Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 2, 289–298
DOI: 10.7546/nntdm.2025.31.2.289-298

Identical equations for multiplicative functions

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Received: 10 April 2025 **Accepted:** 4 May 2025 **Online First:** 9 May 2025

Abstract: We examine identical equations for multiplicative functions and certain special cases, such as totients and quadratics. We confine ourselves to identical equations expressing the value $f(mn)$ (or the value $f(m)f(n)$) nontrivially in terms of the values $f(m/a)f(n/b)$ and $f(mn/(ab))$, where $a \mid m$ and $b \mid n$, and holding for all m and n. Particular attention is paid to Busche–Ramanujan type identities. We characterize all functions that satisfy the identical equations. Quasi-multiplicative functions are central to this discussion.

Keywords: Identical equation, Busche–Ramanujan identity, Quasi-multiplicative function, Multiplicative function, Quadratic, Totient.
2020 Mathematics Subject Classification: 11A25.

1 Introduction

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Furthermore, a multiplicative function f is said to be completely multiplicative if $f(mn) = f(m)f(n)$ for all positive integers m and n. See [18, 28].

An arithmetical function f is said to be quasi-multiplicative [15, p. 184] if $f(1) \neq 0$ and $f(1)f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Quasi-multiplicative functions f with $f(1) = 1$ are the multiplicative functions. It is easy to see that f is quasi-multiplicative if and only if

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$f(1) \neq 0$ and $f/f(1)$ is multiplicative. Furthermore, a quasi-multiplicative function f is said to be completely quasi-multiplicative if $f(1)f(mn) = f(m)f(n)$ for all positive integers m and n. Completely quasi-multiplicative f with $f(1) = 1$ are the completely multiplicative functions. It is easy to see that f is completely quasi-multiplicative if and only if $f(1) \neq 0$ and $f/f(1)$ is completely multiplicative. See [8, 11].

The Dirichlet product of arithmetical functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

The unitary product of f and g is defined by

$$(f \oplus g)(n) = \sum_{d\|n} f(d)g(n/d),$$

where the sum goes over the positive unitary divisors d of n (i.e., $d > 0$, $d|n$, $(d, n/d) = 1$). The function δ defined by $\delta(1) = 1$ and $\delta(n) = 0$ for $n \neq 1$ serves as the identity under Dirichlet and unitary products. The symbol f^{*-1} is used for the Dirichlet inverse of f, and the unitary inverse of f is denoted by $f^{\oplus-1}$. The Dirichlet and unitary inverses exist if and only if $f(1) \neq 0$. See [12, 18].

An arithmetical function f is said to be a rational arithmetical function of order (r, s) (see [28]) if there exist nonnegative integers r and s and completely multiplicative functions $g_1, g_2, \ldots, g_r$ and $h_1, h_2, \ldots, h_s$ such that

$$f = g_1 * g_2 * \cdots * g_r * h_1^{-1} * h_2^{-1} * \cdots * h_s^{-1}.$$

Rational arithmetical functions of order $(1, 1)$ are totients [9], and rational arithmetical functions of order $(2, 0)$ are quadratics or specially multiplicative functions [10]. An arithmetical function f is said to be a quasi-rational arithmetical function of order (r, s) if there exist nonnegative integers r and s and completely quasi-multiplicative functions $g_1, g_2, \ldots, g_r$ and $h_1, h_2, \ldots, h_s$ such that

$$f = g_1 * g_2 * \cdots * g_r * h_1^{-1} * h_2^{-1} * \cdots * h_s^{-1}.$$

An arithmetical function f is a quasi-rational arithmetical function of order (r, s) if and only if $f(1) \neq 0$ and $f/f(1)$ is a rational arithmetical function of order (r, s). Quasi-rational arithmetical functions of order $(1, 1)$ are quasi-totients, and quasi-rational arithmetical functions of order $(2, 0)$ are quasi-quadratics or specially quasi-multiplicative functions. See [8, 16].

Multiplicative functions constitute perhaps the most important subclass of the class of arithmetical functions. For example, Euler's totient function φ is a multiplicative function. In fact, it is a totient function, that is, a rational arithmetical function of order $(1, 1)$. Quasi-multiplicative functions appear also in mathematical literature. For example, the function $S(n) = n\varphi(n)/2$ is a quasi-multiplicative function. To be more precise, $S(n)$ is a quasi-totient. The function $S(n)$ counts the sum of the elements in the set $\{x\colon 1 \leq x \leq n, (x, n) = 1\}$, see [2].

This paper focuses on identical equations for multiplicative functions and certain subclasses thereof. An identical equation is an identity that holds for all functions in the given class.

We consider only identities that express the value $f(mn)$ (or the value $f(m)f(n)$) nontrivially in terms of the values $f(m/a)f(n/b)$ and $f(mn/(ab))$, where $a \mid m$ and $b \mid n$, and hold for all m and n. An identical equation for completely multiplicative functions f is simply $f(mn) = f(m)f(n)$. Multiplicative functions f satisfy $f(mn) = f(m)f(n)$ possibly only for positive integers m and n with $(m, n) = 1$. Therefore, following Vaidyanathaswamy, we say that $f(mn) = f(m)f(n)$ is a restricted identical equation for multiplicative functions.

The starting point of the study of identical equations for multiplicative functions is the identity

$$f(mn) = \sum_{a|m} \sum_{b|n} f(m/a)f(n/b)f^{*-1}(ab)C(a, b), \tag{1}$$

where

$$C(a, b) = \begin{cases} (-1)^{\omega(b)} & \text{if } \gamma(a) = \gamma(b), \\ 0 & \text{otherwise,} \end{cases}$$

for all positive integers m and n, given by R. Vaidyanathaswamy. Here $\omega(n)$ and $\gamma(n)$ denote respectively the number and the product of the distinct prime divisors of $n > 1$, with $\omega(1) = 0$ and $\gamma(1) = 1$. Vaidyanathaswamy [27, 28, p. 645] proved (1) in two different ways, and later A. A. Gioia [6] introduced a third proof. Vaidyanathaswamy refers to (1) as the identical equation for multiplicative functions.

This paper provides a comprehensive survey of identical equations of the type described above for multiplicative functions and characterizes the functions that satisfy them. A central focus is placed on quasi-multiplicative functions, as these play a significant role in the theory. In fact, the identical equations for multiplicative functions in this paper are identities for quasi-multiplicative functions. For example, an arithmetical function f with $f(1) \neq 0$ (and thus possessing the Dirichlet inverse) satisfies (1) if and only if f is quasi-multiplicative. Particular attention is given to identities like the Busche–Ramanujan type and their implications for special classes of rational arithmetical functions, such as quadratics. We also recall two fundamental identities, although they do not fall within the main theme of this paper. In this paper, we consider only arithmetical functions of one variable. For identical equations involving arithmetical functions of several variables, we refer the reader to [17, 26].

2 Identical equations

As noted in the introduction, the study of identical equations begun from Vaidyanathaswamy's identity (1). This identity may be considered a cornerstone of identical equations for multiplicative functions. M. V. Subbarao and A. A. Gioia [25, Theorem 1] gave a generalization of this identity relating to generalized convolutions. Krishna [14] presented another proof for the generalized identity. The generalized identity is given as

$$f(mn) = \sum_{a|m} \sum_{b|n} f(m/a)f(n/b)f^{K-1}(ab)K((mn/ab, ab))K((m/a, n/b))C(a, b), \tag{2}$$

where K is an arithmetical function satisfying $K(1) = 1$ and

$$K((a, b))K((ab, c)) = K((a, bc))K((b, c))$$

for all positive integers a, b, c. Here f^{K-1} is the inverse of f with respect to the K-product [7]. A generalized, structured version of the K-product and equation (2) is given by McCarthy [18, Chapter 4].

M. V. Subbarao and A. A. Gioia [25, Theorem 2] gave also two further identities for multiplicative functions; these identities involve unitary divisors. In fact, they proved that every multiplicative function f satisfies the identities

$$f(mn) = \sum_{\substack{a\|m \\ \gamma(a)|\gamma((m,n))}} \sum_{\substack{b\|n \\ \gamma(b)|\gamma((m,n))}} f(m/a)f(n/b)f^{\oplus-1}(ab)\lambda(a,b), \tag{3}$$

$$f(mn) = \sum_{\substack{a\|m \ b\|n \\ \gamma(a)|\gamma(b)|\gamma((m,n))}} \sum f(m/a)f(n/b)f(ab)(-1)^{\omega(a)+\omega(b)}, \tag{4}$$

where

$$\lambda(a,b) = \begin{cases} (-1)^{\omega(a)} & \text{if } \gamma(a)|\gamma(b), \\ 0 & \text{otherwise.} \end{cases}$$

Haukkanen [8] presented a generalization of (1) in terms of regular convolutions as

$$f(mn) = \sum_{a\in A(m)} \sum_{b\in A(n)} f(m/a)f(n/b)f^{A-1}(ab)C(a,b) \tag{5}$$

for all positive integers m and n with $m \in A(mn)$, where f^{A-1} is the inverse of f with respect to A-convolution. Here A is Narkiewicz's convolution, for details, see [8, 18]. This is a restricted identical equation and it is studied in detail in [8]. We do not consider this equation in this paper.

Sitaramaiah *et al.* [19, 24] gave a generalization of (1) in terms of ψ-convolutions as

$$f(\psi(m,n)) = \sum_{\psi(a,x)=m} \sum_{\psi(b,y)=n} f(x)f(x)f^{\psi-1}(\psi(a,b))C(a,b), \tag{6}$$

where $f^{\psi-1}$ is the inverse of f with respect to ψ-convolution. Equation (6) has been studied rigorously in [19, 24], and it is not an identical equation of the type considered in this paper.

Vaidyanathaswamy [27, 28, p. 645] also considered identical equations for certain subclasses of the class of multiplicative functions. Specifically, he proved that a multiplicative function f is a quadratic if and only if there is a completely multiplicative function f_a such that

$$f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2)f_a(d) \tag{7}$$

for all positive integers m and n. The function f_a is given by $f_a = g_1g_2$ and is termed as the associated completely multiplicative function. Equation (7) is known as the Busche–Ramanujan identity [4, 18, 20]. Totients satisfy a restricted Busche–Ramanujan identity, which means that (7) holds for all positive integers m and n such that m and n do not contain any common prime factor to the same power. Detailed treatments of restricted Busche–Ramanujan identities can be found in [8, 16, 28].

The inverse form of (7) reads

$$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)\mu(d)f_a(d) \tag{8}$$

for all positive integers m and n, where μ is the Möbius function, or

$$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)f_a^{*-1}(d) \tag{9}$$

for all positive integers m and n. Equations (7), (8) and (9) are equivalent. The divisor functions σ_k and Ramanaujan's τ-function are examples of arithmetical functions satisfying Busche–Ramanujan identities. See [18].

Carroll and Gioia [5] proved that an arithmetical function f is a rational arithmetical function of order $(N, 0)$ if and only if f is multiplicative and

$$f(mn) = \sum_{d|(m,\gamma(n)^N)} \sum_{e|(n,\gamma(m)^N/d)} f(m/d)f(n/e)f^{*-1}(de)C(d,e) \tag{10}$$

for all positive integers m and n. It is easy to see that if $f(1) = 1$, then (10) implies that f is multiplicative. Therefore we can say that an arithmetical function f with $f(1) = 1$ is a rational arithmetical function of order $(N, 0)$ if and only if (10) holds. Equation (10) is referred to as a Busche–Ramanujan identity of order N. The N-fold divisor function $\tau_N(n)$ satisfies a Busche–Ramanujan identity of order N. It is defined as the number of ordered N-tuples $(n_1, n_2, \ldots, n_N)$ of positive integers for which $n_1 n_2 \cdots n_N = n$. This function is also known as the Piltz divisor function and is denoted by $d_N(n)$, see [3,21].

Totients possess various identical equations. We write totients f in the form $f = f_t * f_v^{*-1}$, where f_t and f_v are completely multiplicative functions, termed as integral and inverse parts of f. It is known [9] that an arithmetical function f is a totient if and only if there is a completely multiplicative function h such that

$$f(m)f(n) = \sum_{d|(m,n)} f(mn/d)\mu(d)h(d) \tag{11}$$

for all positive integers m and n. In this case $h = f_v$. It is likewise known [9] that an arithmetical function f is a totient if and only if there is a completely multiplicative function h such that

$$f(mn) = f(m) \sum_{\substack{d|n \\ \gamma(d)|m}} f(n/d)h(d) \tag{12}$$

for all positive integers m and n. In this case $h = f_v$. Euler's totient function φ, Jordan's totient function J_k and Dedekind's totient function ψ are typical examples of totients. See [9,13,18].

A multiplicative function f is over-multiplicative [23] if there exists an arithmetical function F with $F(1) = 1$ such that

$$f(mn) = f(m)f(n)F((m,n)) \tag{13}$$

for all positive integers m and n. It is known [13] over-multiplicative functions are totients and vice versa.

Multiplicative functions satisfy the nice formula

$$f(m)f(n) = f((m,n))f([m,n]) \tag{14}$$

for all positive integers m and n, or

$$f(m)f(n) = f((m,n))f(mn/(m,n)) \tag{15}$$

for all positive integers m and n. These equations are, in fact, characterizations of semi-multiplicative functions. For further details, see [11].

We conclude this section with two fundamental identical equations for multiplicative functions. These identities, however, are not of the specific type considered in the main theme of this paper. The first is the multiplicative function identity on prime powers, which states that a multiplicative function is completely determined by its values on prime powers. Specifically, if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the prime factorization of n, then

$$f(n) = f(p_1^{a_1})f(p_2^{a_2}) \cdots f(p_r^{a_r}).$$

This identity leads naturally to the concept of Selberg multiplicative functions [11, 22]. The second is the Euler product expansion [1]: If f is a multiplicative function and the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

converges absolutely for some $s \in \mathbb{C}$, then it can be expressed as

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right),$$

where the product is over all prime numbers p. These identities highlight the connection between the multiplicative nature of f and the fundamental theorem of arithmetic.

3 Characterizations

In this section we characterize all functions satisfying identities (1), (2), (3), (4), (7), (10), (11), (12) and (13). It appears that these identities provide new insights into the structure of quasi-multiplicative functions.

Theorem 3.1. *Suppose that $f(1) \neq 0$. Then f satisfies* (1) *if and only if f is quasi-multiplicative.*

Proof. Suppose that f satisfies (1). Then taking $(m,n)=1$ in (1) gives $f(mn)=f(m)f(n)f^{*-1}(1)$, i.e., $f(1)f(mn) = f(m)f(n)$, hence f is quasi-multiplicative.

Conversely, suppose that f is quasi-multiplicative. Then $f/f(1)$ is multiplicative and consequently it satisfies (1). It is easy to see that $(f/f(1))^{*-1} = f(1)f^{*-1}$. We thus see that f satisfies (1). □

Theorems 3.2 and 3.3 can be proved practically in the same way as Theorem 3.1.

Theorem 3.2. *Suppose that $f(1) \neq 0$. Then f satisfies* (2) *if and only if f is quasi-multiplicative.*

Theorem 3.3. *Suppose that $f(1) \neq 0$. Then f satisfies* (3) *if and only if f is quasi-multiplicative.*

Theorem 3.4. *An arithmetical function f satisfies* (4) *if and only if f is identically 0 or f is quasi-multiplicative with $f(1) = \pm 1$.*

Proof. Suppose that f satisfies (4). Taking $m = n = 1$ in (4) gives $f(1) = f(1)^3$, i.e., $f(1) = 0$ or $f(1) = \pm 1$. If $f(1) = 0$, taking $m = 1$ in (4) gives $f(n) = f(n)f(1)^2 = 0$ for all positive integers n. If $f(1) = \pm 1$, then taking $(m, n) = 1$ we obtain $f(mn) = \pm f(m)f(n)$, i.e., $f(1)f(mn) = f(m)f(n)$, which means that f is quasi-multiplicative.

We then prove the converse. If f is identically 0, then both sides of (4) vanish. If f is quasi-multiplicative with $f(1) = 1$, then it is multiplicative and thus (4) holds. If f is quasi-multiplicative with $f(1) = -1$, then $-f$ is multiplicative and and satisfies (4). Multiplying both sides by -1 we see that f satisfies (4). □

Theorem 3.5. *Suppose that $f(1) \neq 0$. Then f is a quasi-quadratic if and only if there is a completely quasi-multiplicative function f_a such that* (7) *holds for all positive integers m and n.*

Proof. If an arithmetical function f with $f(1) \neq 0$ is a quasi-quadratic, then $g = f/f(1)$ is a quadratic. This means that there is a completely multiplicative function g_a such that

$$g(m)g(n) = \sum_{d|(m,n)} g(mn/d^2)g_a(d) \tag{16}$$

for all positive integers m and n, or

$$f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2)f(1)g_a(d)$$

for all positive integers m and n. Denoting $f_a = f(1)g_a$ we see that (7) holds, where f_a is a completely quasi-multiplicative function.

Conversely, assume that there is a completely quasi-multiplicative function f_a such that (7) holds for all positive integers m and n. Taking $m = n = 1$ in (7) shows that $f_a(1) = f(1)$. Then $g = f/f(1)$ satisfies (16), where $g_a = f_a/f(1)$ a completely multiplicative function. Thus $g = f/f(1)$ is a quadratic and, further, f is a quasi-quadratic. □

Remark 3.1. The inverse form of (7) for quasi-quadratics is

$$f(1)^2 f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)\mu(d)f_a(d) \tag{17}$$

for all positive integers m and n, or,

$$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)f_a^{*-1}(d) \tag{18}$$

for all positive integers m and n. This can be verified, for example, by noting that the Dirichlet inverse of a completely quasi-multiplicative function f_a is $f_a^{*-1} = \mu f_a / f_a(1)^2$ and $f_a(1) = g_1(1)g_2(1) = f(1)$.

Theorem 3.6. *Suppose that $f(1) \neq 0$. Then f is a quasi-rational arithmetical function of order $(N, 0)$ if and only if f satisfies identity* (10).

Proof. An arithmetical function f with $f(1) \neq 0$ is a quasi-rational arithmetical function of order $(N, 0)$ if and only if $g = f/f(1)$ is a rational arithmetical function of order $(N, 0)$. This means that

$$g(mn) = \sum_{d|(m,\gamma(n)^N)} \sum_{e|(n,\gamma(m)^N/d)} g(m/d)g(n/e)g^{*-1}(de)C(d,e) \tag{19}$$

for all positive integers m and n. Taking $g = f/f(1)$ and noting that $g^{*-1} = f(1)f^{*-1}$ in (19) we obtain (10), and, conversely, from (10) we obtain (19). □

Theorem 3.7. *Suppose that $f(1) \neq 0$. Then f is a quasi-totient if and only if there is a completely quasi-multiplicative function h such that* (11) *holds for all positive integers m and n.*

Proof. An arithmetical function f with $f(1) \neq 0$ is a quasi-totient if and only if $g = f/f(1)$ is a totient. This means that there is a completely multiplicative function h' such that

$$g(m)g(n) = \sum_{d|(m,n)} g(mn/d)\mu(d)h'(d)$$

for all positive integers m and n, or

$$f(m)f(n) = \sum_{d|(m,n)} f(mn/d)\mu(d)f(1)h'(d) \tag{20}$$

for all positive integers m and n. Denoting $h(n) = f(1)h'(n)$ for all positive integers n we can write (20) in the form (11), where h is a completely quasi-multiplicative function. □

Remark 3.2. If f is a quasi-totient, then (11) can be written as

$$f(m)f(n) = f(1)^2 \sum_{d|(m,n)} f(mn/d)h^{*-1}(d). \tag{21}$$

This follows from the observations $h^{*-1} = \mu h/h(1)^2$ and $h(1) = f(1)$.

Theorem 3.8. *Suppose that $f(1) \neq 0$. Then f is a quasi-totient if and only if there is a completely quasi-multiplicative function h such that* (12) *holds for all positive integers m and n.*

Proof. An arithmetical function f with $f(1) \neq 0$ is a quasi-totient if and only if $g = f/f(1)$ is a totient. This means that there is a completely multiplicative function h' such that

$$g(mn) = g(m) \sum_{\substack{d|n \\ \gamma(d)|m}} g(n/d)h'(d)$$

for all positive integers m and n, or

$$f(mn) = f(m) \sum_{\substack{d|n \\ \gamma(d)|m}} f(n/d)h'(d)/f(1) \tag{22}$$

for all positive integers m and n. Denoting $h(n) = h'(n)/f(1)$ for all positive integers n we can write (22) in the form (12), where h is a completely quasi-multiplicative function. □

Theorem 3.9. *Suppose that $f(1) \neq 0$. Then f is a quasi-totient if and only if there is an arithmetical function F with $F(1) \neq 0$ such that* (13) *holds for all positive integers m and n.*

Proof. An arithmetical function f with $f(1) \neq 0$ is a quasi-totient if and only if $g = f/f(1)$ is a totient. This means that there exists an arithmetical function F' with $F'(1) = 1$ such that

$$g(mn) = g(m)g(n)F'((m,n))$$

for all positive integers m and n. In other words,

$$f(mn) = f(m)f(n)F'((m,n))/f(1) \tag{23}$$

for all positive integers m and n. Denoting $F(n) = F'(n)/f(1)$ for all positive integers n, we can write (23) in the form (13), where F is an arithmetical function such that $F(1) \neq 0$. □

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