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Two Fibonacci-like sequences

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Abstract: In the present paper we will discuss the two Fibonacci-like sequences

$$s_{2k+1} = s_{2k} - s_{2k-1} + \cdots - s_1 + s_0$$

with $s_0 = a_0, \ldots, s_{2k} = a_{2k}$ and

$$s_{2k+2} = s_{2k+1} - s_{2k} + \cdots + s_1 - s_0$$

with $s_0 = a_0, \ldots, s_{2k} = a_{2k+1}$, where $a_0, \ldots, a_{2k+1}$ are arbitrary numbers. Examples and explicit formulas for s_{2k+1} and s_{2k+2} are given.

Keywords: Arbitrary order recurrence relation, Combined sequence, Fibonacci sequence, Intersection.
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1 Introduction

This paper develops some related pairs of arbitrary order homogeneous linear recurrence relations. One sequence has order $2k + 1$ and its related companion sequence has order $2k + 2$, where k is an arbitrary integer. Three examples with small k are provided to illustrate what is happening in detail in particular cases. While studies of arbitrary order linear recurrence relations are not new [4, 5], these have generally been different extensions of Fibonacci sequences [8], including intersections [8,9]. The novelty of these paired sequences is further illustrated in two propositions and two theorems which develop elegant and counter-intuitive results.

2 Main results

Let us start our exposition with two examples.

For the first example, let the numbers a, b, c be given and let $s_0 = a, s_1 = b, s_2 = c$. Let for $k \geq 3$,

$$s_k = s_{k-1} - s_{k-2} + s_{k-3}.$$

Then we obtain the following values for the members $s_3, s_4, \ldots, s_7$:

$$\begin{aligned}
s_3 &= c - b + a, \\
s_4 &= c - b + a - c + b = a, \\
s_5 &= a - c + b - a + c = b, \\
s_6 &= b - a + c - b + a = c, \\
s_7 &= c - b + a.
\end{aligned}$$

For the second example, let the numbers a, b, c, d be given and let $s_0 = a, s_1 = b, s_2 = c, s_3 = d$. Let for $k \geq 4$,

$$s_k = s_{k-1} - s_{k-2} + s_{k-3} - s_{k-4}.$$

Then we obtain the following values for the members $s_4, s_5, \ldots, s_{13}$:

$$\begin{aligned}
s_4 &= d - c + b - a, \\
s_5 &= d - c + b - a - d + c - b = -a, \\
s_6 &= -a - d + c - b + a + d - c = -b, \\
s_7 &= -b + a + d - c + b - a - d = -c, \\
s_8 &= -c + b - a - d + c - b + a = -d, \\
s_9 &= -d + c - b + a, \\
s_{10} &= -d + c - b + a + d - c + b = a, \\
s_{11} &= a + d - c + b - a - d + c = b, \\
s_{12} &= b - a - d + c - b + a + d = c, \\
s_{13} &= c - b + a + d - c + b - a = d, \\
s_{14} &= d - c + b - a,
\end{aligned}$$

Now, we can formulate and prove the following assertions. The first one of them is related to the first example.

Proposition 1. *For any natural number k:*

$$s_{4k} = a, \qquad s_{4k+1} = b, \qquad s_{4k+2} = c, \qquad s_{4k+3} = c - b + a.$$

Proof. For $k = 0$, the assertion is valid, as we see from the first example. Let us assume that it is valid for some $k \geq 0$. Then we obtain sequentially:

$$\begin{aligned} s_{4k+4} &= s_{4k+3} - s_{4k+2} + s_{4k+1} = c - b + a - c + b = a, \\ s_{4k+5} &= s_{4k+4} - s_{4k+3} + s_{4k+2} = a - c + b - a + c = b, \\ s_{4k+6} &= s_{4k+5} - s_{4k+4} + s_{4k+3} = b - a + c - b + a = c, \\ s_{4k+7} &= s_{4k+6} - s_{4k+5} + s_{4k+4} = c - b + a. \end{aligned}$$

In the same manner we can prove the next assertions. □

A more general assertion than Proposition 1 is the following one.

Theorem 1. *For any even number* $m \geq 0$*, for any numbers* $a_0, \dots, a_m$ *and for any natural number* $k \geq 0$*, if* $s_0 = a_0, \dots, s_m = a_m$ *and for any natural number* $n \geq 0$*,*

$$s_{n+m+1} = s_{n+m} - s_{n+m-1} + \cdots + s_n,$$

it follows that

$$\begin{aligned} s_{(m+2)k} &= a_0, \\ s_{(m+2)k+1} &= a_1, \\ &\vdots \\ s_{(m+2)k+m+1} &= a_m - a_{m-1} + \cdots + a_0. \end{aligned}$$

The next assertion is related to the second example.

Proposition 2. *For any natural number* k*,*

$$\begin{array}{lllll} s_{10k} = a, & s_{10k+1} = b, & s_{10k+2} = c, & s_{10k+3} = d, & s_{10k+4} = d - c + b - a, \\ s_{10k+5} = -a, & s_{10k+6} = -b, & s_{10k+7} = -c, & s_{10k+8} = -d, & s_{10k+9} = -d + c - b + a. \end{array}$$

A more general assertion than Proposition 2 is the following one.

Theorem 2. *For any odd number* $m \geq 0$*, for any numbers* $a_0, \dots, a_m$*, for any natural number* $k \geq 0$*, if* $s_0 = a_0, \dots, s_m = a_m$ *and for any natural number* $n \geq 0$*,*

$$s_{n+m+1} = s_{n+m} - s_{n+m-1} + \cdots + s_n,$$

it follows that

$$\begin{aligned} s_{2(m+2)k} &= a_0, \\ s_{2(m+2)k+1} &= a_1, \\ &\vdots \\ s_{2(m+2)k+m} &= a_m, \\ s_{2(m+2)k+m+1} &= a_m - a_{m-1} + \cdots - a_0, \end{aligned}$$

$$
\begin{aligned}
s_{2(m+2)k+m+2} &= -a_0, \\
s_{2(m+2)k+m+3} &= -a_1, \\
&\vdots \\
s_{2(m+2)k+2m+2} &= -a_m, \\
s_{2(m+2)k+2m+3} &= -a_m + a_{m-1} - \cdots + a_0.
\end{aligned}
$$

We can see that when $m = 1$ we obtain:

$$
\begin{aligned}
s_0 &= a_0, \\
s_1 &= a_1, \\
s_{n+2} &= s_{n+1} - s_n
\end{aligned}
$$

and

$$
\begin{aligned}
s_{6k} &= a_0, \\
s_{6k+1} &= a_1, \\
s_{6k+2} &= a_1 - a_0, \\
s_{6k+3} &= -a_0, \\
s_{6k+4} &= -a_1, \\
s_{6k+5} &= -a_1 + a_0.
\end{aligned}
$$

3 Conclusion

The properties of these two novel inter-related sequences $\{s_{2k+1}\}_{k\geq 0}$ and $\{s_{2k+2}\}_{k\geq 0}$ in terms of arbitrary numbers are new ways of looking at old extensions of the Fibonacci sequences. Although this study is, to some extent, in the older tradition of generalizations of the orders of recursive sequences, it also connects with the newer extensions with properties of pairs of sequences to consider properties such as intersections, pulsated sequences [2], combined sequences [1], and intercalated sequences [3]. While the propositions and theorems in this note are relatively simple, they should suggest to interested readers new ways to explore more about their general terms and generating functions and convolution trees [6].

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