8 = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 0 & & & \\ & & 1 & 1 & 1 & 0 & 0 & & \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \\ & & 0 & 0 & 1 & 1 & 1 & & \\ & & & 0 & 1 & 1 & & & \\ & & & & 1 & & & & \end{array} \right\rangle.$$

(iv) Then, identity rhotrix of R_{10} is

$$I_{10} = \left\langle \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & 1 & 0 & & & & \\ & & & 1 & 1 & 1 & 0 & 0 & & & \\ & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & & \\ & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \\ & & 0 & 0 & 0 & 1 & 1 & 1 & 1 & & \\ & & & 0 & 0 & 1 & 1 & 1 & & & \\ & & & & 0 & 1 & 1 & & & & \\ & & & & & 1 & & & & & \end{array} \right\rangle.$$

(v) The same analysis gives the identity rhotrix of R_{12} as

$$I_{12} = \left\langle \begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & 0 & & & & & \\
 & & & & 1 & 1 & 1 & 0 & 0 & & & & \\
 & & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & & \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 1 & 1 & 1 & & & \\
 & & & & 0 & 0 & 1 & 1 & 1 & & & & \\
 & & & & & 0 & 1 & 1 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array} \right\rangle .$$

Remark 3.4. *RMM brings out the rich symmetric properties inherent in even-dimensional rhotrices. In another paper, the symmetry in even-dimensional rhotrices and application in organic chemistry is being investigated. For example, the structures of Buthane, Cyclobutane and Cyclooctane correspond to the structures of R_2, R_4 and R_6, respectively.*

3.3 Inverse of hl-rhotrices under RMM

The concept of a unique identity hl-rhotrix under RMM guarantees the existence of an inverse hl-rhotrix. If for an hl-rhotrix R_n, we can find another hl-rhotrix S_n, under the multiplication operation, such that $R_n \circ S_n = S_n \circ R_n = I_n$, then S_n is the inverse of R_n. For a particular n, we consider the following:

(i) Let S_4 be the inverse of R_4. That is, given

$$S_4 = \left\langle \begin{array}{ccccc}
 & & b_{11} & & \\
 & b_{21} & d_{11} & b_{12} & \\
b_{31} & d_{21} & & d_{12} & b_{13} \\
 & b_{32} & d_{22} & b_{23} & \\
 & & b_{33} & &
\end{array} \right\rangle \text{ and } R_4 = \left\langle \begin{array}{ccccc}
 & & a_{11} & & \\
 & a_{21} & c_{11} & a_{12} & \\
a_{31} & c_{21} & & c_{12} & a_{13} \\
 & a_{32} & c_{22} & a_{23} & \\
 & & a_{33} & &
\end{array} \right\rangle ,$$

we must have $R_4 \circ S_4 = S_4 \circ R_4 = I_4$. Thus,

$$\left\langle \begin{array}{ccccc}
 & & a_{11} & & \\
 & a_{21} & c_{11} & a_{12} & \\
a_{31} & c_{21} & & c_{12} & a_{13} \\
 & a_{32} & c_{22} & a_{23} & \\
 & & a_{33} & &
\end{array} \right\rangle \circ \left\langle \begin{array}{ccccc}
 & & b_{11} & & \\
 & b_{21} & d_{11} & b_{12} & \\
b_{31} & d_{21} & & d_{12} & b_{13} \\
 & b_{32} & d_{22} & b_{23} & \\
 & & b_{33} & &
\end{array} \right\rangle = \left\langle \begin{array}{ccccc}
 & & 1 & & \\
 & 1 & 1 & 0 & \\
0 & 0 & & 0 & 0 \\
 & 0 & 1 & 1 & \\
 & & 1 & &
\end{array} \right\rangle .$$

Therefore, the inverse R_4^{-1} is

$$\left\langle \begin{array}{ccccc} & & \frac{a_{33}}{A} & & \\ & \frac{a_{23}}{B} & \frac{c_{22}}{C} & \frac{-a_{12}}{B} & \\ \frac{-a_{31}}{A} & \frac{-c_{21}}{C} & & \frac{-c_{12}}{C} & \frac{-a_{13}}{A} \\ & \frac{-a_{32}}{B} & \frac{c_{11}}{C} & \frac{a_{21}}{B} & \\ & & \frac{a_{11}}{A} & & \end{array} \right\rangle,$$

where $A = a_{11}a_{33} - a_{31}a_{13}$, $B = a_{32}a_{12} - a_{21}a_{23}$ and $C = c_{11}c_{22} - c_{21}c_{12}$ for a system in R_4.

(ii) Similarly, the inverse R_6^{-1} is

$$\left\langle \begin{array}{ccccccc} & & & \frac{a_{44}}{A} & & & \\ & & \frac{a_{34}}{B} & \frac{c_{33}}{C} & \frac{-a_{12}}{B} & & \\ & \frac{a_{24}}{E} & \frac{c_{23}}{F} & \frac{a_{33}}{D} & \frac{-c_{12}}{F} & \frac{-a_{13}}{E} & \\ \frac{-a_{41}}{A} & \frac{-c_{31}}{C} & \frac{-a_{32}}{D} & & \frac{-a_{23}}{D} & \frac{-c_{13}}{C} & \frac{-a_{14}}{A} \\ & \frac{-a_{42}}{E} & \frac{-c_{32}}{F} & \frac{a_{22}}{D} & \frac{c_{21}}{F} & \frac{a_{31}}{E} & \\ & & \frac{-a_{43}}{B} & \frac{c_{11}}{C} & \frac{a_{21}}{B} & & \\ & & & \frac{a_{11}}{A} & & & \end{array} \right\rangle,$$

where

$A = a_{11}a_{44} - a_{41}a_{14}$; $B = a_{21}a_{34} - a_{43}a_{12}$; $C = c_{11}c_{33} - c_{31}c_{13}$; $D = a_{22}a_{33} - a_{32}a_{23}$; $E = a_{31}a_{24} - a_{42}a_{13}$; $F = c_{21}c_{23} - c_{32}c_{12}$.

for systems in R_6.

(iii) The inverse R_8^{-1} is

$$\left\langle \begin{array}{ccccccccc} & & & & \frac{a_{55}}{A} & & & & \\ & & & \frac{a_{45}}{B} & \frac{c_{44}}{C} & \frac{-a_{12}}{B} & & & \\ & & \frac{a_{35}}{D} & \frac{c_{34}}{E} & \frac{a_{44}}{F} & \frac{-c_{12}}{E} & \frac{-a_{13}}{D} & & \\ & \frac{a_{25}}{G} & \frac{c_{24}}{H} & \frac{a_{34}}{I} & \frac{c_{33}}{J} & \frac{-a_{23}}{I} & \frac{-c_{13}}{H} & \frac{-a_{14}}{G} & \\ \frac{-a_{51}}{A} & \frac{-c_{41}}{C} & \frac{-a_{42}}{F} & \frac{-c_{32}}{J} & & \frac{-c_{23}}{J} & \frac{-a_{24}}{F} & \frac{-c_{14}}{C} & \frac{-a_{15}}{A} \\ & \frac{-a_{52}}{G} & \frac{-c_{42}}{H} & \frac{-a_{43}}{I} & \frac{c_{22}}{J} & \frac{a_{32}}{I} & \frac{c_{31}}{H} & \frac{a_{41}}{G} & \\ & & \frac{-a_{53}}{D} & \frac{-c_{43}}{E} & \frac{a_{22}}{F} & \frac{c_{21}}{E} & \frac{a_{31}}{D} & & \\ & & & \frac{-a_{54}}{B} & \frac{c_{11}}{C} & \frac{a_{21}}{B} & & & \\ & & & & \frac{a_{11}}{A} & & & & \end{array} \right\rangle,$$

where

$A = a_{11}a_{55} - a_{51}a_{15}$; $B = a_{21}a_{45} - a_{54}a_{12}$; $C = c_{11}c_{44} - c_{41}c_{14}$; $D = a_{31}a_{35} - a_{53}a_{13}$; $E = c_{21}c_{34} - c_{43}c_{12}$; $F = a_{22}a_{44} - a_{42}a_{24}$; $G = a_{41}a_{25} - a_{52}a_{14}$; $H = c_{31}c_{24} - c_{42}c_{13}$; $I = a_{32}a_{34} - a_{43}a_{23}$; $J = c_{22}c_{33} - c_{32}c_{23}$.

for systems in R_8.

(iv) The inverse R_{10}^{-1} is

$$\left\langle \begin{array}{ccccccccccc}
 & & & & & \frac{a_{66}}{A} & & & & & \\
 & & & & \frac{a_{56}}{B} & \frac{c_{55}}{C} & \frac{-a_{12}}{B} & & & & \\
 & & & \frac{a_{46}}{D} & \frac{c_{45}}{E} & \frac{a_{55}}{F} & \frac{-c_{12}}{E} & \frac{-a_{13}}{D} & & & \\
 & & \frac{a_{36}}{G} & \frac{c_{35}}{H} & \frac{a_{45}}{I} & \frac{c_{44}}{J} & \frac{-a_{23}}{I} & \frac{-c_{13}}{H} & \frac{-a_{14}}{G} & & \\
 & \frac{a_{26}}{K} & \frac{c_{25}}{F} & \frac{a_{35}}{M} & \frac{c_{34}}{N} & \frac{a_{44}}{O} & \frac{-c_{23}}{N} & \frac{-a_{24}}{M} & \frac{-c_{14}}{L} & \frac{-a_{15}}{K} & \\
\frac{-a_{61}}{A} & \frac{-c_{51}}{C} & \frac{-a_{52}}{L} & \frac{-c_{42}}{J} & \frac{-a_{43}}{O} & & \frac{-a_{34}}{O} & \frac{-c_{24}}{J} & \frac{-a_{25}}{F} & \frac{-c_{15}}{C} & \frac{-a_{16}}{A} \\
 & \frac{-a_{62}}{K} & \frac{-c_{52}}{L} & \frac{-a_{53}}{M} & \frac{-c_{43}}{N} & \frac{a_{33}}{O} & \frac{c_{32}}{N} & \frac{a_{42}}{M} & \frac{c_{41}}{L} & \frac{a_{51}}{K} & \\
 & & \frac{-a_{63}}{G} & \frac{-c_{53}}{H} & \frac{-a_{54}}{E} & \frac{c_{22}}{J} & \frac{a_{32}}{I} & \frac{c_{31}}{H} & \frac{a_{41}}{G} & & \\
 & & & \frac{-a_{64}}{D} & \frac{-c_{54}}{E} & \frac{a_{22}}{F} & \frac{c_{21}}{E} & \frac{a_{31}}{D} & & & \\
 & & & & \frac{-a_{65}}{B} & \frac{c_{11}}{C} & \frac{a_{21}}{B} & & & & \\
 & & & & & \frac{a_{11}}{A} & & & & &
\end{array} \right\rangle,$$

where

$A = a_{11}a_{66} - a_{61}a_{16}$; $B = a_{21}a_{56} - a_{65}a_{12}$; $C = c_{11}c_{55} - c_{51}c_{15}$; $D = a_{31}a_{46} - a_{64}a_{13}$;
$E = c_{21}c_{45} - c_{54}c_{12}$; $F = a_{22}a_{55} - a_{52}a_{25}$; $G = a_{41}a_{36} - a_{63}a_{14}$; $H = c_{31}c_{35} - c_{53}c_{13}$;
$I = a_{32}a_{45} - a_{54}a_{23}$; $J = c_{22}c_{44} - c_{42}c_{24}$; $K = a_{51}a_{26} - a_{62}a_{15}$; $L = c_{41}c_{25} - c_{52}c_{14}$;
$M = a_{42}a_{35} - a_{53}a_{24}$; $N = c_{32}c_{34} - c_{43}c_{23}$; $O = a_{33}a_{44} - a_{43}a_{34}$.

for the system in R_{10}.

(v) Finally, the inverse R_{12}^{-1} is

$$\left\langle \begin{array}{ccccccccccccc}
 & & & & & & \frac{a_{77}}{A} & & & & & & \\
 & & & & & \frac{a_{67}}{B} & \frac{c_{66}}{C} & \frac{-a_{12}}{B} & & & & & \\
 & & & & \frac{a_{57}}{D} & \frac{c_{56}}{E} & \frac{a_{66}}{F} & \frac{-c_{12}}{E} & \frac{-a_{13}}{D} & & & & \\
 & & & \frac{a_{47}}{G} & \frac{c_{46}}{H} & \frac{a_{56}}{I} & \frac{c_{55}}{J} & \frac{-a_{23}}{I} & \frac{-c_{13}}{H} & \frac{-a_{14}}{G} & & & \\
 & & \frac{a_{37}}{K} & \frac{c_{36}}{L} & \frac{a_{46}}{M} & \frac{c_{45}}{N} & \frac{a_{55}}{O} & \frac{-c_{23}}{N} & \frac{-a_{24}}{S} & \frac{-c_{14}}{L} & \frac{-a_{15}}{K} & & \\
 & \frac{a_{27}}{P} & \frac{c_{26}}{Q} & \frac{a_{36}}{R} & \frac{c_{35}}{S} & \frac{a_{45}}{T} & \frac{c_{44}}{U} & \frac{-a_{34}}{T} & \frac{-c_{24}}{S} & \frac{-a_{25}}{R} & \frac{-c_{15}}{Q} & \frac{-a_{16}}{P} & \\
\frac{-a_{71}}{A} & \frac{-c_{61}}{C} & \frac{-a_{62}}{F} & \frac{-c_{52}}{J} & \frac{-a_{53}}{O} & \frac{-c_{43}}{U} & & \frac{-c_{34}}{U} & \frac{-a_{35}}{O} & \frac{-c_{25}}{J} & \frac{-a_{26}}{F} & \frac{-c_{16}}{C} & \frac{-a_{17}}{A} \\
 & \frac{-a_{72}}{P} & \frac{-c_{62}}{Q} & \frac{-a_{63}}{R} & \frac{-c_{53}}{S} & \frac{-a_{54}}{T} & \frac{c_{33}}{U} & \frac{a_{43}}{T} & \frac{c_{42}}{S} & \frac{a_{52}}{R} & \frac{c_{51}}{Q} & \frac{a_{61}}{P} & \\
 & & \frac{-a_{73}}{K} & \frac{-c_{63}}{L} & \frac{-a_{64}}{M} & \frac{-c_{54}}{N} & \frac{a_{33}}{O} & \frac{c_{32}}{N} & \frac{a_{42}}{M} & \frac{c_{41}}{L} & \frac{a_{51}}{K} & & \\
 & & & \frac{-a_{74}}{G} & \frac{-c_{64}}{H} & \frac{-a_{65}}{I} & \frac{c_{22}}{J} & \frac{a_{32}}{I} & \frac{c_{31}}{H} & \frac{a_{41}}{G} & & & \\
 & & & & \frac{-a_{75}}{D} & \frac{-c_{65}}{E} & \frac{a_{22}}{F} & \frac{c_{21}}{E} & \frac{a_{31}}{D} & & & & \\
 & & & & & \frac{-a_{76}}{B} & \frac{c_{11}}{C} & \frac{a_{21}}{B} & & & & & \\
 & & & & & & \frac{a_{11}}{A} & & & & & &
\end{array} \right\rangle,$$

where

$A = a_{11}a_{77} - a_{71}a_{17}$; $B = a_{21}a_{67} - a_{76}a_{12}$; $C = c_{11}c_{66} - c_{61}c_{16}$; $D = a_{31}a_{57} - a_{75}a_{13}$;
$E = c_{21}c_{56} - c_{65}c_{12}$; $F = a_{22}a_{66} - a_{62}a_{26}$; $G = a_{41}a_{47} - a_{74}a_{14}$; $H = c_{31}c_{46} - c_{64}c_{13}$;
$I = a_{32}a_{56} - a_{65}a_{23}$; $J = c_{22}c_{55} - c_{52}c_{25}$; $K = a_{51}a_{37} - a_{73}a_{15}$; $L = c_{41}c_{36} - c_{63}c_{14}$;
$M = a_{42}a_{46} - a_{64}a_{24}$; $N = c_{32}c_{45} - c_{54}c_{23}$; $O = a_{33}a_{55} - a_{53}a_{35}$; $P = a_{61}a_{27} - a_{72}a_{16}$;
$Q = c_{51}c_{26} - c_{62}c_{15}$; $R = a_{52}a_{36} - a_{63}a_{25}$; $S = c_{42}c_{35} - c_{24}c_{53}$; $T = a_{43}a_{45} - a_{54}a_{34}$;
$U = c_{33}c_{44} - c_{43}c_{34}$.

for the system in R_{12}.

4 Numerical examples

Example 1. Consider some numerical examples of R_4 and R_6, under RMM operation, and their inverses.

(i) Let $A_4 = \left\langle \begin{array}{ccccc} & & -2 & & \\ & 3 & 4 & 5 & \\ 7 & 8 & & -9 & 10 \\ & 1 & 6 & 8 & \\ & & 3 & & \end{array} \right\rangle$ and $B_4 = \left\langle \begin{array}{ccccc} & & 3 & & \\ & -5 & 5 & 2 & \\ 8 & 7 & & 9 & 10 \\ & 6 & -4 & -5 & \\ & & 4 & & \end{array} \right\rangle$

The multiplication of the minors of R_2 and M_2 in R_4 are

$$\left\langle \begin{array}{ccc} & -2 & \\ 7 & & 10 \\ & 3 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 3 & \\ 8 & & 10 \\ & 4 & \end{array} \right\rangle =$$

$$= \left\langle \begin{array}{ccc} & -2 \times 3 + 10 \times 8 & \\ 7 \times 3 + 3 \times 8 & & -2 \times 10 + 10 \times 4 \\ & 7 \times 10 + 3 \times 4 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 74 & \\ 45 & & 20 \\ & 82 & \end{array} \right\rangle,$$

$$\left\langle \begin{array}{ccc} & 4 & \\ 8 & & -9 \\ & 6 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 5 & \\ 7 & & 9 \\ & -4 & \end{array} \right\rangle =$$

$$= \left\langle \begin{array}{ccc} & 4 \times 5 + -9 \times 7 & \\ 8 \times 5 + 6 \times 7 & & 4 \times 9 + -9 \times -4 \\ & 8 \times 9 + 6 \times -4 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & -43 & \\ 82 & & 72 \\ & 48 & \end{array} \right\rangle,$$

and

$$\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix} \circ \begin{bmatrix} -5 & 2 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} 3 \times -5 + 5 \times 6 & 3 \times 2 + 5 \times -5 \\ 1 \times -5 + 8 \times 6 & 1 \times 2 + 8 \times -5 \end{bmatrix} = \begin{bmatrix} 15 & -19 \\ 43 & -38 \end{bmatrix}.$$

Thus,

$$A_4 \circ B_4 = \left\langle \begin{array}{ccccc} & & 74 & & \\ & 15 & -43 & -19 & \\ 45 & 82 & & 72 & 20 \\ & 43 & 48 & -38 & \\ & & 82 & & \end{array} \right\rangle$$

(ii) To solve for A_4^{-1}, we follow the inverse operation of R_4 as presented in 3.3 (i), and obtain:

$$A_4^{-1} = \left\langle \begin{array}{ccccc} & & -\frac{3}{76} & & \\ & \frac{8}{19} & \frac{1}{16} & -\frac{5}{19} & \\ \frac{7}{76} & \frac{1}{12} & & -\frac{3}{32} & \frac{5}{38} \\ & -\frac{1}{19} & \frac{1}{24} & \frac{3}{19} & \\ & & \frac{1}{38} & & \end{array} \right\rangle.$$

Example 2. Some text here?

(i) Let

$$A_6 = \left\langle \begin{array}{ccccccc} & & & 2 & & & \\ & & 0 & 1 & 3 & & \\ & 4 & -1 & -1 & 4 & -2 & \\ 1 & 2 & -1 & & 2 & 1 & 1 \\ & 2 & -4 & 1 & 1 & 2 & \\ & & 2 & 0 & -1 & & \\ & & & 3 & & & \end{array} \right\rangle \text{ and } B_6 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & 1 & 1 & & \\ & 3 & -1 & 1 & 4 & -2 & \\ 1 & 1 & 1 & & 2 & -1 & -1 \\ & -2 & 4 & 0 & 0 & 1 & \\ & & 1 & 0 & -1 & & \\ & & & 1 & & & \end{array} \right\rangle$$

Then,

$$A_6 \circ B_6 = \left\langle \begin{array}{ccccccc} & & & 3 & & & \\ & & 3 & 2 & -3 & & \\ & 16 & 17 & 1 & -4 & -10 & \\ 4 & 2 & 0 & & -2 & -1 & -1 \\ & 2 & 8 & -2 & -16 & -2 & \\ & & 3 & -2 & 3 & & \\ & & & 2 & & & \end{array} \right\rangle$$

(ii) To solve for A_6^{-1}, we follow the inverse operation of R_6 in Subsection 3.3 (ii), and obtain,

$$A_6^{-1} = \left\langle \begin{array}{ccccccc} & & & \frac{3}{5} & & & \\ & & \frac{1}{6} & 0 & \frac{1}{2} & & \\ & \frac{1}{6} & \frac{1}{15} & 1 & -\frac{4}{15} & \frac{1}{6} & \\ -\frac{1}{5} & 1 & 1 & & -2 & \frac{1}{2} & -\frac{1}{5} \\ & -\frac{1}{6} & \frac{4}{15} & -1 & -\frac{1}{15} & \frac{1}{3} & \\ & & \frac{1}{3} & -\frac{1}{2} & 0 & & \\ & & & \frac{2}{5} & & & \end{array} \right\rangle$$

(iii) To obtain the identity element, we have $A_6 \circ A_6^{-1} = I_6$

$$A_6 = \left\langle \begin{array}{ccccccc} & & & 2 & & & \\ & & 0 & 1 & 3 & & \\ & 4 & -1 & -1 & 4 & -2 & \\ 1 & 2 & -1 & & 2 & 1 & 1 \\ & 2 & -4 & 1 & 1 & 2 & \\ & & 2 & 0 & -1 & & \\ & & & 3 & & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccccc} & & & \frac{3}{5} & & & \\ & & \frac{1}{6} & 0 & \frac{1}{2} & & \\ & \frac{1}{6} & \frac{1}{15} & 1 & -\frac{4}{15} & \frac{1}{6} & \\ -\frac{1}{5} & 1 & 1 & & -2 & \frac{1}{2} & -\frac{1}{5} \\ & -\frac{1}{6} & \frac{4}{15} & -1 & -\frac{1}{15} & \frac{1}{3} & \\ & & \frac{1}{3} & -\frac{1}{2} & 0 & & \\ & & & \frac{2}{5} & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 1 & 0 & & \\ & 1 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & \\ & & 0 & 1 & 1 & & \\ & & & 1 & & & \end{array} \right\rangle$$

Example 3. Simple equation under RMM operation.

Find X_4 if $X_4 A_4 = B_4$, where

$$A_4 = \left\langle \begin{array}{ccccc} & & -2 & & \\ & 3 & 4 & 5 & \\ 7 & 8 & & -9 & 10 \\ & 1 & 6 & 8 & \\ & & 3 & & \end{array} \right\rangle \text{ and } B_4 = \left\langle \begin{array}{ccccc} & & 3 & & \\ & -5 & 5 & 2 & \\ 8 & 7 & & 9 & 10 \\ & 6 & -4 & -5 & \\ & & 4 & & \end{array} \right\rangle,$$

$$X_4 = A_4^{-1} B_4.$$

Then,

$$A_4^{-1} = \frac{1}{-153} \left\langle \begin{array}{ccccc} & & 3 & & \\ & 8 & 6 & -5 & \\ -7 & -8 & & -9 & -10 \\ & -1 & 4 & 3 & \\ & & -2 & & \end{array} \right\rangle.$$

Therefore, A_4^{-1} multiplies B_4 gives

$$X_4 = \frac{1}{-153} \left\langle \begin{array}{ccccc} & & 3 & & \\ & 8 & 6 & -5 & \\ -7 & -8 & & -9 & -10 \\ & -1 & 4 & 3 & \\ & & -2 & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & & 3 & & \\ & -5 & 5 & 2 & \\ 8 & 7 & & 9 & 10 \\ & 6 & -4 & -5 & \\ & & 4 & & \end{array} \right\rangle,$$

$$X_4 = \frac{1}{153} \left\langle \begin{array}{ccccc} & & 71 & & \\ & 70 & 33 & -41 & \\ 37 & 12 & & -78 & 10 \\ & -13 & 72 & 17 & \\ & & 78 & & \end{array} \right\rangle.$$

5 Conclusion

This research work developed and examined a new multiplication method called the robust multiplication method (RMM) for higher even-dimensional rhotrices and presented a number of its properties. It also examined the concepts of minor rhotrices and their application in hl-rhotrices. The recent articles [2, 3] on tertions and other algebraic objects are an eye-opener to more properties of even-dimensional rhotrices. All even-dimensional rhotrices have an even cardinality. That is, $|R_n| = \frac{1}{2}(n^2 + 2n)$ for all $n \in 2\mathbb{N}$, for example, a rhotrix of dimension 2 has a cardinality of 4, and this corresponds to the cardinality of the quaternions A- and V-tertions (AV-tertions). However, note that these objects are different from the popular structure of the quaternion group (Q_t) of cardinality 8. The foregoing naturally prompts questions like: *Do all even-dimensional AV-tertions correspond to even-dimensional rhotrices? Do all odd-dimensional*

AV-tertions correspond to heart-based (odd-dimensional) rhotrices? If the answers to these two questions are in the affirmative, then AV-tertions are a generalization of both even- and odd-dimensional rhotrices. At any rate, the quaternion AV-tertions is a meeting point between rhotrices and tertions. Therefore, this is a call for representations of high-dimensional AV-tertions. We project that the Robust Multiplication Method presented in this paper portends a wider application of rhotrices in agriculture, exploration, theoretical chemistry and physics due to its beautiful symmetries, and would be an alternative multiplication method for higher even-dimensional AV-tertions.

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