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On an inequality about Euler's totient function

Cheng-Ting Wang

Independent researcher
2F., No. 382, Daye Rd., Beitou Dist., Taipei City 112029, Taiwan
e-mail: ssssasasaaasrhs.triangle@gmail.com

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Abstract: In this paper, we show that when N_k is a primorial and $\varphi(N_k)$ is Euler's totient function, the inequality $\varphi(N_k) < \frac{N_k}{e^\gamma \log\log N_k}$ holds for all positive integer $k > 44$.

Keywords: Euler's totient function, Prime numbers, Primorials.
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1 Introduction

Properties of the arithmetic functions are a central topic of number theory. Often, it is not easy to calculate the value of many common arithmetic functions; besides, the properties of a given arithmetic function, like the bounds of certain functions for a given value, may lead to important results, and even in case without such consequences, such inequalities could still be useful for researchers. As a result, inequalities of arithmetic functions have become a main research area of number theory, and there have been a number of papers on these inequalities, as summarized by Dimitrov in 2024 [2]. The focus of the paper is about one of such inequalities, which is about the maximal value of the Euler's totient function on a specific subset of positive integers called primorials, defined as the product of all prime numbers below a specific value.

Let $\varphi(n)$ be *Euler's totient function* for a positive integer n and γ be the *Euler–Mascheroni constant*. Nicolas [5] showed that the following inequality:

$$\varphi(n) < \frac{n}{e^{\gamma}\log\log n}, \tag{1}$$

holds for infinitely many n unconditionally; however, it is not known if the inequality holds for all primorials. In this paper, we attempt to prove that the inequality holds for all primorials $n = N_m = \prod_{i=1}^{m} p_i$ with $m > 44$.

Unless otherwise specified, $\log x$ indicates the natural logarithm of x, $\log^a x$ indicates $(\log x)^a$, e denotes the base of the natural logarithm, $\varphi(n) = n\prod_{s=1}^{t}(1 - \frac{1}{p_s})$ denotes Euler's totient function for a positive integer $n = \prod_{s=1}^{t} p_s^{a_s}$ whenever $n \geq 2$, $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - \log n)$ denotes the Euler–Mascheroni constant, whose value is approximately $\gamma \approx 0.577215$, $\vartheta(x) = \prod_{p_i \leq x} \log p_i$ denotes the first Chebyshev function for x, p_k denotes the k-th prime number, $N_k = \prod_{i=1}^{k} p_i$ denotes the k-th primorial, i.e., the product of the first k prime numbers.

2 Main results

Lemma 2.1. *For all $x > e + 1$,*

$$0 < \log\Big(\frac{\log x}{\log x - 1}\Big) < \log\Big(\frac{\log(x-1)}{\log(x-1) - 1}\Big).$$

Proof. This follows immediately from the fact that $\frac{\log x}{\log x - 1} = 1 + \frac{1}{\log x - 1}$ is strictly decreasing, and the fact that $x < x + 1$. □

Lemma 2.2. *Assume that p_n is the n-th prime number, then*

$$\Big(1 - \frac{1}{\log(n-1)}\Big)^{(1+\frac{1}{P_n - 1})}$$

is increasing for all $n \geq 4$.

Proof. Assume that for some $n \geq 4$, we have

$$\Big(1 - \frac{1}{\log n}\Big)^{1+\frac{1}{p_{n+1}-1}} \leq \Big(1 - \frac{1}{\log(n-1)}\Big)^{1+\frac{1}{p_n - 1}},$$

then we have

$$\Big(1 + \frac{1}{p_n - 1}\Big)\log\Big(\frac{\log(n-1)}{\log(n-1)-1}\Big) \leq \Big(1 + \frac{1}{p_{n+1} - 1}\Big)\log\Big(\frac{\log n}{\log n - 1}\Big). \tag{2}$$

Since $n \geq 4 > e + 1$, we have

$$0 < \log\Big(\frac{\log n}{\log n - 1}\Big) < \log\Big(\frac{\log(n-1)}{\log(n-1) - 1}\Big)$$

by Lemma 2.1.

Therefore, we have

$$1 + \frac{1}{p_n - 1} < \Big(1 + \frac{1}{p_n - 1}\Big)\frac{\log\frac{\log(n-1)}{\log(n-1)-1}}{\log\big(\frac{\log n}{\log n - 1}\big)} \leq 1 + \frac{1}{p_{n+1} - 1}. \tag{3}$$

But (3) implies that $p_{n+1} < p_n$, which is impossible.

Therefore, we have

$$\Big(1-\frac{1}{\log(n-1)}\Big)^{1+\frac{1}{p_n-1}} < \Big(1-\frac{1}{\log n}\Big)^{1+\frac{1}{p_{n+1}-1}},$$

whenever $n \geq 4$, thus

$$\Big(1-\frac{1}{\log(n-1)}\Big)^{1+\frac{1}{p_n-1}}$$

is increasing for all $n \geq 4$. □

Corollary 2.1. *It holds that*

$$0.73 < \Big(1-\frac{1}{\log(n-1)}\Big)^{(1+\frac{1}{p_n-1})}$$

for all $n \geq 43$.

Proof. By Lemma 2.2,

$$\Big(1-\frac{1}{\log(n-1)}\Big)^{1+\frac{1}{p_n-1}}$$

is increasing for all $n \geq 4$.

By calculation using a table of prime numbers and *Desmos Graphing Calculator*, the 43-rd prime is 191, and we have

$$\Big(1-\frac{1}{\log(43-1)}\Big)^{1+\frac{1}{p_{43}-1}} = \Big(1-\frac{1}{\log(43-1)}\Big)^{1+\frac{1}{191-1}} > 0.73.$$

Therefore,

$$0.73 < \Big(1-\frac{1}{\log(n-1)}\Big)^{(1+\frac{1}{p_n-1})}$$

for all $n \geq 43$. □

Lemma 2.3. *If* $x \geq 8$, *then* $\frac{0.006788}{0.73 \cdot x^{\frac{1}{x-1}} - 1} < \log x$.

Proof. First, define $f(x) = \frac{0.006788}{0.73 \cdot x^{\frac{1}{x-1}} - 1} - \log x$. Then we have the following claim.

> **<u>Claim:</u>** If $x > 1$, then $x^{\frac{1}{x-1}}$ is decreasing.
> <u>*Proof of the Claim:*</u> We have
>
> $$\frac{d}{dx}x^{\frac{1}{x-1}} = x^{\frac{1}{x-1}}\frac{d}{dx}\frac{\log x}{x-1} = x^{\frac{1}{x-1}}\Big(\frac{1}{x(x-1)} - \frac{\log x}{(x-1)^2}\Big).$$
>
> By the fact that $\frac{x-1}{x} < \log x$ for all $0 < x$, we have $\frac{1}{x(x-1)} - \frac{\log x}{(x-1)^2} < 0$, which implies that $\frac{d}{dx}x^{\frac{1}{x-1}} < 0$ whenever $x > 1$. And the claim is proven.

Since $x^{\frac{1}{x-1}}$ is decreasing for all $x > 0$, and $0.73x^{\frac{1}{x-1}} - 1 = 0$ for some $7.329 < x < 7.33$ by calculation using *Desmos Graphing Calculator*, this implies that $0.73 \cdot x^{\frac{1}{x-1}} - 1 < 0$ whenever $x \geq 8$.

Combining this and the fact that $\log x > 0$ for all $x > 1$ implies that $f(x) < 0$ whenever $x \geq 8$. □

Lemma 2.4. *If* $m \geq 43$*, then* $(m - \frac{m}{\log m} + 1) \log p_{m+1} < \vartheta(p_m)$.

Proof. First, define $g(x) = x \frac{\log\log x}{4\log^2 x}$. Then we have the following claim.

Claim: $g(x)$ is increasing when $x > e^e$.
Proof of the Claim: We have

$$g'(x) = \frac{d}{dx}g(x) = \frac{(\log\log x + \frac{1}{\log x})}{4\log^2 x} - \frac{(\log\log x)}{4}\Big(\frac{2}{\log^3 x}\Big) = \frac{(\log\log x)(\log x - 2) + 1}{4\log^3 x}.$$

When $x > e^e$, we have $\log x > e$, which implies that $\log x - 2 > e - 2 > 0$ and $\log\log x > \log e = 1$, thus whenever $x > e^e$, we have

$$g'(x) = \frac{(\log\log x)(\log x - 2) + 1}{4\log^3 x} > \frac{(1)(-1+1)}{4\log^3 x} = 0.$$

Therefore, we have $g'(x) > 0$ for all $x > e^e$, which implies that $g(x)$ is increasing for all $x > e^e$, thus proving the claim.

Also, for $m > 6$, we have the following (Ghosh, [4]):

$$\Big(m - \frac{m}{\log m} + m\frac{\log\log m}{4\log^2 m}\Big)\log p_{m+1} < \vartheta(p_m). \tag{4}$$

Since $g(m)$ is increasing for all $m = x > e^e > 6$, and we have $m\frac{\log\log m}{4\log^2 m} > 1$ for $m \geq 43 > e^e$ by calculation using *Desmos Graphing Calculator*, thus for all $m \geq 43$, we have

$$\Big(m - \frac{m}{\log m} + 1\Big)\log p_{m+1} < \Big(m - \frac{m}{\log m} + m\frac{\log\log m}{4\log^2 m}\Big)\log p_{m+1} < \vartheta(p_m). \tag{5}$$

This completes the proof. □

Theorem 2.1. *If* $m > 44$*, then* $\varphi(N_m) < \frac{N_m}{e^\gamma \log\log N_m}$.

Proof. Assume that $m > 44$ and m is the smallest number such that $\varphi(N_m) > \frac{N_m}{e^\gamma \log\log N_m}$.

Then by definition of m, we have $\varphi(N_{m-1}) < \frac{N_{m-1}}{e^\gamma \log\log N_{m-1}}$.

By the multiplicativity of Euler's totient function, we have

$$\frac{N_m}{e^\gamma \log\log N_m} < \varphi(N_m) = \varphi(N_{m-1})\varphi(p_m) = \varphi(N_{m-1})(p_m - 1) < \frac{N_{m-1}(p_m - 1)}{e^\gamma \log\log N_{m-1}},$$

which implies that

$$\frac{p_m}{e^\gamma \log\log N_m} < \frac{p_m - 1}{e^\gamma \log\log N_{m-1}}. \tag{6}$$

From (6), we have

$$\Big(1 + \frac{1}{p_m - 1}\Big)\log\log N_{m-1} < \log\log N_m.$$

Since $\log N_k = \log \prod_{i=1}^{k} p_i = \sum_{i=1}^{k} \log p_i = \vartheta(p_k)$ for all positive integer k, we have

$$(\vartheta(p_{m-1}))^{(1+\frac{1}{p_m-1})} = (\log N_{m-1})^{(1+\frac{1}{p_m-1})} < \log N_m = \vartheta(p_m). \tag{7}$$

Since $m > 44$, by Lemma 2.3, the fact that $m-1 \geq 44 > 6$, $k \log k < p_k$ when $k \geq 1$ (Rosser, [6]), $p_k < k(\log k + \log\log k)$ when $k \geq 6$ (Rosser, [7]) and $k\Big(1 - \frac{1}{\log k} + \frac{\log\log k}{4\log^2 k}\Big)\log p_{k+1} < \vartheta(p_k)$ when $k \geq 6$ (Ghosh, [4]), (7) implies that

$$\begin{aligned}
\vartheta(p_m) &> (\vartheta(p_{m-1}))^{(1+\frac{1}{p_m-1})} \\
&> \Big(\Big(m - 1 - \frac{m-1}{\log(m-1)} + (m-1)\frac{\log\log(m-1)}{4\log^2(m-1)}\Big)\log p_m\Big)^{1+\frac{1}{p_m-1}} \\
&> \Big(\Big(m - 1 - \frac{m-1}{\log(m-1)} + 1\Big)\log p_m\Big)^{1+\frac{1}{p_m-1}} \\
&> \Big(\Big(m - \frac{m}{\log(m-1)}\Big)(\log m + \log\log m)\Big)^{1+\frac{1}{p_m-1}} \\
&= \Big(\Big(1 - \frac{1}{\log(m-1)}\Big)m(\log m + \log\log m)\Big)^{1+\frac{1}{p_m-1}} \\
&> \Big(\Big(1 - \frac{1}{\log(m-1)}\Big)p_m\Big)^{1+\frac{1}{p_m-1}}.
\end{aligned} \tag{8}$$

Also, since we have $|x - \vartheta(x)| \leq \frac{0.006788x}{\log x}$ for all $x \geq 10544111$ (Dusart, [3]), and since $\vartheta(x) < x$ for all $x < 10^{19}$ (Broadbent *et al.*, [1]), we have $\vartheta(x) \leq x(1 + \frac{0.006788}{\log x})$ for every positive integer x. Thus (8) implies

$$\begin{aligned}
&\Big(\Big(1 - \frac{1}{\log(m-1)}\Big)p_m\Big)^{1+\frac{1}{p_m-1}} < \vartheta(p_m) \leq p_m\Big(1 + \frac{0.006788}{\log p_m}\Big) \\
\implies\ & 0.73 \cdot p_m^{\frac{1}{p_m-1}} < \Big(1 - \frac{1}{\log(m-1)}\Big)^{1+\frac{1}{p_m-1}} p_m^{\frac{1}{p_m-1}} \leq 1 + \frac{0.006788}{\log p_m}.
\end{aligned} \tag{9}$$

But (9) implies that

$$\log p_m < \frac{0.006788}{0.73 \cdot p_m^{\frac{1}{p_m-1}} - 1}. \tag{10}$$

Since $m > 44$, we have $p_m > 193 > 8$, (10) leads to a contradiction with Lemma 2.3.
Therefore, if $m > 44$, then $\varphi(p_m\#) < \frac{p_m\#}{e^\gamma \log\log p_m\#}$. $\square$

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