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Floor and ceiling functions for Pell numbers

İsmail Sulan [1] **and Mustafa Aşçı** [2]

[1] Department of Mathematics, Faculty of Science, Pamukkale University
20160, Kınıklı, Denizli, Türkiye
e-mail: ismailsulan@gmail.com

[2] Department of Mathematics, Faculty of Science, Pamukkale University
20160, Kınıklı, Denizli, Türkiye
e-mail: masci@pau.edu.tr

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Abstract: The analytical study of the Pell number and the role of floor and ceiling functions into their computation is examined. Closed expressions of Pell numbers were initially derived using Binet's formula, followed by an asymptotic behavior study of the sequence using this formula. Taking into account the decreasing trend in the term $|\beta|^n = |1-\sqrt{2}|^n$ for large values of n, a formula that closely approximates Pell numbers has been developed. In this context, relationships between numbers are clarified using floor and ceiling functions. The accuracy with which various theorems and lemmas mathematically prove these approximations is also included. The study also looks at limit processes with emphasis placed upon the determining influence that the ratio $\alpha = 1+\sqrt{2}$ has on the growth rate of the sequence.
Keywords: Pell numbers, Floor function, Ceiling function, Recurrence relation, Binet's formula.
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1 Introduction

The recursive sequences have provided deep links among mathematical structures. The Pell numbers among these have seen continuous exploration over the ages by many mathematicians, mainly due to their link to Diophantine equations. From a number-theoretical point of view, the Pell numbers are famous for being used in determining integer solutions to equations such as $x^2 - 2y^2 = 1$ [1, 2].

The interaction between floor and ceiling functions with special sequences has caught the idea of researchers in recent years. In this regard, T. Koshy has dealt very well with the work of these functions in the relation of Fibonacci and Lucas numbers with integers [7]. It is, therefore, noted that these studies have greatly influenced the use of similar approaches for Pell numbers and demonstrated a great advantage of these functions in estimating numbers and learning relations among integers. These functions are widely used in number theory, especially in their analysis and prediction over relationships among integers [3, 8].

This paper intends to proceed further into the relationship of Pell numbers with the floor and ceiling function. Numerical closure relationships yield numerical approximations that can significantly enhance the precision of high-interval Pell numbers where n is large. Their solution will open new roads at the intersection of number theory and analytical methods. They show how these methods can be adapted to a much wider arena of social applications.

This work significantly adds to the understanding of recursive structures in number theory, aiming at initially providing new ideas directed toward mathematical modeling and developing algorithms.

2 Preliminaries

In this section, we will include the definitions and theorems that will be used throughout our study.

Definition 2.1. ([5,6]). *The **floor** value of a real number x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The **ceiling** value of x, denoted by $\lceil x \rceil$, is the smallest integer greater than or equal to x. The floor value rounds x down, while the ceiling value rounds it up. If $x \notin \mathbb{Z}$, the floor value of x is the nearest integer on the left of x on the number line, and the ceiling value is the nearest integer on the right. The **floor function** is represented as $f(x) = \lfloor x \rfloor$, and the **ceiling function** as $g(x) = \lceil x \rceil$, known respectively as the **greatest integer function** and the **smallest integer function**.*

Theorem 2.1. ([5]). *Let x be any real number, and let n be any integer. Then:*

1. $\lfloor n \rfloor = n = \lceil n \rceil$;
2. $\lceil x \rceil = \lfloor x \rfloor + 1 \quad (x \notin \mathbb{Z})$;
3. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$;
4. $\lceil x + n \rceil = \lceil x \rceil + n$;
5. $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}, \quad$ *if n is odd*;
6. $\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}, \quad$ *if n is odd.*

Definition 2.2. ([4]). *Pell numbers are defined as a sequence P_n obtained through the following recurrence relation:*

$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \quad (n \geq 2).$$

This definition shows that the first few terms of the sequence are $0, 1, 2, 5, 12, 29, 70, \ldots$.

Pell numbers can be expressed in a closed form using the Binet formula [4]:

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}},$$

where $\alpha = 1 + \sqrt{2},\ \beta = 1 - \sqrt{2}$.

3 Main results

Since $|\beta| < 1$, the term β^n tends to 0 for large values of n. Therefore, for large n, the following approximation can be obtained:

$$P_n \approx \frac{\alpha^n}{2\sqrt{2}}.$$

Now, let us calculate $\frac{\alpha^n}{2\sqrt{2}}$ for the first ten values of n and observe if there is a pattern:

$$\frac{\alpha^1}{2\sqrt{2}} = 0.853553390593274, \quad \frac{\alpha^2}{2\sqrt{2}} = 2.06066017177982, \quad \frac{\alpha^3}{2\sqrt{2}} = 4.97487373415292,$$
$$\frac{\alpha^4}{2\sqrt{2}} = 12.0104076400857, \quad \frac{\alpha^5}{2\sqrt{2}} = 28.9956890143242, \quad \frac{\alpha^6}{2\sqrt{2}} = 70.0017856687341,$$
$$\frac{\alpha^7}{2\sqrt{2}} = 168.999260351792, \quad \frac{\alpha^8}{2\sqrt{2}} = 408.000306372319, \quad \frac{\alpha^9}{2\sqrt{2}} = 984.99987309643,$$
$$\frac{\alpha^{10}}{2\sqrt{2}} = 2378.00005256518.$$

Let us add $\frac{1}{2}$ to the expression $\frac{\alpha^n}{2\sqrt{2}}$ to see if there is a pattern:

$$\frac{\alpha^1}{2\sqrt{2}} + \frac{1}{2} \approx 1.353553390593274, \qquad \frac{\alpha^2}{2\sqrt{2}} + \frac{1}{2} \approx 2.56066017177982,$$
$$\frac{\alpha^3}{2\sqrt{2}} + \frac{1}{2} \approx 5.47487373415292, \qquad \frac{\alpha^4}{2\sqrt{2}} + \frac{1}{2} \approx 12.5104076400857,$$
$$\frac{\alpha^5}{2\sqrt{2}} + \frac{1}{2} \approx 29.4956890143242, \qquad \frac{\alpha^6}{2\sqrt{2}} + \frac{1}{2} \approx 70.5017856687341,$$
$$\frac{\alpha^7}{2\sqrt{2}} + \frac{1}{2} \approx 169.499260351792, \qquad \frac{\alpha^8}{2\sqrt{2}} + \frac{1}{2} \approx 408.500306372319,$$
$$\frac{\alpha^9}{2\sqrt{2}} + \frac{1}{2} \approx 985.49987309643, \qquad \frac{\alpha^{10}}{2\sqrt{2}} + \frac{1}{2} \approx 2378.50005256518.$$

Now, by rounding down the above expression for each n, let us try to find a pattern:

$$\left\lfloor \frac{\alpha^1}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 1, \qquad \left\lfloor \frac{\alpha^2}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 2,$$
$$\left\lfloor \frac{\alpha^3}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 5, \qquad \left\lfloor \frac{\alpha^4}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 12,$$
$$\left\lfloor \frac{\alpha^5}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 29, \qquad \left\lfloor \frac{\alpha^6}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 70,$$
$$\left\lfloor \frac{\alpha^7}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 169, \qquad \left\lfloor \frac{\alpha^8}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 408,$$
$$\left\lfloor \frac{\alpha^9}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 985, \qquad \left\lfloor \frac{\alpha^{10}}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 2378.$$

Therefore, we can argue that:

$$\left\lfloor \frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = P_n.$$

The following theorem will confirm this result. We will need the lemma below for the proof.

Lemma 3.1. *It holds that:*

$$0 < \frac{\beta^n}{2\sqrt{2}} + \frac{1}{2} < 1$$

Proof. As known, $\beta = 1 - \sqrt{2}$. Thus:

$$-\frac{1}{2} < \beta < 0.$$

As a result:

$$0 < |\beta| < \frac{1}{2} \quad \text{and} \quad 0 < \frac{|\beta^n|}{2\sqrt{2}} < \frac{1}{2}.$$

Case1: When n is even

$$0 < \frac{\beta^n}{2\sqrt{2}} < \frac{1}{2}.$$

Therefore:

$$\frac{1}{2} < \frac{\beta^n}{2\sqrt{2}} + \frac{1}{2} < 1 \quad \Rightarrow \quad 0 < \frac{\beta^n}{2\sqrt{2}} + \frac{1}{2} < 1.$$

Case2: When n is odd

$$0 < -\frac{\beta^n}{2\sqrt{2}} < \frac{1}{2}.$$

This results in:

$$-\frac{1}{2} < \frac{\beta^n}{2\sqrt{2}} < 0 \quad \Rightarrow \quad 0 < \frac{\beta^n}{2\sqrt{2}} + \frac{1}{2} < \frac{1}{2}.$$

Hence, in both cases:

$$0 < \frac{\beta^n}{2\sqrt{2}} + \frac{1}{2} < 1.$$

Thus, the proof is complete. □

Theorem 3.1. *It holds that:*
$$P_n = \left\lfloor \frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} \right\rfloor$$

Proof. According to the Binet formula,
$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}.$$
This implies:
$$P_n = \left(\frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2}\right) - \left(\frac{\beta^n}{2\sqrt{2}} + \frac{1}{2}\right).$$
Therefore,
$$\frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} = P_n + \left(\frac{\beta^n}{2\sqrt{2}} + \frac{1}{2}\right).$$
According to Lemma 3.1, it holds that
$$0 < \frac{\beta^n}{2\sqrt{2}} + \frac{1}{2} < 1.$$
Therefore,
$$P_n < \frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} < P_n + 1.$$
Thus, we arrive at:
$$P_n = \left\lfloor \frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} \right\rfloor. \qquad \square$$

For example,
$$\frac{\alpha^{10}}{2\sqrt{2}} + \frac{1}{2} \approx 2378.50005256518.$$
This results in:
$$\left\lfloor \frac{\alpha^{10}}{2\sqrt{2}} + \frac{1}{2} \right\rfloor = 2378 = P_{10}.$$
As expected, this outcome is correct. For real numbers x that are not integers, according to Theorem 2.1(2), $\lfloor x \rfloor = \lceil x \rceil - 1$ holds. Therefore, we derive:
$$P_n = \left\lceil \frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} \right\rceil - 1.$$
However, for integers, according to Theorem 2.1(4) $\lceil x + n \rceil = \lceil x \rceil + n$ holds,
$$P_n = \left\lceil \frac{\alpha^n}{2\sqrt{2}} + \frac{1}{2} - 1 \right\rceil = \left\lceil \frac{\alpha^n}{2\sqrt{2}} - \frac{1}{2} \right\rceil.$$
This leads to the following corollary.

Corollary 3.1. *It holds that:*
$$P_n = \left\lceil \frac{\alpha^n}{2\sqrt{2}} - \frac{1}{2} \right\rceil.$$

This formula is used for calculating P_n for Pell numbers:

$$\left\lceil \frac{\alpha^1}{2\sqrt{2}} \right\rceil = P_1, \qquad \left\lfloor \frac{\alpha^2}{2\sqrt{2}} \right\rfloor = P_2,$$
$$\left\lceil \frac{\alpha^3}{2\sqrt{2}} \right\rceil = P_3, \qquad \left\lfloor \frac{\alpha^4}{2\sqrt{2}} \right\rfloor = P_4,$$
$$\left\lceil \frac{\alpha^5}{2\sqrt{2}} \right\rceil = P_5, \qquad \left\lfloor \frac{\alpha^6}{2\sqrt{2}} \right\rfloor = P_6,$$
$$\left\lceil \frac{\alpha^7}{2\sqrt{2}} \right\rceil = P_7, \qquad \left\lfloor \frac{\alpha^8}{2\sqrt{2}} \right\rfloor = P_8,$$
$$\left\lceil \frac{\alpha^9}{2\sqrt{2}} \right\rceil = P_9, \qquad \left\lfloor \frac{\alpha^{10}}{2\sqrt{2}} \right\rfloor = P_{10}.$$

The following corollary contains this observation.

Corollary 3.2. *It holds that:*

$$\left\lfloor \frac{\alpha^{2n}}{2\sqrt{2}} \right\rfloor = P_{2n} \quad \textit{and} \quad \left\lceil \frac{\alpha^{2n+1}}{2\sqrt{2}} \right\rceil = P_{2n+1}.$$

Proof. According to Lemma 3.1, we know:

$$-\frac{1}{2} < \frac{-\beta^{2n}}{2\sqrt{2}} < 0.$$

This leads to:

$$\frac{\alpha^{2n}}{2\sqrt{2}} - \frac{1}{2} < \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{2}} < \frac{\alpha^{2n}}{2\sqrt{2}}.$$

Since $P_{2n} \in \mathbb{Z}$, we have:

$$P_{2n} = \left\lfloor \frac{\alpha^{2n}}{2\sqrt{2}} \right\rfloor.$$

Similarly, when $n \to 2n+1$:

$$0 < \frac{-\beta^{2n+1}}{2\sqrt{2}} < \frac{1}{2},$$

leading to:

$$\frac{\alpha^{2n+1}}{2\sqrt{2}} < \frac{\alpha^{2n+1} - \beta^{2n+1}}{2\sqrt{2}} < \frac{\alpha^{2n+1}}{2\sqrt{2}} + \frac{1}{2}.$$

Thus, since $P_{2n+1} \in \mathbb{Z}$:

$$P_{2n+1} = \left\lceil \frac{\alpha^{2n+1}}{2\sqrt{2}} \right\rceil.$$

Thus, the proof is complete. □

Theorem 3.2. *It holds that:*

$$P_{n+1} = \left\lfloor \alpha P_n + \frac{1}{2} \right\rfloor.$$

Proof. We have that:

$$
\begin{aligned}
\alpha P_n &= \alpha \cdot \frac{\alpha^n - \beta^n}{2\sqrt{2}} = \frac{\alpha^{n+1} - \alpha\beta^n}{2\sqrt{2}} \\
&= \frac{\alpha^{n+1} - \beta^{n+1} + \beta^{n+1} - \alpha\beta^n}{2\sqrt{2}} \\
&= \frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{2}} + \frac{\beta^{n-1}(\beta^2 - \overbrace{\alpha \cdot \beta}^{-1})}{2\sqrt{2}} \\
&= P_{n+1} + \frac{\beta^{n-1}(\overbrace{\beta^2 + 1}^{-2\sqrt{2}\cdot\beta})}{2\sqrt{2}} \\
&= P_{n+1} - \beta^n.
\end{aligned}
$$

Now, adding $\frac{1}{2}$ to both sides of the equation $\alpha P_n = P_{n+1} - \beta^n$, we get:

$$\alpha P_n + \frac{1}{2} = P_{n+1} + \left(\frac{1}{2} - \beta^n\right).$$

Since $0 < \frac{1}{2} - \beta^n < \frac{1}{2}$, we have:

$$\left\lfloor \alpha P_n + \frac{1}{2} \right\rfloor = \left\lfloor P_{n+1} + \left(\frac{1}{2} - \beta^n\right) \right\rfloor.$$

Thus, we conclude:

$$P_{n+1} = \left\lfloor \alpha P_n + \frac{1}{2} \right\rfloor.$$

□

Corollary 3.3. *It holds that:*

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \alpha.$$

Proof. From Theorem 3.2, we know that $P_{n+1} = \left\lfloor \alpha P_n + \frac{1}{2} \right\rfloor$. Therefore, when calculating the limit:

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \lim_{n\to\infty} \frac{\alpha P_n + \frac{1}{2} + \theta}{P_n},$$

where $\theta \in (0, 1)$, meaning that θ is a fixed error term. Continuing with the expression:

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \lim_{n\to\infty} \left(\alpha + \frac{1}{2P_n} + \frac{\theta}{P_n}\right).$$

As P_n approaches infinity, the terms $\frac{1}{2P_n}$ and $\frac{\theta}{P_n}$ approach zero. The limit becomes:

$$\lim_{n\to\infty} (\alpha + 0 + 0) = \alpha.$$

Thus, the proof is complete. □

4 Conclusion

In the present paper, a analysis of the analytic framework of the Pell number sequence is carried out. It greatly illustrates several amazing relationships between these numbers and the floor and ceiling functions. First, the closed forms derived through the Binet formula allowed for the prediction of Pell numbers with precision for great values of n. The made analysis gave an easy explanation of how parameters $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ result in the behavior of the sequence. More precisely, the decaying trend of the term $|\beta|^n$ with dominant participation of α in the asymptotic limit process gives the important information concerning the growth rate and other characteristics of the sequence.

The theorems and lemmas introduced in the given work exemplify the exact calculation of the Pell number series with the help of floor and ceiling functions, where both methodologies have their mathematical justification. The outcome addresses certain fundamental problems in number theory and provides an extensive platform for understanding the recurrence relations while processing such sequences using analytic methods. Therefore, the paper constitutes a worthy addition to the related literature regarding asymptotic analysis performed on sequences defined by recurrence, as in the case of the Pell numbers.

Future work can be done by seeing how similar methods apply to other recurrence relations or to more complicated sequences. New directions not only are opened in theoretical mathematics but in all the fields where such sequences occur, such as cryptography, algorithmic analysis, and combinatorial number theory

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