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An algorithm for complex factorization of the bi-periodic Fibonacci and Lucas polynomials

Baijuan Shi [1] and **Can Kızılateş** [2]

[1] School of Science, Xi'an University of Posts and Telecommunications
Xi'an, Shaanxi P. R. China
e-mail: 593800425@qq.com

[2] Department of Mathematics, Faculty of Science,
Zonguldak Bulent Ecevit University, Türkiye
e-mail: cankizilates@gmail.com

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Abstract: In this paper, we consider the factorization of generalized sequences, by employing a method based on trigonometric identities. The new method is of reduced complexity and represents an improvement compared to existing results. We establish a connection between the bi-periodic Fibonacci and Lucas polynomials and tridiagonal matrices, which exploits the calculation of eigenvalues of associated tridiagonal matrices.

Keywords: Bi-periodic Fibonacci polynomials, Bi-periodic Lucas polynomials, Tridiagonal matrices, Trigonometric identities, Eigenvalues, Complex factorizations.
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1 Introduction and preliminaries

Recently, the recursive integer numbers and some tridiagonal matrices have attracted continued interest in variety of fields, including algebra and number theory. Usually, meeting of apparently different fields of mathematics leads to new perspectives and reveals some unexpected results. In past decades, many research papers focused on the connection between tridiagonal matrices and integer sequences. For instance, by employing the properties of Chebyshev polynomials, Cahill *et al.* in [5] obtained a series of complex factorization equations of Fibonacci and Lucas numbers by computing eigenvalues and determinants of associated tridiagonal matrices.

In addition, a plethora of research results regarding the factorization, determinants, inversion etc. of tridiagonal matrices have been obtained. (For more details the reader is referred to [1, 2, 4–11, 13, 15–18] and references therein.) In particular, famous Fibonacci numbers and their properties have been studied through properties of tridiagonal matrices. In [5] Cahill *et al.* provided an $n \times n$ tridiagonal matrix $M(n)$ of the form:

$$M(n) = \begin{pmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}_{n\times n},$$

such that $\det(M(n)) = F_{n-1}$. By employing this matrix and constructing additional tridiagonal matrices, they also proved that

$$F_n = \prod_{j=1}^{n-1} \left(1 - 2i\cos\left(\frac{j\pi}{n}\right)\right), n \geq 2,$$

where F_n is Fibonacci number and i is the imaginary number. Later, they considered sets of tridiagonal matrices whose determinants generate linear subsequences of Fibonacci and Lucas numbers. Meanwhile, they focused on complex factorizations of these subsequences (see for example [5, 6]) that led to

$$F_{2n} = \prod_{j=1}^{n-1} \left(3 - 2\cos\left(\frac{j\pi}{n}\right)\right),$$

which was improved by

$$F_{2mn} = F_{2m} \prod_{j=1}^{n-1} \left(L_{2m} - 2\cos\left(\frac{j\pi}{n}\right)\right).$$

For other sequences, such as Lucas or Pell numbers, tridiagonal matrices

$$D(n) = \begin{pmatrix} \frac{1}{2} & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}_{n\times n}, N(n) = \begin{pmatrix} 2i & 1 & & & \\ 1 & 2i & 1 & & \\ & 1 & 2i & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2i \end{pmatrix}_{n\times n}$$

were provided with the property that their determinants form Lucas and Pell numbers, respectively.

By choosing different entries of tridiagonal matrices, similarly other sequences can be derived. Inspired by the aforementioned reference, we change the entries of $M(n)$ as follows:

$$P(n) = \begin{pmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{pmatrix}_{n\times n}.$$

In this case, we obtain that $\det(P(n)) = F_{2n+2}$.

In [17] Strang designed a family of matrices that can result in a series of number sequences. In 2011, the authors of [4] proposed the complex factorization of generalized Lucas sequence by same tridiagonal matrix methods. Jun [14] got the complex factorizations of the generalized Fibonacci sequences. In short, there is a relationship between the determinants of tridiagonal matrices and famous sequences of numbers, and similar methods have been used for calculations. Therefore, it makes sense to establish a different method as well as an algorithm with reduced complexity. In the light of above mentioned articles, in this paper, we will fuse techniques from the areas of both number theory and linear algebra to consider a method for complex factorization of bi-periodic Fibonacci and Lucas numbers. Our method is based on a trigonometric identity and it is essentially different from those of [4, 14, 15, 17, 18]. Furthermore, by choosing different elements in tridiagonal matrices, it can be used to illustrate some properties of second-order linear equations.

Subsequently, we review various of preliminaries regarding the bi-periodic Fibonacci and Lucas polynomials, tridiagonal matrices, trigonometric identities, respectively (see [19]).

Definition 1.1. The bi-periodic Fibonacci polynomial $q_n(x)_{n=0}^{\infty}$ and the bi-periodic Lucas polynomial $l_n(x)_{n=0}^{\infty}$ are respectively defined in the following forms:

$$q_n(x) = \begin{cases} axq_{n-1}(x) + q_{n-2}(x), & n \text{ is even} \\ bxq_{n-1}(x) + q_{n-2}(x), & n \text{ is odd} \end{cases} \quad n \geq 2,$$

$$l_n(x) = \begin{cases} bxl_{n-1}(x) + l_{n-2}(x), & n \text{ is even} \\ axl_{n-1}(x) + l_{n-2}(x), & n \text{ is odd} \end{cases} \quad n \geq 2,$$

with initial values $q_0(x) = 0, q_1(x) = 1, l_0(x) = 2, l_1(x) = ax$.

In particular,

$$\begin{aligned} q_n(x) &= a^{1-\xi(n)}b^{\xi(n)}xq_{n-1}(x) + q_{n-2}(x), \\ l_n(x) &= a^{\xi(n)}b^{1-\xi(n)}xl_{n-1}(x) + l_{n-2}(x), \end{aligned}$$

where and $\xi(n) = \frac{1-(-1)^n}{2}$ is the parity function, namely, when n is an even number, $\xi(n) = 0$, otherwise, $\xi(n) = 1$. Furthermore, when $x = 1$, we can obtain the classical bi-periodic Fibonacci and Lucas numbers defined in [3, 12]. We calculate the following identities:

$$\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n(x) = \sqrt{ab}x\left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} q_{n-1}(x) + \left(\frac{b}{a}\right)^{\frac{\xi(n-3)}{2}} q_{n-2}(x),$$
$$\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n(x) = \sqrt{ab}x\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1}(x) + \left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} l_{n-2}(x).$$

By setting $\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n(x) = Q_n(x)$, $\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n(x) = L_n(x)$, we obtain:

$$Q_n(x) = \sqrt{ab}xQ_{n-1}(x) + Q_{n-2}(x), Q_0(x) = q_0(x) = 0, Q_1(x) = q_1(x) = 1;$$

$$L_n(x) = \sqrt{ab}xL_{n-1}(x) + L_{n-2}(x), L_0(x) = l_0(x) = 2, L_1(x) = \sqrt{ab}x.$$

Thus, the study of bi-periodic Fibonacci and Lucas polynomials $q_n(x), l_n(x)$ is converted to an analogous problem of polynomials $Q_n(x)$ and $L_n(x)$. In the sequel, we study how this property affects the complex factorization of bi-periodic polynomials.

Lemma 1.1 ([19]). *The bi-periodic Fibonacci and Lucas polynomials satisfy the following properties:*

$$\begin{aligned}
q_n(x) &= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}\right),\\
l_n(x) &= \frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}(\alpha^n(x)+\beta^n(x)),\\
q_{n+1}(x)+q_{n-1}(x) &= l_n(x),\\
q_n(x)l_n(x) &= q_{2n}(x),
\end{aligned}$$

with $\alpha(x), \beta(x)$ being the roots of $\lambda^2 - abx\lambda - ab = 0$.

Lemma 1.2. *The following recurrence relations always hold:*

$$\begin{aligned}
q_{2mn}(x) &= l_{2m}(x)q_{2m(n-1)}(x) - q_{2m(n-2)}(x),\\
l_{2mn}(x) &= l_{2m}(x)l_{2m(n-1)}(x) - l_{2m(n-2)}(x).
\end{aligned}$$

Proof. Observe that by Binet formula $\alpha(x)\beta(x) = -ab$, and therefore

$$\begin{aligned}
q_{2mn}(x) &= \frac{a}{(ab)^{mn}}\left(\frac{\alpha^{2mn}(x)-\beta^{2mn}(x)}{\alpha(x)-\beta(x)}\right),\\
l_{2m}(x) &= \frac{1}{(ab)^m}(\alpha^{2m}(x)+\beta^{2m}(x)),
\end{aligned}$$

as well as

$$\begin{aligned}
l_{2m}q_{2m(n-1)} &= \frac{1}{(ab)^m}(\alpha^{2m}(x)+\beta^{2m}(x))\frac{a}{(ab)^{m(n-1)}}\left(\frac{\alpha^{2m(n-1)}(x)-\beta^{2m(n-1)}(x)}{\alpha(x)-\beta(x)}\right)\\
&= \frac{a}{(ab)^{mn}}\left(\frac{\alpha^{2mn}(x)-\beta^{2mn}(x)}{\alpha(x)-\beta(x)}\right) + \frac{a}{(ab)^{m(n-2)}}\left(\frac{\alpha^{2m(n-2)}(x)-\beta^{2m(n-2)}(x)}{\alpha(x)-\beta(x)}\right)\\
&= q_{2mn}(x) + q_{2m(n-2)}(x).
\end{aligned}$$

Hence the identity can be represented in the following form:

$$q_{2mn}(x) = l_{2m}(x)q_{2m(n-1)}(x) - q_{2m(n-2)}(x).$$

In a similar way, we obtain that $l_{2mn}(x)$ satisfies:

$$l_{2mn}(x) = l_{2m}(x)l_{2m(n-1)}(x) - l_{2m(n-2)}(x). \qquad \square$$

Lemma 1.3. *Let*

$$\mathbb{Q}_n = \begin{pmatrix} \sqrt{abx} & i & & & \\ i & \sqrt{abx} & i & & \\ & i & \sqrt{abx} & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & \sqrt{abx} \end{pmatrix},$$

where i is the imaginary number. Then $\det(\mathbb{Q}_n) = Q_{n+1}(x)$.

Lemma 1.4. *Let*

$$\mathbb{Q}_m(n) = \begin{pmatrix} q_{2m}(x) & 0 & & & \\ 0 & l_{2m}(x) & 1 & & \\ & 1 & l_{2m}(x) & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & l_{2m}(x) \end{pmatrix},$$

where i is the imaginary number. Then $\det(\mathbb{Q}_m(n)) = q_{2mn}(x)$.

2 Main results

In this part of our paper we give complex factorizations and their proofs for bi-periodic Fibonacci and bi-periodic Lucas polynomials. Moreover, we provide some examples to verify our main results.

Theorem 2.1. *The bi-periodic Fibonacci polynomial $q_n(x)$ satisfies the identity*

$$q_n(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} \prod_{k=1}^{n-1} \left(\sqrt{abx} + 2i\cos\left(\frac{k\pi}{n}\right)\right), \tag{1}$$

where $i = \sqrt{-1}$.

Proof. From the relation between tridiagonal matrices and polynomials $Q_n(x)$, we can obtain a sequence of tridiagonal matrices $\mathbb{Q}_n(x)$ of the form:

$$\mathbb{Q}_n(x) = \begin{pmatrix} \sqrt{abx} & i & & & \\ i & \sqrt{abx} & i & & \\ & i & \sqrt{abx} & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & \sqrt{abx} \end{pmatrix}.$$

Furthermore, by Lemma 1.3, we have

$$Q_{n+1}(x) = \det(\mathbb{Q}_n(x)) = \prod_{k=1}^{n} \lambda_k, n \geq 1.$$

Next we apply a new method to compute the eigenvalues λ_k of $\mathbb{Q}_n(x)$. Firstly, we present an efficient formula for computing the eigenvalues of tridiagonal matrix $\mathbb{Q}_n(x)$. If $\mathbb{Q}_n(x)u = \lambda u$, then

$$iu_{j-1} + \sqrt{ab}xu_j + iu_{j+1} = \lambda u_j, u_0 = u_{n+1} = 0, 1 \leq j \leq n. \tag{2}$$

The initial values are due to the first and the last row of $\mathbb{Q}_n(x)$. Taking into account the trigonometric identity

$$\sin(j+1)\alpha + \sin(j-1)\alpha = 2\sin j\alpha \cos\alpha,$$

we first multiply by i both sides, and then we add $\sqrt{ab}x \sin j\alpha$. This leads to

$$\begin{aligned} i\sin(j+1)\alpha + i\sin(j-1)\alpha + \sqrt{ab}x\sin j\alpha &= 2i\sin j\alpha\cos\alpha + \sqrt{ab}x\sin j\alpha \\ &= (\sqrt{ab}x + 2i\cos\alpha)\sin j\alpha. \end{aligned}$$

Hence the equation (2) always holds if

$$\lambda_j = \sqrt{ab}x + 2i\cos\alpha, u_j = \sin j\alpha.$$

The boundary condition $u_{n+1} = 0$ implies that $\alpha = \frac{k\pi}{n+1}, 1 \leq k \leq n$.

Since u is a nonzero vector, then n eigenvalues of $\mathbb{Q}_n(x)$ are of the form,

$$\lambda_k = \sqrt{ab}x + 2i\cos\left(\frac{k\pi}{n+1}\right), 1 \leq k \leq n.$$

Thus, we obtain

$$Q_{n+1}(x) = \prod_{k=1}^{n} \left(\sqrt{ab}x + 2i\cos\left(\frac{k\pi}{n+1}\right)\right).$$

It can be also expressed as

$$Q_n(x) = \prod_{k=1}^{n-1} \left(\sqrt{ab}x + 2i\cos\left(\frac{k\pi}{n}\right)\right) = \prod_{k=1}^{n-1} \left(\sqrt{ab}x - 2i\cos\left(\frac{k\pi}{n}\right)\right).$$

The corresponding factorization of the bi-periodic Fibonacci polynomial $q_n(x)$ is constructed as follows

$$q_n(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} Q_n(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n-1)}{2}} \prod_{k=1}^{n-1} \left(\sqrt{ab}x + 2i\cos\left(\frac{k\pi}{n}\right)\right).$$

This completes the proof. $\square$

Example 2.1. By substituing $n = 3$ in (1), we obtain $q_3(x)$ as

$$\begin{aligned} q_3(x) &= \left(\frac{a}{b}\right)^{\frac{\xi(2)}{2}} \prod_{k=1}^{2} \left(\sqrt{ab}x + 2i\cos\left(\frac{k\pi}{3}\right)\right) \\ &= \left(\sqrt{ab}x + 2i\cos\left(\frac{\pi}{3}\right)\right)\left(\sqrt{ab}x + 2i\cos\left(\frac{2\pi}{3}\right)\right) \\ &= \left(\sqrt{ab}x + i\right)\left(\sqrt{ab}x - i\right) \\ &= abx^2 + 1. \end{aligned}$$

Remark 2.1. When we set $a = b = 1$ in Example 2.1, the third term of the Fibonacci polynomial is obtained.

Theorem 2.2. *The bi-periodic Lucas polynomial $l_n(x)$ satisfies the following identity*

$$l_n(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} \prod_{k=1}^{n} \left(\sqrt{ab}x + 2i\cos\frac{(k-\frac{1}{2})\pi}{n}\right), \tag{3}$$

where $i = \sqrt{-1}$.

Proof. Since $\det(\mathbb{L}_n(x)) = L_n(x)$ for

$$\mathbb{L}_n(x) = \begin{pmatrix} \sqrt{ab}x & 2i & & & \\ i & \sqrt{ab}x & i & & \\ & i & \sqrt{ab}x & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & \sqrt{ab}x \end{pmatrix}_{n\times n},$$

we compute the eigenvalues of $\mathbb{L}_n(x)$. From $\mathbb{L}_n(x)v = \lambda v$, we obtain

$$iv_{j-1} + \sqrt{ab}xv_j + iv_{j+1} = \lambda v_j, 1 \leq j \leq n.$$

The boundary conditions are $v_0 = v_2, v_{n+1} = 0$. Taking into account the trigonometric identity

$$\sin(n-j+1-1)\alpha + \sin(n-j+1+1)\alpha = 2\sin(n-j+1)\alpha\cos\alpha,$$

we first multiply by i both sides, and then we add $\sqrt{ab}x\sin(n-j+1)\alpha$. This leads to

$$\begin{aligned} & i\sin(n-j+1-1)\alpha + i\sin(n-j+1+1)\alpha + \sqrt{ab}x\sin(n-j+1)\alpha \\ = {} & 2i\sin(n-j+1)\alpha\cos\alpha + \sqrt{ab}x\sin(n-j+1)\alpha \\ = {} & (\sqrt{ab}x + 2i\cos\alpha)\sin(n-j+1)\alpha. \end{aligned}$$

To ensure the initial values $v_0 = v_2, v_{n+1} = 0$, taking $v_j = \sin(n-j+1)\alpha,\ 1 \leq j \leq n$, and then we obtain

$$\lambda_k = \sqrt{ab}x + 2i\cos\frac{(k-\frac{1}{2})\pi}{n},$$

because $v_0 = v_2$ means $\sin(n-1)\alpha = \sin(n+1)\alpha$, that is $\alpha = \frac{(k-\frac{1}{2})\pi}{n}$.

Thus we obtain

$$L_n(x) = \prod_{k=1}^{n}\left(\sqrt{ab}x + 2i\cos\frac{(k-\frac{1}{2})\pi}{n}\right).$$

The bi-periodic Lucas polynomial $l_n(x)$ satisfies the identity

$$l_n(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} L_n(x) = \left(\frac{a}{b}\right)^{\frac{\xi(n)}{2}} \prod_{k=1}^{n}\left(\sqrt{ab}x + 2i\cos\frac{(k-\frac{1}{2})\pi}{n}\right). \qquad \square$$

Example 2.2. Substituting $n = 3$ in (3), we can get the complex factorization of bi-periodic Lucas polynomials $l_3(x)$ as below

$$
\begin{aligned}
l_3(x) &= \left(\frac{a}{b}\right)^{\frac{\xi(3)}{2}} \prod_{k=1}^{3}\left(\sqrt{ab}x + 2i\cos\frac{(k-\frac{1}{2})\pi}{3}\right) \\
&= \sqrt{\frac{a}{b}}\left(\sqrt{ab}x + 2i\cos\left(\frac{\pi}{6}\right)\right)\left(\sqrt{ab}x + 2i\cos\left(\frac{\pi}{2}\right)\right)\left(\sqrt{ab}x + 2i\cos\left(\frac{5\pi}{6}\right)\right) \\
&= \sqrt{\frac{a}{b}}\left(\sqrt{ab}x + i\sqrt{3}\right)\sqrt{ab}x\left(\sqrt{ab}x - i\sqrt{3}\right) \\
&= ax\left(abx^2 + 3\right).
\end{aligned}
$$

Remark 2.2. When we take $a = b = 1$ in Example 2.1, the third term of the Lucas polynomial can be derived.

Theorem 2.3. *The bi-periodic Fibonacci polynomial $q_{2mn}(x)$ satisfies the identity*

$$
q_{2mn}(x) = q_{2m}(x)\prod_{k=1}^{n-1}\left(l_{2m}(x) + 2\cos\left(\frac{k\pi}{n}\right)\right).
$$

Proof. By Lemma 1.4 and Theorem 2.1, it is clear that the following identity holds

$$
q_{2mn}(x) = q_{2m}(x)\prod_{k=1}^{n-1}\left(l_{2m}(x) + 2\cos\left(\frac{k\pi}{n}\right)\right).
$$

□

Theorem 2.4. *The bi-periodic Lucas polynomial $l_{2mn}(x)$ satisfies the identity*

$$
l_{2mn}(x) = \prod_{k=1}^{n}\left(l_{2m}(x) + 2\cos\left(\frac{(k-\frac{1}{2})\pi}{n}\right)\right).
$$

Proof. Consider the matrix

$$
\mathbb{L}_m(n) = \begin{pmatrix}
l_{2m}(x) & \sqrt{2} & & & \\
\sqrt{2} & l_{2m}(x) & 1 & & \\
 & 1 & l_{2m}(x) & \ddots & \\
 & & \ddots & \ddots & 1 \\
 & & & 1 & l_{2m}(x)
\end{pmatrix}.
$$

Then

$$
\begin{aligned}
&\det(\mathbb{L}_m(1)) = l_{2m}(x), \\
&\det(\mathbb{L}_m(2)) = l_{2m}^2(x) - 2 = l_{4m}(x), \\
&\det(\mathbb{L}_m(n)) = l_{2m}(x)\det(\mathbb{L}_m(n-1)) - \det(\mathbb{L}_m(n-2)).
\end{aligned}
$$

with $\det(\mathbb{L}_m(n)) = l_{2mn}(x)$. Due to Lemma 1.2 and Theorem 2.2, we present in the form:

$$
l_{2mn}(x) = \prod_{k=1}^{n}\left(l_{2m}(x) + 2\cos\left(\frac{(k-\frac{1}{2})\pi}{n}\right)\right).
$$

□

Remark 2.3. Consider the matrices,

$$\mathbb{F}_m(n) = \begin{pmatrix} F_{2m} & 0 & & & \\ 0 & L_{2m} & 1 & & \\ & 1 & L_{2m} & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{pmatrix},$$

and

$$\mathbb{N}_m(n) = \begin{pmatrix} L_{2m} & 2 & & & \\ 1 & L_{2m} & 1 & & \\ & 1 & L_{2m} & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & L_{2m} \end{pmatrix},$$

where F_n and L_n are the Fibonacci and Lucas numbers, respectively. Then we obtain $\det(\mathbb{F}_m(n)) = F_{2mn}$ and $\det(\mathbb{N}_m(n)) = L_{2mn}$. With a similar approach as in this article, we obtain

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{k\pi}{n} \right)$$

and

$$L_{2mn} = \prod_{k=1}^{n} \left(L_{2m} + 2\cos\frac{(2k-1)\pi}{2n} \right).$$

Remark 2.4. Consider the matrix

$$\mathbb{L}_{m,p}(n) = \begin{pmatrix} l_{2m+p}(x) & \sqrt{l_p(x)} & & & \\ \sqrt{l_p(x)} & l_{2m}(x) & 1 & & \\ & 1 & l_{2m}(x) & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & l_{2m}(x) \end{pmatrix}.$$

Using this method, we obtain the complex factorization of the bi-periodic Fibonacci and Lucas polynomials $q_{2mn+p}(x)$, $l_{2mn+p}(x)$, $q_{\alpha k+\beta}(x)$, and $l_{\alpha k+\beta}(x)$. For example, we have $\det(\mathbb{L}_{m,p}(n)) = l_{2mn+p}(x)$. Moreover, readers can check the complex factorization of $l_{2m+p}(x)$ using the method mentioned in this paper.

Remark 2.5. Employing a methodology analogous to that utilized in this study, it is possible to derive complex factorizations of well-known numerical sequences, including the Fibonacci, k-Fibonacci, Pell, Lucas, k-Lucas, and Pell-Lucas number sequences, among others.

Remark 2.6. Utilizing the methodology presented in this paper, readers can analyze the eigenvalues of a special tridiagonal matrix in the following form:

$$\mathbb{D}(n) = \begin{pmatrix} d & 2 & & & & \\ 1 & d & 1 & & & \\ & 1 & d & & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 2 & d \end{pmatrix}.$$

3 Conclusion

In this study, combining matrix theory and number theory, we have given complex factorizations for bi-periodic Fibonacci and Lucas polynomials with the aid of tridiagonal matrices. The proposed method consists of a simpler and shorter algorithm than of those previously studied. Complex factorization of many polynomials and numbers can be obtained according to the special cases of the polynomial of this study. As a result, our results provide a more general approach.

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