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On arithmetic Heilbronn supercharacters

Mircea Cimpoeaş [1,2]

[1] National University of Science and Technology Politehnica Bucharest
Splaiul Independenceei no. 313, Bucharest, Romania

[2] Institute of Mathematics of the Romanian Academy
Calea Grivitei Street no. 21, Bucharest, Romania
e-mail: mircea.cimpoeas@imar.ro

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Abstract: In this note, we introduce arithmetic Heilbronn supercharacters that generalize the notions of arithmetic Heilbronn characters and Heilbronn supercharacters and discuss several properties of them.
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1 Introduction

Let $\mathbb{Q} \subset K$ be a number field. In order to study the zeros of the Dedekind zeta function $\zeta_K(s)$, Heilbronn [5] introduced what are now called Heilbronn characters, which allowed him to give a simple proof of the famous Aramata and Brauer Theorem [1, 3], that is $\zeta_K(s)/\zeta(s)$ is entire.

More generally, if $K \subset L$ is a number field externsion, the problem if $\zeta_L(s)/\zeta_K(s)$ is entire is open, and it would be a consequence of the Artin conjecture for L-functions [2]. This connection suggests that the method of Heilbronn is useful in studying L-functions and, indeed, it was used by several authors; see for instance [6].

In [8], P.-J. Wong introduced the so called arithmetic Heilbronn characters which generalize the classical Heilbronn characters and, at the same time, catch almost all properties of them. The aim of this note is to further generalize the arithmetic Heilbronn characters, in the framework of the supercharacter theory, introduced by Diaconis and Isaacs [4].

In Definition 3.2 we introduce the notion of arithmetic Heilbronn supercharacters, which generalizes both the arithmetic Heilbronn characters and the Heilbronn supercharacters; see [7, Section 4.1]. Using the supercharacter theoretic formalism, we prove several generalizations of some classical results: Artin–Takagi decomposition (Theorem 3.6), Heilbronn–Stark Lemma (Theorem 3.7) and Uchida–van der Waall Theorem (Theorem 3.8).

2 Preliminaries

Definition 2.1. *Let G be a finite group. Let $\mathcal{K}$ be a partition of G and let $\mathcal{X}$ be a partition of* $\mathrm{Irr}(G)$*. The ordered pair $C := (\mathcal{X}, \mathcal{K})$ is a* supercharacter theory *if:*

1. *$\{1\} \in \mathcal{K}$,*
2. *$|\mathcal{X}| = |\mathcal{K}|$, and*
3. *for each $X \in \mathcal{X}$, the character $\sigma_X = \sum_{\psi \in X} \psi(1)\psi$ is constant on each $K \in \mathcal{K}$.*

The characters σ_X are called supercharacters*, and the elements K in $\mathcal{K}$ are called* superclasses*. We denote by* $\mathrm{Sup}(G)$ *the set of supercharacter theories of G.*

Diaconis and Isaacs showed their theory enjoys properties similar to the classical character theory. For example, every superclass is a union of conjugacy classes in G; see [4, Theorem 2.2]. The irreducible characters and conjugacy classes of G give a supercharacter theory of G, which will be referred to as the *classical theory* of G.

Also, as noted in [4], every group G admits a non-classical theory with only two supercharacters 1_G and $\mathrm{Reg}(G) - 1_G$ and superclasses $\{1\}$ and $G \setminus \{1\}$, where 1_G denotes the trivial character of G and

$$\mathrm{Reg}(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$$

is the regular character of G. This theory will be called the *maximal theory* of G.

We recall that a character λ of G is called *linear*, if $\lambda(1) = 1$, that is $\lambda : G \to \mathbb{C}^*$ is a group homomorphism. Also, if χ is a character of G, then χ is written uniquely as a linear combination

$$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_r\chi_r,$$

where a_i are non-negative integers and $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$. The constituents of χ are those irreducible characters χ_i for which $a_i > 0$, where $1 \leq i \leq r$. We mention also that $\mathrm{Irr}(G)$ span the space of complex-valued functions on G which are constant on the conjugacy classes of G.

We note that from the definition, every supercharacter is a character. Moreover, according to [4, Theorem 2.2], if $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory, then the characters σ_X for $X \in \mathcal{X}$ span the space of all complex-valued functions on G that are constant on the member of $\mathcal{K}$.

Let $C_G \in \mathrm{Sup}(G)$ be a supercharacter theory of G, $C_G = (\mathcal{X}_G, \mathcal{K}_G)$. Let $g \in G$. We denote by $\mathrm{SCl}_G(g)$, the superclass of G which contains g.

Definition 2.2. ([7, Definition 2.7]) *Let G be a finite group and H be a subgroup of G. Let $C_G \in \operatorname{Sup}(G)$ be a supercharacter theory of G and $C_H \in \operatorname{Sup}(H)$ be a supercharacter theory of H. We say that C_G and C_H are compatible if for any $h \in H$, we have*

$$\operatorname{SCl}_H(h) \subseteq \operatorname{SCl}_G(h).$$

Moreover, if C_H and C_G are compatible and $\Phi : H \to \mathbb{C}$ is a superclass function of H, i.e., a function constant on superclasses of H, then the superinduction $\Phi^G : G \to \mathbb{C}$ is defined by

$$\operatorname{SInd}_H^G \Phi^G(g) = \frac{|G|}{|H| \cdot |\operatorname{SCl}_G(g)|} \sum_{x \in \operatorname{SCl}_G(g)} \Phi^0(x),$$

where $\Phi^0(x)$ denotes $\Phi(x)$ if $x \in H$ and zero otherwise.

Remark 2.3. Let $H_1 \subset H_2 \subset G$ be a chain of subgroups of G and let C_{H_1}, C_{H_2} and C_G be supercharacter theories of H_1, H_2 and, respectively, G. If C_{H_1} and C_{H_2} are compatible and C_{H_2} and C_G are compatible, then C_{H_1} and C_G are also compatible.

We recall the following result, see [7, Proposition 2.14]:

Proposition 2.4 (Super Frobenius Reciprocity)**.** *Let G be a finite group and H be a subgroup of G. Let $C_G \in \operatorname{Sup}(G)$ and $C_H \in \operatorname{Sup}(H)$ such that C_G and C_H are compatible. For all superclass functions Φ on H and all superclass functions θ on G,*

$$\langle \operatorname{SInd}_H^G \Phi^G, \theta \rangle = \langle \Phi, \theta|_H \rangle,$$

where $\theta|_H$ is the restriction of θ from G to H.

As it was noted in [7], the superinduction is unique, in the sense that it satisfies the Super Frobenius Reciprocity. More precisely, if $\Phi \to \Phi^{(G)}$ is another arbitrary map sending superclass functions of H to superclass functions of G such that

$$\langle \Phi^{(G)}, \theta \rangle = \langle \Phi, \theta|_H \rangle,$$

for any super class function θ of G, it follows that $\Phi^{(G)} = \operatorname{SInd}_H^G \Phi^G$.

3 Main results

Definition 3.1. *Let G be a finite group. Let $\mathcal{C} := \{C_H \in \operatorname{Sup}(H) \ : \ H \leqslant G\}$ be a family of supercharacter theories on the all subgroups of G. We say that $\mathcal{C}$ is compatible if, for any subgroups $H_1 \subset H_2$ of G, C_{H_1} and C_{H_2} are compatible.*

We introduce the following generalization of [8, Definition 3.1], in the framework of the supercharacter theory:

Definition 3.2. *Let G be a finite group and $\mathcal{C}$ be a compatible family of supercharacter theories on the subgroups of G. Let*

$$I(G,\mathcal{C}) = \{(H,\sigma) \ : \ \sigma \text{ is a supercharacter of } H\}.$$

Suppose that there is a set of integers $\{n(H,\sigma) \ : \ (H,\sigma) \in I(G,\mathcal{C})\}$ satisfying the following three properties:

$ACH1$ $n(H,\sigma_1+\sigma_2) = n(H,\sigma_1)+n(H,\sigma_2)$ *for any subgroup H of G and any supercharacters σ_1 and σ_2 of H.*

$ACH2$ $n(G,\operatorname{SInd}_H^G \sigma^G) = n(H,\sigma)$ *for any subgroup H of G and any supercharacter σ of H.*

$ACH3$ $n(H,\sigma) \geq 0$ *for any supercharacter σ of a subgroup H of G with linear constituents, that is $\sigma = \lambda_1 + \cdots + \lambda_m$, where λ_i are linear characters of H.*

Then the arithmetic Heilbronn supercharacter of a subgroup H of G associated to $n(H,\sigma)$'s is

$$\Theta_H := \sum_{X\in\mathcal{X}_H} \frac{n(H,\sigma_X)}{\sigma_X(1)}\sigma_X,$$

where $C_H = (\mathcal{X}_H,\mathcal{K}_H)$ and $\sigma_X = \sum_{\chi\in X}\chi(1)\chi$.

Proposition 3.3. *With the above notations, we have that*

$$\Theta_H = \sum_{X\in\mathcal{X}_H} \frac{n(G,\operatorname{SInd}_H^G \sigma_X^G)}{\sigma_X(1)}\sigma_X.$$

Proof. It follows immediately from $ACH2$. □

Example 3.4. Let $K/\mathbb{Q}$ be a Galois extension with Galois group $G := \operatorname{Gal}(K/\mathbb{Q})$. Let $\mathcal{C}$ be a compatible family of supercharacter theories on the subgroups of G. For any subgroup H of G and any supercharacter σ of H, we define:

$$n(H,\sigma) := \operatorname{ord}_{s=s_0} L(s,\sigma,K/K^H),$$

where $L(s,\sigma,K/K^H)$ is the L-Artin function associated to the extension $K^H \subset K$ and $s_0 \in \mathbb{C}\setminus\{1\}$ is a fixed point.

Then the integers $n(H,\sigma)$ satisfy the conditions $ACH1$, $ACH2$ and $ACH3$.

Lemma 3.5. *Let $C = (\mathcal{X},\mathcal{K})$ be a supercharacter theory on G. Then, for any $X\in\mathcal{X}$ we have*

$$\langle\sigma_X,\sigma_X\rangle = \sigma_X(1).$$

Proof. Without any loss of generality, assume that $X = \{\chi_1,\ldots,\chi_m\} \subset \operatorname{Irr}(G) = \{\chi_1,\ldots,\chi_r\}$, that is $\sigma_X = \chi_1(1)\chi_1 + \cdots + \chi_m(1)\chi_m$. It follows that $\sigma_X(1) = \chi_1(1)^2 + \cdots + \chi_m(1)^2$.

On the other hand, we have

$$\langle\sigma_X,\sigma_X\rangle = \sum_{i=1}^m\sum_{j=1}^m \chi_i(1)\chi_j(1)\langle\chi_i,\chi_j\rangle.$$

Since $\operatorname{Irr}(G)$ is an orthonormal basis in the space of complex valued functions, i.e., $\langle\chi_i,\chi_j\rangle = \delta_{ij}$, for all $1\leq i,j\leq m$, we get the required solution. □

Theorem 3.6 (Artin–Takagi decomposition)**.** *We have that:*

$$n(G, \mathrm{Reg}(G)) = \sum_{X \in \mathcal{X}_G} n(G, \sigma_X) = \sum_{X \in \mathcal{X}_G} \frac{n(G, \sigma_X)}{\sigma_X(1)} \langle \sigma_X, \sigma_X \rangle.$$

Proof. Since $\mathrm{Reg}(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi = \sum_{X \in \mathcal{X}_G} \sigma_X$, the formula follows from $AHC1$ and Lemma 3.5. □

Theorem 3.7 (Heilbronn–Stark Lemma)**.** *For every subgroup H of G, one has $\Theta_G|_H = \Theta_H$.*

Proof. By Super Frobenius Reciprocity we have

$$\begin{aligned}
\Theta_G|_H &= \sum_{X \in \mathcal{X}_G} \frac{n(G, \sigma_X)}{\sigma_X(1)} \sigma_X|_H = \sum_{X \in \mathcal{X}_G} \frac{n(G, \sigma_X)}{\sigma_X(1)} \sum_{Y \in \mathcal{X}_H} \langle \sigma_X|_H, \sigma_Y \rangle \sigma_Y \\
&= \sum_{X \in \mathcal{X}_G} \frac{n(G, \sigma_X)}{\sigma_X(1)} \sum_{Y \in \mathcal{X}_H} \langle \sigma_X, \mathrm{SInd}_H^G \sigma_Y^G \rangle \sigma_Y \\
&= \sum_{Y \in \mathcal{X}_H} \left(\sum_{X \in \mathcal{X}_G} \frac{n(G, \sigma_X)}{\sigma_X(1)} \langle \sigma_X, \mathrm{SInd}_H^G \sigma_Y^G \rangle \right) \sigma_Y.
\end{aligned}$$

On the other hand, from $AHC1$ it follows that

$$\begin{aligned}
\sum_{X \in \mathcal{X}_G} \frac{n(G, \sigma_X)}{\sigma_X(1)} \langle \sigma_X, \mathrm{SInd}_H^G \sigma_Y^G \rangle &= n\left(G, \sum_{X \in \mathcal{X}_G} \frac{1}{\sigma_X(1)} \langle \sigma_X, \mathrm{SInd}_H^G \sigma_Y^G \rangle \sigma_X \right) \\
&= n(G, \mathrm{SInd}_H^G \sigma_Y^G).
\end{aligned}$$

Hence, the conclusion follows from Proposition 3.3. □

Theorem 3.8 (Uchida–van der Waall Theorem)**.** *Let H be a subgroup of G such that*

$$\mathrm{SInd}_H^G 1_H = 1_G + \sum_{i \in I} \mathrm{SInd}_{H_i}^G \sigma_i,$$

where H_i are subgroups of G, σ_i are supercharacters of H_i with linear constituents, and I is a finite set of indices. Then

$$n(H, 1_H) \geq n(G, 1_G).$$

Proof. From $ACH1$ and the hypothesis, it follows that

$$n(G, \mathrm{SInd}_H^G 1_H) = n(G, 1_G) + \sum_{i \in I} n(G, \mathrm{SInd}_{H_i}^G \sigma_i). \tag{1}$$

From $ACH2$ and (1), it follows that

$$n(H, 1_H) = n(G, 1_G) + \sum_{i \in I} n(H_i, \sigma_i). \tag{2}$$

The conclusion follows from (2) and $ACH3$. □

Remark 3.9. *If we consider the classical theory on all the subgroups of G, then the hypothesis of Theorem 3.8 is satisfied when G is solvable, according to* [6, Lemma 2.4].

4 Conclusion

We introduced the notion of arithmetic Heilbronn supercharacters and we generalized several previous results in the framework of supercharacter theory, namely Artin–Takagi decomposition, Heilbronn–Stark Lemma and Uchida–van der Waall Theorem.

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