Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2025, Volume 31, Number 2, 280–288
DOI: 10.7546/nntdm.2025.31.2.280-288

Weighted sum of the sixth powers of Horadam numbers

Kunle Adegoke [1], **Chiachen Hsu** [2] **and Olawanle Layeni** [3]

[1] Department of Physics and Engineering Physics, Obafemi Awolowo University
220005, Ile-Ife, Nigeria
e-mail: adegoke00@gmail.com

[2] No. 605, Daxue S. Rd., Nanzi District,
Kaohsiung City, Taiwan
e-mail: mcatch7269@gmail.com

[3] Department of Mathematics, Obafemi Awolowo University
220005, Ile-Ife, Nigeria
e-mail: olayeni@oauife.edu.ng

Received: 23 November 2024 **Accepted:** 7 May 2025 **Online First:** 9 May 2025

Abstract: Ohtsuka and Nakamura found simple formulas for $\sum_{j=1}^{n} F_j^6$ and $\sum_{j=1}^{n} L_j^6$, where F_k and L_k are the k-th Fibonacci and Lucas numbers, respectively. In this note we extend their results to a general second order sequence by deriving a formula for $\sum_{j=1}^{n} (-1/q^3)^j w_{j+t}^6$, where $(w_j(w_0, w_1; p, q))$ is the Horadam sequence defined by w_0, w_1; $w_j = pw_{j-1} - qw_{j-2}\,(j \geq 2)$; where t is an arbitrary integer and w_0, w_1, p and q are arbitrary complex numbers, with $p \neq 0$ and $q \neq 0$. As a by-product we establish a divisibility property for the generalized Fibonacci sequence.

Keywords: Fibonacci number, Lucas number, Horadam sequence, Summation identity, Sixth power, Divisibility property.
2020 Mathematics Subject Classification: 11B39, 11B37.

1 Introduction

Let F_k and L_k be the k-th Fibonacci and Lucas numbers.

In the year 2010, Ohtsuka and Nakamura [6] derived the following formulas for the sum of the sixth powers of Fibonacci and Lucas numbers.

$$\sum_{j=1}^{n} F_j^6 = \frac{F_n^5 F_{n+3} + F_{2n}}{4}, \tag{1.1}$$

$$\sum_{j=1}^{n} L_j^6 = \frac{L_n^5 L_{n+3} + 125F_{2n}}{4} - 32. \tag{1.2}$$

We will extend (1.1) and (1.2) as follows:

$$\sum_{j=1}^{n} F_{j+t}^6 = \frac{F_{n+t}^5 F_{n+t+3} - F_t^5 F_{t+3}}{4} + \frac{F_{2n+2t} - F_{2t}}{4}, \tag{1.3}$$

$$\sum_{j=1}^{n} L_{j+t}^6 = \frac{L_{n+t}^5 L_{n+t+3} - L_t^5 L_{t+3}}{4} + \frac{125(F_{2n+2t} - F_{2t})}{4}. \tag{1.4}$$

Our goal in this paper is to develop a formula for

$$\sum_{j=1}^{n} \left(\frac{1}{-q^3}\right)^j w_{j+t}^6,$$

where $(w_j)_{j\in\mathbb{Z}} = (w_j(w_0, w_1; p, q))$ is the Horadam sequence, defined for all integers and arbitrary complex numbers w_0, w_1, $p \neq 0$ and $q \neq 0$, by the recurrence relation

$$w_j = pw_{j-1} - qw_{j-2}, \quad j \geq 2, \tag{1.5}$$

with $w_{-j} = (pw_{-j+1} - w_{-j+2})/q$.

Associated with (w_j) are the Lucas sequences of the first kind, $(u_j(p,q)) = (w_j(0,1;p,q))$, and of the second kind, $(v_j(p,q)) = (w_j(2,p;p,q))$; that is

$$u_0 = 0,\ u_1 = 1, \quad u_j = pu_{j-1} - qu_{j-2}, \quad j \geq 2, \tag{1.6}$$

and

$$v_0 = 2,\ v_1 = p, \quad v_j = pv_{j-1} - qv_{j-2}, \quad j \geq 2, \tag{1.7}$$

with $u_{-j} = (pu_{-j+1} - u_{-j+2})/q$ and $v_{-j} = (pv_{-j+1} - v_{-j+2})/q$.

The closed formula for $w_j(w_0, w_1; p, q)$ in the non-degenerate case, $p^2 - 4q > 0$ is

$$w_j = \frac{A\sigma^j - B\tau^j}{\sigma - \tau}, \tag{1.8}$$

where

$$A = w_1 - w_0\tau, \quad B = w_1 - w_0\sigma, \tag{1.9}$$

and σ and τ are given by

$$\sigma = \sigma(p,q) = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad \tau = \tau(p,q) = \frac{p - \sqrt{p^2 - 4q}}{2}; \tag{1.10}$$

so that

$$\sigma + \tau = p, \quad \sigma - \tau = \sqrt{p^2 - 4q}, \quad \text{and } \sigma\tau = q. \tag{1.11}$$

In particular,

$$u_j = \frac{\sigma^j - \tau^j}{\sigma - \tau}, \quad v_j = \sigma^j + \tau^j. \tag{1.12}$$

Further results on Horadam sequence can be found in the survey paper [5]. Properties of Lucas sequences can be found in [7, Chapter 1].

We require the following telescoping summation identity (see [1]):

$$\sum_{j=1}^{n} \left(\frac{\gamma}{\lambda}\right)^j (\gamma f_{j+1} - \lambda f_j) = \frac{\gamma^{n+1}}{\lambda^n} f_{n+1} - \gamma f_1; \tag{1.13}$$

where (f_j) is a sequence and γ and λ are arbitrary parameters.

As by-products, we will establish the following Fibonacci divisibility properties

$$4 \mid F_{3n}, \text{if } n \text{ is even}; \quad 4 \mid L_{3n}, \text{if } n \text{ is odd}$$

and evaluate the following sums:

$$\sum_{j=1}^{n} G_{6j+t}$$

and

$$\sum_{j=1}^{n} G_{j+t}^5 (G_{j+t+1} + G_{j+t-1}),$$

where G_j is a generalized Fibonacci number, defined by the recurrence relation

$$G_j = G_{j-1} + G_{j-2}, \quad (j \geq 2),$$

where G_0 and G_1 are arbitrary integers.

2 Results

Our main result, an explicit expression for the weighted sum of the sixth powers of Horadam numbers is given in Theorem 2.1, but first we state a couple of required lemmata.

Lemma 2.1. [4, Equations (3.16), (4.5)] *If j and r are integers, then*

$$\begin{aligned} w_{j-r}w_{j+r} &= w_j^2 + eq^{j-r}u_r^2, \\ w_{j+r} + q^r w_{j-r} &= v_r w_j, \end{aligned}$$

where $e = p\,w_0 w_1 - q\,w_0^2 - w_1^2$.

In particular, we have

$$w_{j-1}w_{j+1} = w_j^2 + eq^{j-1}, \tag{2.1}$$

$$w_{j-2}w_{j+2} = w_j^2 + eq^{j-2}p^2, \tag{2.2}$$

$$w_{j+3} + q^3 w_{j-3} = p(p^2 - 3q)w_j. \tag{2.3}$$

Lemma 2.2. *If n and t are integers, then*

$$p\sum_{j=1}^{n}\left(\frac{1}{-q}\right)^{j}w_{j+t}^{2}=\left(\frac{1}{-q}\right)^{n}w_{n+t}w_{n+t+1}-w_{t}w_{t+1}, \tag{2.4}$$

and

$$\begin{aligned}&(p^2-2q)\sum_{j=1}^{n}\left(\frac{1}{-q^2}\right)^{j}w_{j+t}^{4}\\&=\left(\frac{1}{-q^2}\right)^{n}w_{n+t}^{2}w_{n+t+1}^{2}-w_{t}^{2}w_{t+1}^{2}+\frac{2eq^t}{p}\left(\left(\frac{1}{-q}\right)^{n}w_{n+t}w_{n+t+1}-w_{t}w_{t+1}\right).\end{aligned} \tag{2.5}$$

Proof. The recurrence relation of the Horadam sequence allows the following arrangement:

$$pw_{j+t}^{2}=w_{j+t}w_{j+t+1}+qw_{j+t-1}w_{j+t}. \tag{2.6}$$

Use $\gamma=1$, $\lambda=-q$ and $f_j=w_{j+t-1}w_{j+t}$ in (1.13) to obtain

$$\begin{aligned}p\sum_{j=1}^{n}\left(\frac{1}{-q}\right)^{j}w_{j+t}^{2}&=\sum_{j=1}^{n}\left(\frac{1}{-q}\right)^{j}\left(w_{j+t}w_{j+t+1}+qw_{j+t-1}w_{j+t}\right)\\&=\left(\frac{1}{-q}\right)^{n}w_{n+t}w_{n+t+1}-w_{t}w_{t+1}.\end{aligned}$$

Square both sides of (2.6) and use the equation $w_{j+t-1}w_{j+t+1}=w_{j+t}^{2}+eq^{j+t-1}$ to obtain

$$(p^2-2q)w_{j+t}^{4}=w_{j+t}^{2}w_{j+t+1}^{2}+q^{2}w_{j+t-1}^{2}w_{j+t}^{2}+2eq^{j+t}w_{j+t}^{2}.$$

Use $\gamma=1$, $\lambda=-q^2$ and $f_j=w_{j+t-1}^{2}w_{j+t}^{2}$ in (1.13) to obtain

$$\begin{aligned}&(p^2-2q)\sum_{j=1}^{n}\left(\frac{1}{-q^2}\right)^{j}w_{j+t}^{4}-2eq^{t}\sum_{j=1}^{n}\left(\frac{1}{-q}\right)^{j}w_{j+t}^{2}\\&=\sum_{j=1}^{n}\left(\frac{1}{-q^2}\right)^{j}\left(w_{j+t}^{2}w_{j+t+1}^{2}+q^{2}w_{j+t-1}^{2}w_{j+t}^{2}\right)\\&=\left(\frac{1}{-q^2}\right)^{n}w_{n+t}^{2}w_{n+t+1}^{2}-w_{t}^{2}w_{t+1}^{2}.\end{aligned}$$

It follows from (2.4) that

$$\begin{aligned}&(p^2-2q)\sum_{j=1}^{n}\left(\frac{1}{-q^2}\right)^{j}w_{j+t}^{4}\\&=\left(\frac{1}{-q^2}\right)^{n}w_{n+t}^{2}w_{n+t+1}^{2}-w_{t}^{2}w_{t+1}^{2}+\frac{2eq^t}{p}\left(\left(\frac{1}{-q}\right)^{n}w_{n+t}w_{n+t+1}-w_{t}w_{t+1}\right).\end{aligned}$$

This completes the proof. ☐

Theorem 2.1. *If n and t are integers, then*

$$\begin{aligned}\sum_{j=1}^{n}\left(\frac{1}{-q^3}\right)^j w_{j+t}^6 &= \frac{w_{n+t}w_{n+t+3}}{(-q^3)^n p(p^2-3q)}(w_{n+t}^4 + eq^{n+t-2}(p^2+q)w_{n+t}^2 + e^2q^{2n+2t-3}p^2)\\ &\quad - \frac{w_t w_{t+3}}{p(p^2-3q)}(w_t^4 + eq^{t-2}(p^2+q)w_t^2 + e^2q^{2t-3}p^2)\\ &\quad - \frac{eq^{t-2}(p^2+q)}{p^2-2q}\left(\left(\frac{1}{-q^2}\right)^n w_{n+t}^2 w_{n+t+1}^2 - w_t^2 w_{t+1}^2\right)\\ &\quad - \frac{e^2q^{2t-3}(p^4+2q^2)}{p(p^2-2q)}\left(\left(\frac{1}{-q}\right)^n w_{n+t}w_{n+t+1} - w_t w_{t+1}\right).\end{aligned}$$

Proof. Use $\gamma = 1$, $\lambda = -q^3$ and $f_j = w_{j+t-3}w_{j+t-2}w_{j+t-1}w_{j+t}w_{j+t+1}w_{j+t+2}$ in (1.13) to obtain

$$\begin{aligned}&\sum_{j=1}^{n}\left(\frac{1}{-q^3}\right)^j w_{j+t}\left(w_{j+t+3} + q^3 w_{j+t-3}\right) w_{j+t-1}w_{j+t+1}w_{j+t-2}w_{j+t+2}\\ &\quad = \left(\frac{1}{-q^3}\right)^n w_{n+t}w_{n+t+3}w_{n+t-1}w_{n+t+1}w_{n+t-2}w_{n+t+2} - w_t w_{t+3}w_{t-1}w_{t+1}w_{t-2}w_{t+2},\end{aligned}$$

which becomes

$$\begin{aligned}&p(p^2-3q)\sum_{j=1}^{n}\left(\frac{1}{-q^3}\right)^j w_{j+t}^2(w_{j+t}^2 + eq^{j+t-1})(w_{j+t}^2 + eq^{j+t-2}p^2)\\ &\quad = \left(\frac{1}{-q^3}\right)^n w_{n+t}w_{n+t+3}(w_{n+t}^2 + eq^{n+t-1})(w_{n+t}^2 + eq^{n+t-2}p^2)\\ &\qquad - w_t w_{t+3}(w_t^2 + eq^{t-1})(w_t^2 + eq^{t-2}p^2),\end{aligned}$$

with the use of (2.1), (2.2), and (2.3).

Thus,

$$\begin{aligned}&\sum_{j=1}^{n}\left(\frac{1}{-q^3}\right)^j (w_{j+t}^6 + eq^{j+t-2}(p^2+q)w_{j+t}^4 + e^2q^{2j+2t-3}p^2 w_{j+t}^2)\\ &\quad = \frac{w_{n+t}w_{n+t+3}}{(-q^3)^n p(p^2-3q)}(w_{n+t}^4 + eq^{n+t-2}(p^2+q)w_{n+t}^2 + e^2q^{2n+2t-3}p^2)\\ &\qquad - \frac{w_t w_{t+3}}{p(p^2-3q)}(w_t^4 + eq^{t-2}(p^2+q)w_t^2 + e^2q^{2t-3}p^2);\end{aligned}$$

so that

$$\begin{aligned}\sum_{j=1}^{n}\left(\frac{1}{-q^3}\right)^j w_{j+t}^6 &= \frac{w_{n+t}w_{n+t+3}}{(-q^3)^n p(p^2-3q)}(w_{n+t}^4 + eq^{n+t-2}(p^2+q)w_{n+t}^2 + e^2q^{2n+2t-3}p^2)\\ &\quad - \frac{w_t w_{t+3}}{p(p^2-3q)}(w_t^4 + eq^{t-2}(p^2+q)w_t^2 + e^2q^{2t-3}p^2)\\ &\quad - eq^{t-2}(p^2+q)\sum_{j=1}^{n}\left(\frac{1}{-q^2}\right)^j w_{j+t}^4 - e^2q^{2t-3}p^2\sum_{j=1}^{n}\left(\frac{1}{-q}\right)^j w_{j+t}^2;\end{aligned}$$

which upon inserting (2.4) and (2.5) gives the stated result. □

3 Application

We now apply the results of the previous section to derive closed formulas for the finite sums $\sum_{j=1}^{n} G_{j+t}^{6}$, $\sum_{j=1}^{n} G_{6j+t}$ and $\sum_{j=1}^{n} G_{j+t}^{5}\left(G_{j+t+1}+G_{j+t-1}\right)$. We will also establish a couple of Fibonacci divisibility properties.

Let $(G_j(G_0,G_1))_{j\in\mathbb{Z}} = (w_j(G_0,G_1;1,-1))$ be the generalized Fibonacci sequence, the so-called Gibonacci sequence (a name that was coined by Benjamin & Quinn [3, p. 17]), having the same recurrence relation as the Fibonacci sequence but starting with arbitrary initial values; that is, let

$$G_j = G_{j-1} + G_{j-2}, \quad (j \geq 2), \tag{3.1}$$

where G_0 and G_1 are arbitrary numbers (here we shall assume that they are integers) not both zero; with

$$G_{-j} = G_{-(j-2)} - G_{-(j-1)}.$$

The terms of the generalized Fibonacci sequence can be accessed directly through the Binet formula:

$$G_j = \frac{A\alpha^j - B\beta^j}{\alpha - \beta}, \quad j \in \mathbb{Z},$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, and $A = G_1 - G_0\beta$ and $B = G_1 - G_0\alpha$.

It is clear that with the substitution of $p = 1$ and $q = -1$, the identity stated in Theorem 2.1 reduces to the following:

Proposition 3.1. *If n and t are integers, then*

$$\begin{aligned}\sum_{j=1}^{n} G_{j+t}^{6} &= \frac{G_{n+t}^{5}G_{n+t+3} - G_t^5 G_{t+3}}{4} \\ &\quad + \frac{e_G^2\left(G_{n+t}(G_{n+t+1}+G_{n+t-1}) - G_t(G_{t+1}+G_{t-1})\right)}{4},\end{aligned} \tag{3.2}$$

where $e_G = G_0^2 - G_1^2 + G_0G_1$.

Note that, in obtaining the final form of (3.2), we used

$$4G_{s+1} - G_{s+3} = G_{s+1} + G_{s-1}.$$

The Fibonacci and Lucas numbers stated in the Introduction correspond to $(G_j(0,1))_{j\in\mathbb{Z}}$ and $(G_j(2,1))_{j\in\mathbb{Z}}$, respectively.

Our next result, a certain sum of generalized Fibonacci numbers with indices in arithmetic progression is an immediate consequence of (3.2).

Proposition 3.2. *If n and t are integers, then*

$$\sum_{j=1}^{n} G_{6j+t} = \frac{G_{6n+t+3} - G_{t+3}}{4}. \tag{3.3}$$

Proof. Since the sequences $(\alpha^j)_{j\in\mathbb{Z}}$ and $(\beta^j)_{j\in\mathbb{Z}}$ are each a generalized Fibonacci sequence, setting $G_m = \alpha^m$ and $G_m = \beta^m$, in turn, in (3.2) gives

$$\sum_{j=1}^{n} \alpha^{6j+t} = \frac{\alpha^{6n+t+3} - \alpha^{t+3}}{4}$$

and

$$\sum_{j=1}^{n} \beta^{6j+t} = \frac{\beta^{6n+t+3} - \beta^{t+3}}{4},$$

where we wrote t for $6t$ since t is arbitrary and $6t$ occurs in all the exponents. Combining these, using the Binet formula yields (3.3). Note that

$$e_\alpha = \left(\alpha^0\right)^2 - \left(\alpha^1\right)^2 + \alpha^0\alpha^1 = 1 - \alpha^2 + \alpha = 0 = e_\beta. \qquad \square$$

Since the left-hand side of (3.3) is always an integer, we deduce the following divisibility rule.

Proposition 3.3. *If n and t are integers, then*

$$4 \mid G_{6n+t+3} - G_{t+3}. \tag{3.4}$$

Taking (G_j) to be the Fibonacci sequence and using the fact that

$$F_{6n+t+3} - F_{t+3} = \begin{cases} L_{3n+t+3}F_{3n}, & \text{if } n \text{ is even} \\ F_{3n+t+3}L_{3n}, & \text{if } n \text{ is odd} \end{cases},$$

we have

$$4 \mid L_{3n+t+3}F_{3n}, \text{if } n \text{ is even}; \quad 4 \mid F_{3n+t+3}L_{3n}, \text{if } n \text{ is odd}; \tag{3.5}$$

and, since t is an arbitrary integer which can be chosen as $t = -3n - 2$, this gives

$$4 \mid F_{3n}, \text{if } n \text{ is even}; \quad 4 \mid L_{3n}, \text{if } n \text{ is odd}. \tag{3.6}$$

Of course, the same result can be obtained by taking (G_j) to be the sequence of Lucas numbers and using the fact that

$$L_{6n+t+3} - L_{t+3} = \begin{cases} 5F_{3n+t+3}F_{3n}, & \text{if } n \text{ is even} \\ L_{3n+t+3}L_{3n}, & \text{if } n \text{ is odd} \end{cases},$$

so that

$$4 \mid F_{3n+t+3}F_{3n}, \text{if } n \text{ is even}; \quad 4 \mid L_{3n+t+3}L_{3n}, \text{if } n \text{ is odd}, \tag{3.7}$$

giving again (3.6).

Based on a method developed in [2], stated here in Lemma 3.1, we will now derive, from (3.2), a formula for $\sum_{j=1}^{n} G_{j+t}^5 \left(G_{j+t+1} + G_{j+t-1}\right)$.

Lemma 3.1. *Let $g(x)$ be the infinite times differentiable, complex-valued Gibonacci function defined by*

$$g(x) = \frac{A\alpha^x - B\beta^x}{\alpha - \beta}, \quad x \in \mathbb{R}, \tag{3.8}$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, and $A = G_1 - G_0\beta$ and $B = G_1 - G_0\alpha$.

Then

$$g(x)|_{x=j\in\mathbb{Z}} = G_j; \tag{3.9}$$

and

$$\Re\left(\frac{d}{dx}g(x)\bigg|_{x=j\in\mathbb{Z}}\right) = \frac{G_{j+1}+G_{j-1}}{\sqrt{5}}\,\ln\alpha, \tag{3.10}$$

where $\Re(y)$ *denotes the real part of* y.

Proposition 3.4. *If* n *and* t *are integers, then*

$$\begin{aligned}
&\sum_{j=1}^{n} G_{j+t}^5\left(G_{j+t+1}+G_{j+t-1}\right)\\
&\quad=\frac{e_G^2}{24}\left(5G_{n+t}^2+\left(G_{n+t+1}+G_{n+t-1}\right)^2-5G_t^2-\left(G_{t+1}+G_{t-1}\right)^2\right)\\
&\quad\quad+\frac{1}{24}G_{n+t}^4\left(G_{n+t}\left(G_{n+t+4}+G_{n+t+2}\right)+5G_{n+t+3}\left(G_{n+t+1}+G_{n+t-1}\right)\right)\\
&\quad\quad-\frac{1}{24}G_t^4\left(G_t\left(G_{t+4}+G_{t+2}\right)+5G_{t+3}\left(G_{t+1}+G_{t-1}\right)\right).
\end{aligned}\tag{3.11}$$

Proof. The Gibonacci function form of (3.2) is

$$\begin{aligned}
\sum_{j=1}^{n} g(j+t)^6 &= \frac{g(n+t)^5g(n+t+3)-g(t)^5g(t+3)}{4}\\
&\quad+\frac{e_G^2\left(g(n+t)(g(n+t+1)+g(n+t-1))-g(t)(g(t+1+g(t-1)))\right)}{4},
\end{aligned}$$

Differentiating this with respect to t and making use of Lemma 3.1 gives (3.11). ☐

In particular,

$$\begin{aligned}
\sum_{j=1}^{n} F_{j+t}^5L_{j+t} &= \frac{1}{12}\left(L_{2n+2t}-L_{2t}+F_{n+t}^4\left(F_{2n+2t+3}+2F_{n+t+3}L_{n+t}\right)\right)\\
&\quad-\frac{F_t^4}{12}\left(F_{2t+3}+2F_{t+3}L_t\right),
\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
\sum_{j=1}^{n} L_{j+t}^5F_{j+t} &= \frac{25}{12}\left(L_{2n+2t}-L_{2t}\right)+\frac{1}{12}L_{n+t}^4\left(F_{2n+2t+3}+2L_{n+t+3}F_{n+t}\right)\\
&\quad-\frac{L_t^4}{12}\left(F_{2t+3}+2L_{t+3}F_t\right);
\end{aligned}\tag{3.13}$$

with the special values

$$\sum_{j=1}^{n} F_j^5L_j = \frac{1}{12}\left(L_{2n}+F_n^4\left(F_{2n+3}+2F_{n+3}L_n\right)\right)-\frac{1}{6}, \tag{3.14}$$

$$\sum_{j=1}^{n} L_j^5F_j = \frac{1}{12}\left(25L_{2n}+L_n^4\left(F_{2n+3}+2L_{n+3}F_n\right)\right)-\frac{41}{6}. \tag{3.15}$$

Note that, in deriving (3.12) and (3.13), we used

$$L_n^2 + 5F_n^2 = 2L_{2n},$$

and

$$F_m L_n + L_n F_m = 2F_{m+n}.$$

Acknowledgements

The authors thank the anonymous referee for a careful reading and useful suggestions.

References

[1] Adegoke, K. (2018). Weighted sums of some second-order sequences. *The Fibonacci Quarterly*, 56(3), 252–262.

[2] Adegoke, K. (2024). Fibonacci identities via Fibonacci functions. *Journal of Integer Sequences*, 27, Article 24.6.2.

[3] Benjamin, A. T. & Quinn, J. J. (2003). *Proofs that Really Count: The Art of Combinatorial Proof*. The Mathematical Association of America.

[4] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, 3(3), 161–176.

[5] Larcombe, P. J. (2017). Horadam sequences: A survey update and extension. *Bulletin of the Institute of Combinatorics and its Applications*, 80, 99–118.

[6] Ohtsuka, H. & Nakamura, S. (2010). A new formula for the sum of the sixth powers of Fibonacci numbers. *Congressus Numerantium. Proceedings of the thirteenth conference on Fibonacci numbers and their applications*, 201, 297–300.

[7] Ribenboim, P. (2000). *My Numbers, My Friends*. Springer, New York.