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On the generalized p-periodic linear recursive sequences via the Fibonacci–Horner decomposition of the matrix powers

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Abstract: In this study, we investigate the matrix formulation of the generalized p-periodic linear recursive sequences. To reach our goal, we consider the properties of the Fibonacci–Horner decomposition of the matrix powers and those of the weighted linear recursive sequence of Fibonacci type. We provide the linear, the combinatorial, and the analytic representations of the generalized p-periodic linear recursive sequences. For illustrating our general results, properties of some special cases are studied and numerical example are furnished.

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1 Introduction

The usual sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by the well-known recursive relation of order 2, given by $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Since its appearance in connection with the famous problem of the evolution of the population of rabbits, in the seminal work of Fibonacci, the sequence of Fibonacci numbers has been widely studied. Several properties and applications of the Fibonacci numbers have been established in various papers of the literature (see, for instance, [7, 8]). Moreover, this sequence has been the subject of many generalizations, and it has been the source of several identities in additive number theory. Among the generalizations of the sequence of Fibonacci numbers that have been proposed in the literature, there is the connection with a periodicity condition linked to a particular parametrization.

Indeed, Edson and Yayenie in [5] introduce a new generalization for the sequence of Fibonacci numbers, labelled *bi-periodic Fibonacci sequence*, which is defined as follows. Let a and b be two non-zero real numbers and consider the sequence $\{F_n^{(a,b)}\}_{n\geq 0}$ defined by

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if } n \text{ is even,} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

for $n \geq 2$, with initial conditions given by $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. When $a = b = 1$, we have the classical Fibonacci sequence, and when $a = b = 2$, we get the Pell numbers. If we set $a = b = k$ for some positive integer k, we get the k-Fibonacci numbers, representing a generalization of the classical Fibonacci numbers. The sequence of bi-periodic Fibonacci numbers has been studied with the aid of several interesting approaches (see, for example, [4, 5, 10]), and k-periodic binary recurrences (see [6]). Recently, in [10] a matrix formulation approach of Expression (1) allows to obtain various properties of the bi-periodic Fibonacci numbers, especially their analytic and combinatorial formulations.

In the present study, we are interested in the generalization of the bi-periodic Fibonacci sequence (1) as follows. Let $p \geq 2$ be an integer number and s_j $(0 \leq j \leq p-1)$, a_{jk} $(0 \leq j \leq p-1, 1 \leq k \leq p-1)$ be two finite sequences of real numbers, and set $[s] = [s_j]$ and $[a] = [a_{j,k}]$. For every fixed j $(0 \leq j \leq p-1)$, let $\{G_{np+j}^{([s],[a])}\}_{n\geq 0}$ be the sequence, whose general term $G_{np+j}^{([s],[a])}$ is defined as

$$G_{np+j}^{([s],[a])} = a_{j,1}G_{np+j-1}^{([s],[a])} + a_{j,2}G_{np+j-2}^{([s],[a])} + \cdots + a_{j,p-1}G_{np+j-p+1}^{([s],[a])} + s_jG_{np+j-p}^{([s],[a])}, \tag{2}$$

for $n \geq 1$, where $G_k^{([s],[a])}$ $(0 \leq k \leq p-1)$ are the initial conditions, more explicitly,

$$G_m = \begin{cases} a_{0,1}G_{np-1}^{([s],[a])} + a_{0,2}G_{np-2}^{([s],[a])} + \ldots + a_{0,p-1}G_{np-p+1}^{([s],[a])} + s_0 G_{np-p}^{([s],[a])}, & \text{for } m = np, \\ a_{1,1}G_{np}^{([s],[a])} + a_{1,2}G_{np-1}^{([s],[a])} + \ldots + a_{1,p-1}G_{np-p+2}^{([s],[a])} + s_1 G_{np-p+1}^{([s],[a])}, & \text{for } m = np+1, \\ \vdots & \\ a_{p-1,1}G_{np+p-2}^{([s],[a])} + a_{p-1,2}G_{np+p-3}^{([s],[a])} + \ldots + a_{p-1,p-1}G_{np}^{([s],[a])} + s_{p-1}G_{np-1}^{([s],[a])}, & \\ \qquad\qquad \text{for } m = np+p-1. & \end{cases}$$

Our main goal is to provide some properties of the generalized p-periodic linear recursive sequences $\{G_n^{([s],[a])}\}_{n\geq 0}$, using the matrix formulation of Expression (2). Properties of these sequences are provided, using the Rachidi *et. al.* methods (see, [2, 3, 10]). More specifically, the matrix formulation of Expression (2) leads to the computation of the matrix powers. Therefore, our fundamental tool is based on some properties of the Fibonacci–Horner decomposition for computing the matrix powers, in connection with a specific weighted linear recursive sequence of Fibonacci type. Therefore, the analytic Binet representation and the combinatorial formula, as well as some identities of the p-periodic Fibonacci numbers are established. For illustrating our general results, properties of some special cases are studied and illustrative numerical example are given.

This paper is organized as follows. In Section 2 we are concerned with the matrix formulation of the p-periodic Fibonacci numbers. Section 3 is devoted to the linear and combinatorial formulation of p-periodic Fibonacci numbers. Section 4 is devoted to the analytic representation of the sequence (2). In Section 5, we provide some properties of the 3-periodic Fibonacci sequences. Some concluding remarks and perspectives are exhibited in Section 6.

2 Matrix formulation of the bi-periodic Fibonacci sequence

2.1 Two special cases

For reason of clarity and conciseness, we present in this sub-section some special cases on the matrix formulation of the p-periodic Fibonacci numbers for $p = 2,\ 3$, respectively.

Special case $p = 2$. For $p = 2$ we show easily that Expression (2) is reduced to the bi-periodic Fibonacci sequence (1). That is, let us consider the integers s_0, s_1 and $a_{01} = a$, $a_{11} = b$, and set $[a] = [a_{01}, a_{11}]$. Then, Expression (2) takes the form

$$G_n^{(2,[a])} = \begin{cases} aG_{n-1}^{(2,[a])} + s_0 G_{n-2}^{(2,[a])}, & \text{if } n \text{ is even}, \\ bG_{n-1}^{(2,[a])} + s_1 G_{n-2}^{(2,[a])}, & \text{if } n \text{ is odd}. \end{cases} \tag{3}$$

Let us consider the vector columns $Y_n^{(2,[a])} = \left[G_{2n}^{(2,[a])}, G_{2n-1}^{(2,[a])}\right]^T$, $Z_n^{(2,[a])} = \left[G_{2n+1}^{(2,[a])}, G_{2n}^{(2,[a])}\right]^T$, (here and in the sequel the notation $[c_1, \ldots, c_p]^T$ means the transpose of the vector line $[c_1, \ldots, c_p]$.) and the two matrices $A_0 = \begin{bmatrix} a & s_0 \\ 1 & 0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} b & s_1 \\ 1 & 0 \end{bmatrix}$. We can show that the sequence of numbers $\{G_n^{(2,[a])}\}_{n\geq 0}$ defined as in Expression (3) (or Expression (1)) is equivalent to the following two matrix equations $Y_n^{(2,[a])} = A_0 Z_{n-1}^{(2,[a])}$ and $Z_n^{(2,[a])} = A_1 Y_n^{(2,[a])}$. Therefore, we have $Z_n^{(2,[a])} = A_1 Y_n^{(2,[a])} = A_1 A_0 Z_{n-1}^{(2,[a])}$. This implies that we have $Z_n^{(2,[a])} = A Z_{n-1}^{(2,[a])}$, where $A = A_1 A_0 = \begin{bmatrix} ab + s_1 & bs_0 \\ a & s_0 \end{bmatrix}$.

Special case $p = 3$. Let s_j $(0 \leq j \leq 2)$ and a_{jk} $(0 \leq j \leq 2,\ 1 \leq k \leq 2)$ be two sequences of real numbers. Then, the related sequence $\{G_n^{(3,[a])}\}_{n\geq 0}$ is given by

$$G_n^{(3,[a])} = \begin{cases} a_{01} G_{n-1}^{(3,[a])} + a_{02} G_{n-2}^{(3,[a])} + s_0 G_{n-3}^{(3,[a])} & \text{if } n = 3k, \\ a_{11} G_{n-1}^{(3,[a])} + a_{12} G_{n-2}^{(3,[a])} + s_1 G_{n-3}^{(3,[a])} & \text{if } n = 3k+1, \\ a_{21} G_{n11}^{(3,[a])} + a_{22} G_{n-2}^{(3,[a])} + s_2 G_{n-3}^{(3,[a])} & \text{if } n = 3k+2. \end{cases} \tag{4}$$

Let us consider the following vector columns

$$X_n^{(3,[a])} = \left[G_{3n}^{(3,[a])}, G_{3n-1}^{(3,[a])}, G_{3n-2}^{(3,[a])}\right]^T,$$
$$Y_n^{(3,[a])} = \left[G_{3n+1}^{(3,[a])}, G_{3n}^{(3,[a])}, G_{3n-1}^{(3,[a])}\right]^T,$$
$$Z_n^{(3,[a])} = \left[G_{3n-1}^{(3,[a])} G_{3n-2}^{(3,[a])}, G_{3n-3}^{(3,[a])}\right]^T,$$

and the square matrices

$$A_0 = \begin{bmatrix} a_{01} & a_{02} & s_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_{11} & a_{12} & s_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{21} & a_{22} & s_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence, the matrix formulation of the three expressions of (4) can be written under the following matrix equation $X_n^{(3,[a])} = A_0 Z_n^{(3,[a])}$, $Y_n^{(3,[a])} = A_1 X_n^{(3,[a])}$ and $Z_{n+1}^{(3,[a])} = A_2 Y_n^{(3,[a])}$. The first two preceding matrix formulas show that $Y_n^{(3,[a])} = A_1 X_n^{(3,[a])} = A_1 A_0 Z_n^{(3,[a])}$. Therefore, for every $n \geq 0$, we obtain $Z_{n+1}^{(3,[a])} = A_2 Y_n^{(3,[a])} = A_2 A_1 A_0 Z_n^{(3,[a])}$, or equivalently

$$Z_{n+1}^{(3,[a])} = A Z_n^{(3,[a])}, \text{ where } A = A_2 A_1 A_0. \tag{5}$$

These special cases will be used in the next sections for providing results on the p-periodic Fibonacci sequences of order r.

2.2 General setting: Matrix formulation of (2)

Let $\{G_n^{([s],[a])}\}_{n\geq 0}$ be the p-periodic Fibonacci sequence defined as in (2). We consider the following column matrices $X_n^{(j)} = [G_{np+j}^{([s],[a])}, G_{np+j-1}^{([s],[a])}, \ldots, G_{np+j-p+1}^{([s],[a])}]^T$, where $-1 \leq j \leq p-1$. For $j = -1$ in the former expression, we have $X_n^{(-1)} = X_{n-1}^{(p-1)}$. Let us consider the square matrix

$$A_j = \begin{bmatrix} a_{j1} & a_{j2} & \ldots & a_{jp-1} & s_j \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ldots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

Taking into account Expression (2), we show that $X_n^{(j)} = A_j X_n^{(j-1)}$. Therefore, we have $X_n^{(0)} = A_0 X_n^{(-1)} = A_0 X_{n-1}^{(p-1)}$, $X_n^{(1)} = A_1 X_n^{(0)} = A_1 A_0 X_{n-1}^{(p-1)}$, and $X_n^{(2)} = A_2 X_n^{(1)} = A_2 A_1 A_0 X_{n-1}^{(p-1)}$. By iteration of the preceding process, we get

$$X_n^{(k)} = A_k X_n^{(k-1)} = \left[\prod_{j=0}^{*,k} A_j\right] X_{n-1}^{(p-1)},$$

where $\prod_{j=0}^{*,k} A_j = A_k \cdots A_1 A_0$. The notation " $\prod_{j=0}^{*,p-1}$ " is due to the non-commutativity of the matrix product. Therefore, we have the following proposition.

Proposition 2.1. *Under the previously formulated conditions, for every $n \geq 1$, we have*

$$X_n^{(p-1)} = A X_{n-1}^{(p-1)}, \text{ where } A = \prod_{j=0}^{*,p-1} A_j, \tag{7}$$

The matrix formulation (7) of the p-periodic Fibonacci sequence (2), as well as, the matrix formulation of its special cases, will be used in the next sections for studying analytic and combinatorial expressions of the p-periodic Fibonacci sequence and the bi-periodic Leonardo sequence. Moreover, some new identities related to these sequences will be provided.

3 The linear and combinatorial expression of sequences (2) via the Fibonacci–Horner decomposition

Let $\{G_n^{([s],[a])}\}_{n\geq 0}$ be the generalized p-periodic linear recursive sequence defined as in (2). Expression (7) implies that we have

$$X_n^{(p-1)} = A^n X_0^{(p-1)}, \text{ where } A = \prod_{j=0}^{*,p-1} A_j, \tag{8}$$

where $X_0^{(p-1)}$ is the vector column $X_0^{(p-1)} = [G_{p-1}^{([s],[a])}, G_{p-2}^{([s],[a])}, \ldots, G_0^{([s],[a])}]^T$. In addition, the computation of the entries of the powers A^n of the matrix $A = \prod_{j=0}^{*,p-1} A_j$, can be also obtained from the so-called Fibonacci–Horner decomposition introduced in [2, 9].

Lemma 3.1. (Rachidi et al. [2,9]) *Let M be a square $p \times p$ matrix and*

$$P(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$$

its characteristic polynomial. Then, the Fibonacci–Horner decomposition of M^n for $n \geq r$, is given by

$$\begin{cases} M^n = u_n M_0 + u_{n-1} M_1 + \cdots + u_{n-p+1} M_{p-1}, \ \textit{for every} \ n \geq p, \\ M_0 = I_p\,; \ M_i = M^i - a_0 M^{i-1} - \cdots - a_{i-1} I_p, \textit{ for every } i = 1, \cdots, p-1, \end{cases} \tag{9}$$

where I_p is the identity matrix of order $p \times p$ and the sequence $\{u_n\}_{n\geq 0}$ is defined by

$$u_n = \sum_{k_0+2k_1+\cdots+pk_{p-1}=n-p+1} \binom{k_0+k_1+\cdots+k_{p-1}}{k_0, k_1, \ldots, k_{p-1}} a_0^{k_0} a_1^{k_1} \cdots a_{p-1}^{k_{p-1}}, \tag{10}$$

for every $n \geq 1$, with $u_{p-1} = 1$ and $u_n = 0$ for $0 \leq n \leq p-2$, and

$$\binom{k_0+k_1+\cdots+k_{p-1}}{k_0, k_1, \ldots, k_{p-1}} = \frac{(k_0+k_1+\cdots+k_{p-1})!}{k_0!k_1!\cdots k_{p-1}!} a_0^{k_0} a_1^{k_1} \cdots a_{p-1}^{k_{p-1}}.$$

It was established in [9] that the sequence $\{u_n\}_{n\geq 0}$ defined by (10) satisfies the following linear recurrence relation

$$u_{n+1} = a_0 u_n + a_1 u_{n-1} + \cdots + a_{p-1} u_{n-p+1}, \text{ for every } n \geq p-1, \tag{11}$$

in other words, $\{u_n\}_{n\geq 0}$ is a recurrence of order p. It is usually, denoted by $u_n = \rho(n+1, p)$.

Application of Lemma 3.1, namely Expression (9), for computing of the matrix powers A^n of $A = \prod_{j=0}^{*,p-1} A_j$, allows us to obtain the following result.

Proposition 3.1. *Let $A = \prod_{j=0}^{*,p-1} A_j$ be the matrices defined by* (7) *and*

$$P(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$$

its characteristic polynomial. Then, the Fibonacci–Horner decomposition of A^n, for $n \geq p$, is given by

$$\begin{cases} A^n = u_n A^{(0)} + u_{n-1} A^{(1)} + \cdots + u_{n-p+1} A^{(p-1)}, & \textit{for every } n \geq p, \\ A^{(0)} = I_p\,;\ A^{(i)} = A^i - a_0 A^{i-1} - \cdots - a_{i-1} I_p, & \textit{for } i = 1, \cdots, p-1, \end{cases} \tag{12}$$

where I_p is the identity matrix of order $p \times p$ and the sequence $\{u_n\}_{n\geq 0}$ is defined by (10)–(11).

For every j $(1 \leq j \leq p-1)$ we set $A^j = (\gamma_{h,k}^{(j)})_{1\leq h,k\leq p}$ and $A^{(j)} = (\alpha_{h,k}^{(j)})_{1\leq h,k\leq p}$. Following Expression (12), we get

$$\alpha_{h,k}^{(j)} = \gamma_{h,k}^{(j)} - a_0 \gamma_{h,k}^{(j-1)} - \cdots - a_{j-2} \gamma_{h,k}^{(1)} - a_{j-1} \delta_{hk}, \tag{13}$$

where δ_{hk} is the Kronecker symbol. Combining Expressions (8), namely, $X_n^{(p-1)} = A^n X_0^{(p-1)}$, and (12) we obtain

$$X_n^{(p-1)} = u_n A^{(0)} X_0^{(p-1)} + u_{n-1} A^{(1)} X_0^{(p-1)} + \cdots + u_{n-r+1} A^{(p-1)} X_0^{(p-1)}, \tag{14}$$

for every $n \geq r$, where $A = \prod_{j=0}^{*,p-1} A_j$ and $\{u_n\}_{n\geq 0}$ is the sequence defined by (10)–(11). For every j $(1 \leq j \leq p-1)$, we have

$$A^{(j)} X_0^{(p-1)} = \begin{bmatrix} \alpha_{1,1}^{(j)} & \alpha_{1,2}^{(j)} & \ldots & \alpha_{1,p}^{(j)} \\ \alpha_{2,1}^{(j)} & \alpha_{2,2}^{(j)} & \ldots & \alpha_{2,p}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{p,1}^{(j)} & \alpha_{1,2}^{(j)} & \ldots & \alpha_{p,p}^{(j)} \end{bmatrix} \begin{bmatrix} G_{p-1}^{([s],[a])} \\ G_{p-2}^{([s],[a])} \\ \vdots \\ G_0^{([s],[a])} \end{bmatrix} = \begin{bmatrix} \omega_1^{(j)} \\ \omega_2^{(j)} \\ \vdots \\ \omega_p^{(j)} \end{bmatrix},$$

where

$$\omega_h^{(j)} = \sum_{k=1}^{p} \alpha_{h,k}^{(j)} G_{p-k}^{([s],[a])}, \tag{15}$$

with the $\alpha_{h,k}^{(j)}$ given by (13), namely $\alpha_{h,k}^{(j)} = \gamma_{h,k}^{(j)} - \sum_{r=0}^{j-1} a_r \gamma_{r,k}^{(j-r-1)}$ with $\gamma_{h,k}^{(0)} = \delta_{hk}$. Therefore, we get the following matrix equation

$$X_n^{(p-1)} = A^n X_0^{(p-1)} = u_n X_0^{(p-1)} + \sum_{j=1}^{p-1} u_{n-j} A^{(j)} X_0^{(p-1)}.$$

Since $X_n^{(p-1)} = [G_{np+p-1}^{([s],[a])}, G_{np+p-2}^{([s],[a])}, \ldots, G_{np}^{([s],[a])}]^T$, we have

$$\begin{bmatrix} G_{np+p-1}^{([s],[a])} \\ G_{np+p-2}^{([s],[a])} \\ \vdots \\ G_{np}^{([s],[a])} \end{bmatrix} = u_n \begin{bmatrix} G_{p-1}^{([s],[a])} \\ G_{p-2}^{([s],[a])} \\ \vdots \\ G_0^{([s],[a])} \end{bmatrix} + \sum_{j=1}^{p-1} u_{n-j} \begin{bmatrix} \omega_1^{(j)} \\ \omega_2^{(j)} \\ \vdots \\ \omega_p^{(j)} \end{bmatrix} = \begin{bmatrix} u_n G_{p-1}^{([s],[a])} + \sum_{j=1}^{p-1} u_{n-j}\omega_1^{(j)} \\ u_n G_{p-2}^{([s],[a])} + \sum_{j=1}^{p-1} u_{n-j}\omega_2^{(j)} \\ \vdots \\ u_n G_0^{([s],[a])} + \sum_{j=1}^{p-1} u_{n-j}\omega_p^{(j)} \end{bmatrix}.$$

In summary, we get the following result.

Theorem 3.1. *Let $\{G_n^{([s],[a])}\}_{n\geq 0}$ be the generalized p-periodic linear recursive sequence defined as in* (2)*. Then, for every h ($0 \leq h \leq p-1$), we have*

$$G_{np+p-h}^{([s],[a])} = u_n G_{p-h}^{([s],[a])} + \sum_{j=1}^{p-1} u_{n-j}\omega_h^{(j)}, \tag{16}$$

where $\{u_n\}_{n\geq 0}$ is the sequence defined by (10)–(11) *and $\omega_h^{(j)}$ are given by* (15)*.*

As a consequence of Expression (10) and Theorem 3.1, namely formula (16), we derive the combinatorial expression of the p-periodic Fibonacci sequence (2).

Corollary 3.1. *Let $\{G_n^{([s],[a])}\}_{n\geq 0}$ be the generalized p-periodic linear recursive sequence defined as in* (2)*. Then, for every h ($0 \leq h \leq p-1$), the following combinatorial formula holds*

$$G_{np+p-h}^{([s],[a])} = \rho(n+1,r) G_{p-h}^{([s],[a])} + \sum_{j=1}^{p-1} \rho(n-j+1,r)\omega_h^{(j)}, \tag{17}$$

where $\rho(n+1,r)$ is given by (10) *and $\omega_h^{(j)}$ are given by* (15) *(and* (13)*).*

In the next section we illustrate the general results of this section by studying the special case $p = 2$, namely, the special case of bi-periodic Fibonacci numbers.

3.1 Special case $p = 2$

In the special case $p = 2$, we can apply the Fibonacci–Horner decomposition (12) to the related matrix $A = A_1 A_0 = \begin{bmatrix} ab + s_1 & bs_0 \\ a & s_0 \end{bmatrix}$, whose characteristic polynomial is $P(z) = z^2 - a_0 z - a_1$, where $a_0 = ab + s_0 + s_1$ and $a_1 = -s_0 s_1$. As a corollary of Proposition 3.1, we get the following corollary.

Corollary 3.2. *For every $n \geq 2$, we have $A^n = u_n A^{(0)} + u_{n-1} A^{(1)}$, where*

$$\begin{cases} A^{(0)} = I_2 \text{ and } A^{(1)} = A - a_0 I_2, \\ u_{n+1} = a_0 u_n + a_1 u_{n-1}, \end{cases} \tag{18}$$

where I_2 is the 2×2 identity matrix, with $u_0 = 0$, $u_1 = 1$ and $a_0 = ab + s_0 + s_1, a_1 = -s_0 s_1$ are the coefficients of the characteristic polynomial of the matrix $A = A_1 A_0$.

Following Expression (8), namely $X_n^{(p-1)} = A^n.X_0^{(p-1)}$, where $A = \prod_{j=0}^{*,p-1} A_j$, we obtain a linear formula for $Z_n^{(2,[a])} =^t [G_{2n+1}^{(2,[a])}, G_{2n}^{(2,[a])}]$ as follows

$$Z_n^{(2,[a])} = (u_n A^{(0)} + u_{n-1}A^{(1)})Z_0^{(2,[a])},$$

for every $n \geq 2$, where $A^{(0)}$, $A^{(1)}$ and $\{u_n\}_{n\geq 0}$ are given as in (18) and $Z_0^{(2,[a])} =^t [G_1^{(2,[a])}, G_0^{(2,[a])}]$. Since $A^n = (u_n - a_0u_{n-1})I_2 + u_{n-1}A$, we get

$$Z_n^{(2,[a])} = A^n Z_0^{(2,[a])} = (u_n - a_0u_{n-1})Z_0^{(2,[a])} + u_{n-1}AZ_0^{(2,[a])},$$

for every $n \geq 2$. A direct computation implies

$$\begin{bmatrix} G_{2n+1}^{(2,[a])} \\ G_{2n}^{(2,[a])} \end{bmatrix} = (u_n - a_0u_{n-1}) \begin{bmatrix} G_1^{(2,[a])} \\ G_0^{(2,[a])} \end{bmatrix} + u_{n-1} \begin{bmatrix} (ab+s_1)G_1^{(2,[a])} + bs_0G_0^{(2,[a])} \\ aG_1^{(2,[a])} + s_0G_0^{(2,[a])} \end{bmatrix}.$$

In addition, application of Expression (16) allows us to obtain

$$\begin{cases} G_{2n+1}^{(2,[a])} = bs_0u_{n-1}G_0^{(2,[a])} + (u_n + (ab - a_0 + s_1)u_{n-1})G_1^{(2,[a])} \\ G_{2n}^{(2,[a])} = (u_n + (s_0 - a_0)u_{n-1})G_0^{(2,[a])} + au_{n-1}G_1^{(2,[a])}. \end{cases}$$

In summary, since $a_0 = ab + s_0 + s_1$, a direct computation allows us to the following result.

Proposition 3.2. *Under the previously formulated conditions, the linear formula of the bi-periodic Fibonacci numbers is given by*

$$\begin{cases} G_{2n+1}^{(2,[a])} = bs_0u_{n-1}G_0^{(2,[a])} + (u_n - s_0u_{n-1})G_1^{(2,[a])} \\ G_{2n}^{(2,[a])} = (u_n - (ab + s_1)u_{n-1})G_0^{(2,[a])} + au_{n-1}G_1^{(2,[a])}. \end{cases}$$

The combinatorial expression of the bi-periodic Fibonacci numbers is derived as in Corollary 3.1, namely, Expression (17). That is, since $u_n = \rho(n+1,2)$, for every $n \geq 2$, with $u_0 = 0$ and $u_1 = 1$, we have the following corollary.

Corollary 3.3. *Let $\{G_n^{(2,[a])}\}_{n\geq 0}$ be the bi-periodic Fibonacci sequence defined as in* (1)*, of where initial conditions $G_j^{(2,[a])}$ ($0 \leq j \leq 2$). Then, the combinatorial formula of the bi-periodic Fibonacci numbers is given by*

$$\begin{cases} G_{2n+1}^{(2,[a])} = bs_0\rho(n,2)G_0^{(2,[a])} + [\rho(n+1,2) - s_0\rho(n,2)]G_1^{(2,[a])} \\ G_{2n}^{(2,[a])} = [\rho(n+1,2) - (ab + s_1)\rho(n,2)]G_0^{(2,[a])} + a\rho(n,2)G_1^{(2,[a])}, \end{cases}$$

where $\rho(n+1,2) = \sum_{k_0+2k_1=n-1} \binom{k_0+k_1}{k_0,k_1}(ab+s_0+s_1)^{k_0}(-s_0s_1)^{k_1}$, *with* $\rho(1,2) = 0, \rho(2,2) = 1.$

4 Analytic expression of (2) via the Fibonacci–Horner decomposition: The case of simple roots

4.1 General setting

Let $\{G_n^{([s],[a])}\}_{n\geq 0}$ be the generalized p-periodic linear recursive sequence (2) and consider its matrix formulation given by Expression (7), namely, $X_n^{(p-1)} = AX_{n-1}^{(p-1)}$. This former expression implies that we have Expression (8), i.e., $X_n^{(p-1)} = A^n X_0^{(p-1)}$, where $A = \prod_{j=0}^{*,p-1} A_j$ of characteristic polynomial $P(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$, with A_j is the companion matrix (6). Let us consider the Fibonacci–Horner decomposition (12) of the powers A^n, for $n \geq p$, namely

$$\begin{cases} A^n = u_n A^{(0)} + u_{n-1}A^{(1)} + \cdots + u_{n-r+1}A^{(p-1)}, & \text{for every } \ n \geq p, \\ A^{(0)} = I_p \,;\ A^{(i)} = A^i - a_0 A^{i-1} - \cdots - a_{i-1} I_p, & \text{for every } i = 1, \ldots, p-1, \end{cases}$$

where I_p is the identity matrix of order $p \times p$ and the sequence $\{u_n\}_{n\geq 0}$ is defined by (10)–(11). Note that $P(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$ is also the characteristic polynomial of the sequence $\{u_n\}_{n\geq 0}$.

Suppose that the roots of the polynomial $P(z) = z^p - a_0 z^{p-1} - \cdots - a_{p-1}$ are simple. In the aim to provide an explicit formula of the analytic of the fundamental solution, we will use the result of [1] and [3, Theorem 2.2]. Indeed, the analytic formula of $v_n^{(r)}$ is given in the following lemma.

Lemma 4.1. (Rachidi *et al.*) *Suppose that the roots $\lambda_1, \ldots, \lambda_r$ of the characteristic polynomial $P(z) = z^p - a_1 z^{p-1} - \cdots - a_{p-2}z - a_{r-1}$ ($a_{p-1} \neq 0$) satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, the analytic Binet formula of the general term u_n of the sequence $\{u_n\}_{n\geq 0}$ is given by*

$$u_n = \sum_{i=1}^{p} \frac{\lambda_i^n}{P'\,(\lambda_i)} = \sum_{i=1}^{p} \frac{\lambda_i^n}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \quad \textit{for every } \ n \geq p, \tag{19}$$

with $u_{p-1} = 1,\ \ u_n = 0$ for $0 \leq n \leq p-2$, where $P'(z)$ is the derivative of $P(z)$.

Application of Lemma 4.1 to result of Theorem 3.1, namely to Expression (16) we can formulate the analytic formula of the p-periodic Fibonacci sequence defined as in (2) as follows.

Theorem 4.1. *Let $\{G_n^{([s],[a])}\}_{n\geq 0}$ be the generalized p-periodic linear recursive sequence defined as in* (2)*. Then, for every s ($0 \leq s \leq p-1$), we have*

$$G_{np+p-h}^{([s],[a])} = \left[\sum_{i=1}^{p} \frac{\lambda_i^n}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}\right] G_{p-h}^{([s],[a])} + \sum_{j=1}^{p-1}\left[\sum_{i=1}^{p} \frac{\lambda_i^{n-j}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}\right]\omega_h^{(j)}, \tag{20}$$

where $\{u_n\}_{n\geq 0}$ is the sequence defined by (10)–(11) *and the $\omega_s^{(j)}$ are given by* (15).

Let us illustrate the general result of Theorem 4.1 by studying the special case of the bi-periodic Fibonacci numbers.

4.2 Special case $p = 2$

Let $\{G_n^{(2,[a])}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers defined by (1); here we use the notation $[a] = (a, b)$. Result of Proposition 3.2 shows that the linear formula of the bi-periodic Fibonacci numbers is given by

$$\begin{cases} G_{2n+1}^{(2,[a])} = bs_0 u_{n-1} G_0^{(2,[a])} + (u_n - s_0 u_{n-1}) G_1^{(2,[a])} \\ G_{2n}^{(2,[a])} = (u_n - (ab + s_1) u_{n-1}) G_0^{(2,[a])} + a u_{n-1} G_1^{(2,[a])}, \end{cases}$$

where $\{u_n\}_{n\geq 0}$ is the recursive sequence (11), defined by a recursive relation of order 2, namely, $u_{n+1} = a_0 u_n + a_1 u_{n-1},$ for $n \geq 1$, with $u_0 = 0$ and $u_1 = 1$, where $a_0 = ab+s_0+s_1$, $a_1 = -s_0 s_1$. For $\Delta_{a,b}(s_0, s_1) = (ab + s_0 + s_1)^2 + 4s_0 s_1 \geq 0$, the two distinct roots of the characteristic polynomial $P(z) = z^2 - (ab + s_0 + s_1)z - s_0 s_1$ are given by $\lambda_1 = \dfrac{ab + s_0 + s_1 + \sqrt{\Delta_{a,b}(s_0, s_1)}}{2}$ and $\lambda_2 = \dfrac{ab + s_0 + s_1 - \sqrt{\Delta_{a,b}(s_0, s_1)}}{2}$. Then, by applying formula (19), we show that the analytic Binet formula of the sequence $\{u_n\}_{n\geq 0}$ is given by $u_n = \dfrac{1}{\sqrt{\Delta_{a,b}(s_0, s_1)}}[\lambda_1^n - \lambda_2^n]$, for every $n \geq 0$. When $\Delta_{a,b}(s_0, s_1) = 0$ the unique double root is $\lambda = \lambda_1 = \lambda_2 = \dfrac{ab + s_0 + s_1}{2}$. Hence, the analytic Binet formula of the sequence $\{u_n\}_{n\geq 0}$ is given by $u_n = n\lambda^{n-1},$ for every $n \geq 1$.

Therefore, the preceding discussion, combined with Proposition 3.2, allows us to formulate the analytic Binet formula of the bi-periodic Fibonacci numbers as follows. For $\Delta_{a,b}(s_0, s_1) \neq 0$, we have

$$G_{2n+1}^{(2,[a])} = \frac{bs_0}{\sqrt{\Delta_{a,b}(s_0, s_1)}} \left[\lambda_1^{n-1} - \lambda_2^{n-1}\right] G_0^{(2,[a])} + \frac{1}{\sqrt{\Delta_{a,b}(s_0, s_1)}}([\lambda_1^n - \lambda_2^n] - s_0 \left[\lambda_1^{n-1} - \lambda_2^{n-1}\right]) G_1^{(2,[a])}$$

$$G_{2n}^{(2,[a])} = \frac{1}{\sqrt{\Delta_{a,b}(s_0, s_1)}}([\lambda_1^n - \lambda_2^n] - (ab + s_1) \left[\lambda_1^{n-1} - \lambda_2^{n-1}\right]) G_0^{(2,[a])} + \frac{a}{\sqrt{\Delta_{a,b}(s_0, s_1)}} \left[\lambda_1^{n-1} - \lambda_2^{n-1}\right] G_1^{(2,[a])}.$$

For $\Delta_{a,b}(s_0, s_1) = 0$, we have

$$G_{2n+1}^{(2,[a])} = bs_0 n\lambda^{n-1} G_0^{(2,[a])} + (n\lambda^{n-1} - s_0(n-1)\lambda^{n-2}) G_1^{(2,[a])},$$

$$G_{2n}^{(2,[a])} = (n\lambda^{n-1} - (ab + s_1)(n-1)\lambda^{n-2}) G_0^{(2,[a])} + a(n-1)\lambda^{n-2} G_1^{(2,[a])}.$$

In summary, we have the following proposition.

Proposition 4.1. *Let $\{G_n^{(2,[a])}\}_{n\geq 0}$ be the sequence of bi-periodic Fibonacci numbers defined by* (1)*, and $\Delta_{a,b}(s_0, s_1) = (ab+s_0+s_1)^2+4s_0 s_1$. Then, for $\Delta_{a,b}(s_0, s_1) = (ab+s_0+s_1)^2+4s_0 s_1 > 0$, the analytic Binet formula of the bi-periodic Fibonacci numbers is given as follows*

$$\begin{cases} G_{2n+1}^{(2,[a])} = \frac{bs_0}{\sqrt{\Delta_{a,b}(s_0,s_1)}} \Phi_{n-1}(\lambda_1, \lambda_2) G_0^{(2,[a])} + \frac{1}{\sqrt{\Delta_{a,b}(s_0,s_1)}} \Delta_n(\lambda_1, \lambda_2) G_1^{(2,[a])} \\ G_{2n}^{(2,[a])} = \frac{1}{\sqrt{\Delta_{a,b}(s_0,s_1)}} \Omega_n(\lambda_1, \lambda_2) G_0^{(2,[a])} + \frac{a}{\sqrt{\Delta_{a,b}(s_0,s_1)}} \Phi_n(\lambda_1, \lambda_2) G_1^{(2,[a])}. \end{cases}$$

where $\lambda_1 = \dfrac{ab + 1 + \sqrt{\Delta_{a,b}(s_0, s_1)}}{2}$, $\lambda_2 = \dfrac{ab + 1 - \sqrt{\Delta_{a,b}(s_0, s_1)}}{2}$, $\Phi_n(\lambda_1, \lambda_2) = \lambda_1^n - \lambda_2^n$, $\Delta_n(\lambda_1, \lambda_2) = \Phi_n(\lambda_1, \lambda_2) - s_0 \Phi_{n-1}(\lambda_1, \lambda_2) = (\lambda_1 - s_0)\lambda_1^{n-1} - (\lambda_2 - s_0)\lambda_2^{n-1}$ and $\Omega_n(\lambda_1, \lambda_2) = \Phi_n(\lambda_1, \lambda_2) - (ab + s_1)\Phi_{n-1}(\lambda_1, \lambda_2) = (\lambda_1 - ab - s_1)\lambda_1^{n-1} - (\lambda_2 - ab - s_1)\lambda_2^{n-1}$.

When $\Delta_{a,b}(s_0, s_1) = 0$, *we have*

$$\begin{cases} G_{2n+1}^{(2,[a])} = bs_0 n\lambda^{n-1} G_0^{(2,[a])} + (n\lambda^{n-1} - s_0(n-1)\lambda^{n-2}) G_1^{(2,[a])}, \\ G_{2n}^{(2,[a])} = (n\lambda^{n-1} - (ab+s_1)(n-1)\lambda^{n-2}) G_0^{(2,[a])} + a(n-1)\lambda^{n-2} G_1^{(2,[a])}, \end{cases}$$

where $\lambda = \dfrac{ab + s_0 + s_1}{2}$.

5 Some considerations on the 3-periodic linear recursive sequences

Let $\{G_n^{(3,[a])}\}_{n\geq 0}$ be the generalized 3-periodic linear recursive sequence defined as in (4). It was established in Section 2 that the matrix formulation of expression the 3-periodic Fibonacci sequence $\{G_n^{(3,[a])}\}_{n\geq 0}$ is given by (5). More precisely, this matrix expression is given by $Z_{n+1}^{(3,[a])} = AZ_n^{(3,[a])}$, where $A = A_2 A_1 A_0$. For reason of simplicity, we set $a_{01} = \alpha_1$, $a_{02} = \beta_1$, $a_{11} = \alpha_2$, $a_{12} = \beta_2$ and $a_{21} = \alpha_3$, $a_{22} = \beta_3$. Then, a direct matrix computation permits us to get

$$A = A_2.A_1.A_0 = \begin{bmatrix} (\alpha_2\alpha_1 + \beta_2)\alpha_3 + \alpha_1\beta_3 + 1 & (\alpha_2\beta_1 + 1)\alpha_3 + \beta_1\beta_3 & \alpha_2\alpha_3 \\ \alpha_1\alpha_2 + \beta_2 & \alpha_2\beta_1 + 1 & \alpha_2 \\ \alpha_1 & \beta_1 & 1 \end{bmatrix}.$$

We have the following corollary of Proposition 3.1.

Corollary 5.1. *For every* $n \geq 3$, *we have* $A^n = u_n A^{(0)} + u_{n-1} A^{(1)} + u_{n-2} A^{(2)}$, *where*

$$\begin{cases} A^{(0)} = I_3 \text{ and } A^{(1)} = A - a_0 I_3, A^{(2)} = A^2 - a_0 A - a_1 I_3 \\ u_{n+1} = a_0 u_n + a_1 u_{n-1} + a_2 u_{n-2}, \quad \text{for } n \geq 2, \end{cases} \tag{21}$$

where I_3 *is the* 3×3 *identity matrix,* $u_0 = u_1 = 0$ *and* $u_2 = 1$ *are the initial conditions and* a_0, a_1, a_2 *the coefficients of the characteristic polynomial* $P(z) = z^3 - a_0 z^2 - a_1 z - a_2$ *of the matrix* $A = A_2 A_1 A_0$.

A direct computation, using Expression (21), implies that we have

$$A^n = a_2 u_{n-3} I_3 + (u_{n-1} - a_0 u_{n-2})A + u_{n-3} A^2,$$

for every $n \geq 3$. For reason of convenience, we set $A = (\gamma_{h,k})_{1\leq h,k\leq p}$, $A^2 = (\gamma_{h,k}^{(2)})_{1\leq h,k\leq p}$ and $A^n = (\theta_{h,k}(n))_{1\leq h,k\leq p}$, for $n \geq 3$. Therefore, a straightforward computation permits to get

$$\begin{cases} \theta_{h,h}(n) = a_2 u_{n-3} + (u_{n-1} - a_0 u_{n-2})\gamma_{h,h} + u_{n-2}\gamma_{h,h}^{(2)}, & \text{for } 1 \leq h \leq 3 \\ \theta_{h,k}(n) = (u_{n-1} - a_0 u_{n-2})\gamma_{h,k} + u_{n-2}\gamma_{h,k}^{(2)}, & \text{for } 1 \leq h \neq k \leq 3 \end{cases} \tag{22}$$

Moreover, Expression (5) shows that $Z_n^{(3,[a])} = A^n Z_0^{(3,[a])}$, for every $n \geq 0$, where $Z_0^{(3,[a])} = \left[G_2^{(3,[a])}, G_1^{(3,[a])}, G_0^{(3,[a])}\right]^T$. In summary, we get the following property.

Proposition 5.1. *Let* $\{G_n^{(3,[a])}\}_{n\geq 0}$ *be the generalized* 3*-periodic linear recursive sequence defined as in* (4)*, of initial conditions* $G_2^{(3,[a])}$, $G_1^{(3,[a])}$, $G_0^{(3,[a])}$. *Then, the linear recursive formulas of the numbers* $G_{3n+2}^{(3,[a])}$, $G_{3n+1}^{(3,[a])}$ *and* $G_{3n}^{(3,[a])}$ *are given as*

$$\begin{cases} G_{3n+2}^{(3,[a])} = \theta_{1,1}(n)G_2^{(3,[a])} + \theta_{1,2}(n)G_1^{(3,[a])} + \theta_{1,3}(n)G_0^{(3,[a])}, \\ G_{3n+1}^{(3,[a])} = \theta_{2,1}(n)G_2^{(3,[a])} + \theta_{2,2}(n)G_1^{(3,[a])} + \theta_{2,3}(n)G_0^{(3,[a])}, \\ G_{3n}^{(3,[a])} = \theta_{3,1}(n)G_2^{(3,[a])} + \theta_{3,2}(n)G_1^{(3,[a])} + \theta_{3,3}(n)G_0^{(3,[a])}, \end{cases}$$

where the $\theta_{h,k}(n)$ $(1 \leq h, k \leq 3)$ *are given as in* (22)*.*

Moreover, using Expression (10), we obtain the combinatorial formula of the sequence defined in Expression (21) as follows

$$u_n = \rho(n+1,3) = \sum_{k_0+2k_1+3k_2=n-2} \frac{(k_0+k_1+k_2)!}{k_0!k_1!k_2!} a_0^{k_0} a_1^{k_1} a_2^{k_2}, \tag{23}$$

for every $n \geq 1$, with $u_2 = 1$ and $u_0 = u_1 = 0$. Therefore, Expression (23) and Proposition 5.1 permit us to have the following proposition.

Proposition 5.2. *Let* $\{G_n^{(3,[a])}\}_{n\geq 0}$ *be the generalized* 3*-periodic linear recursive sequence defined as in* (4)*, of initial conditions* $G_2^{(3,[a])}$, $G_1^{(3,[a])}$, $G_0^{(3,[a])}$. *Then, the combinatorial recursive formulas of the numbers* $G_{3n+2}^{(3,[a])}$, $G_{3n+1}^{(3,[a])}$ *and* $G_{3n}^{(3,[a])}$ *are given as*

$$\begin{cases} G_{3n+2}^{(3,[a])} = \theta_{1,1}(n)G_2^{(3,[a])} + \theta_{1,2}(n)G_1^{(3,[a])} + \theta_{1,3}(n)G_0^{(3,[a])}, \\ G_{3n+1}^{(3,[a])} = \theta_{2,1}(n)G_2^{(3,[a])} + \theta_{2,2}(n)G_1^{(3,[a])} + \theta_{2,3}(n)G_0^{(3,[a])}, \\ G_{3n}^{(3,[a])} = \theta_{3,1}(n)G_2^{(3,[a])} + \theta_{3,2}(n)G_1^{(3,[a])} + \theta_{3,3}(n)G_0^{(3,[a])}, \end{cases}$$

such that $\theta_{h,h}(n) = a_2\rho(n-2,3) + (\rho(n,3) - a_0\rho(n-1,3))\gamma_{h,h} + \rho(n-1,3)\gamma_{h,h}^{(2)}$ *and* $\theta_{h,k}(n) = (\rho(n-2,3) - a_0\rho(n-3,3))\gamma_{h,k} + \rho(n-3,3)\gamma_{h,k}^{(2)}$*, for* $1 \leq h \neq k \leq 3$*, where* $\rho(n,3)$ *is given as in Expression* (23)*.*

Furthermore, the analytical formula of the generalized 3-periodic linear recursive sequences can be provided, using the roots λ_1, λ_2 and λ_3 of the characteristic polynomial $P(z) = z^3 - a_0z^2 - a_1z - a_2$ of the matrix $A = A_2A_1A_0$. Let consider that the roots of $P(z)$ are simple. Then, the term $u_n = \rho(n+1,3)$ is expressed under the form

$$u_n = \rho(n+1,3) = \sum_{k=1}^{3} \frac{1}{P'(\lambda_k)}\lambda_k^n = \sum_{k=1}^{3} \frac{1}{\prod_{f\neq k}(\lambda_k - \lambda_f)}\lambda_k^n,$$

for every $n \geq 3$, where $P'(z) = \frac{dP}{dz}(z)$ (see, for example, [1,3]). Therefore, have the result.

Proposition 5.3. *Let $\{G_n^{(3,[a])}\}_{n\geq 0}$ be the generalized 3-periodic linear recursive sequence defined as in* (4)*, of initial conditions $G_2^{(3,[a])}$, $G_1^{(3,[a])}$, $G_0^{(3,[a])}$. Then, the analytical recursive formulas of the numbers $G_{3n+2}^{(3,[a])}$, $G_{3n+1}^{(3,[a])}$ and $G_{3n}^{(3,[a])}$ are given by*

$$\begin{cases} G_{3n+2}^{(3,[a])} = \theta_{1,1}(n)G_2^{(3,[a])} + \theta_{1,2}(n)G_1^{(3,[a])} + \theta_{1,3}(n)G_0^{(3,[a])}, \\ G_{3n+1}^{(3,[a])} = \theta_{2,1}(n)G_2^{(3,[a])} + \theta_{2,2}(n)G_1^{(3,[a])} + \theta_{2,3}(n)G_0^{(3,[a])}, \\ G_{3n}^{(3,[a])} = \theta_{3,1}(n)G_2^{(3,[a])} + \theta_{3,2}(n)G_1^{(3,[a])} + \theta_{3,3}(n)G_0^{(3,[a])}, \end{cases}$$

such that

$$\theta_{h,k}(n) = \sum_{d=1}^{3} \frac{\lambda_d^{n-4}}{P'(\lambda_d)} \left[(\lambda_d + a_0)\gamma_{h,k} + \gamma_{h,k}\right],$$

for $1 \leq h,k \leq 3$, with $h \neq k$ and

$$\theta_{h,h}(n) = \sum_{d=1}^{3} \frac{\lambda_d^{n-3}}{P'(\lambda_d)} \left[a_2 + \lambda_d(\lambda_d + a_0)\gamma_{h,h} + \lambda_d\gamma_{h,h}^{(2)}\right],$$

where $\lambda_1, \lambda_2, \lambda_3$ are the simple roots of the characteristic polynomial $P(z) = z^3 - a_0z^2 - a_1z - a_2$ of the matrix $A = A_2A_1A_0$.

The general setting when the roots λ_1, λ_2 and λ_3 of the characteristic polynomial $P(z) = z^3 - a_0z^2 - a_1z - a_2$ (of $A = A_2A_1A_0$) are not all simple, can be studied.

6 Discussion and some considerations

In the previous sections, we have introduced and studied the generalized p-periodic linear recursive sequences, which are defined by means of the higher order of recursiveness and two parameter sequences. We had provided their matrix formulation, where two special cases allows us to illustrate the main role of this formulation. Then, by utilizing the Fibonacci–Horner decomposition we have derived several fundamental properties of these sequences, especially, the combinatorial and the analytical formulations. All results obtained in the study can be varied according to the different values of the integer p and the two parameters sequences. For example, in the main results we have obtained, the special case of $p = 2$ gives various results about the generalized bi-periodic Fibonacci, Lucas, Jacobsthal–Pell sequences studied in [5, 10–13].

In addition, with the generalized 3-periodic linear recursive sequence defined as in (4), we can derive other other 3-periodic linear recursive sequence. Indeed, when $a_{01} = a_{11} = a_{21} = 0$ in (4) and initial conditions $G_0^{(3,[a])} = G_1^{(3,[a])} = G_2^{(3,[a])} = 1$, we deal with the sequence of generalized 3-periodic Padovan numbers, and initial conditions $G_0^{(3,[a])} = 3$, $G_1^{(3,[a])} = 0$, and $G_2^{(3,[a])} = 2$, we obtain the sequence of generalized 3-periodic Perrin numbers, as well as, with initial conditions $G_0^{(3,[a])} = 1$, $G_1^{(3,[a])} = 0$, and $G_2^{(3,[a])} = 1$, we obtain the sequence of generalized 3-periodic Van der Laan numbers.

7 Concluding remarks and perspectives

In the present study, we have generalized the bi-periodic Fibonacci numbers to p-periodic Fibonacci numbers and provide their matrix formulation. Using the Fibonacci–Horner decomposition of the powers of their related matrix, we have established the linear, the combinatorial and the analytical expressions of the sequence of p-periodic linear recursive sequences. The first illustrative case of our results of the general setting, we have considered the special case of bi-periodic Fibonacci numbers. Moreover, to show the efficiency of our general results, we had studied the generalized 3-periodic linear recursive sequences.

To the best of our knowledge, our results are not current in the literature. In addition, with the aid of our approach and method, various Fibonacci-like type algebraic structures can be generalized and their properties exhibited.

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