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Common values of two k-generalized Pell sequences

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Abstract: Let $k \geq 2$ and let $(P_n^{(k)})_{n\geq 2-k}$ be the k-generalized Pell sequence defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}$$

for $n \geq 2$ with initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_{-1}^{(k)} = P_0^{(k)} = 0, \text{ and } P_1^{(k)} = 1.$$

In this study, we look at the equation $P_n^{(k)} = P_m^{(l)}$ in positive integers n, m, k, l such that $2 \leq l < k$ and show that it has only trivial solution, namely $n = m$.

Keywords: Baker's method, Exponential Diophantine equation, Fibonacci numbers, Lucas numbers, Linear forms in logarithms.
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1 Introduction

Let k be integer with $k \geq 2$. The k-generalized Pell sequence $(P_n^{(k)})_{n\geq 2-k}$ is defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}$$

for $n \geq 2$ with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_{-1}^{(k)} = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. This sequence has been studied recently in various papers (see, for example, [3–5, 15]). In particular, what is of interest are Diophantine equations with members of such a sequence. A similar sequence is the sequence of k-generalized Fibonacci numbers denoted by $(F_n^{(k)})_{n\geq 2-k}$ and defined by

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$$

for $n \geq 2$ with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_{-1}^{(k)} = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. Clearly, for $k = 2$ we obtain the classical Fibonacci sequence $(F_n)_{n\geq 0}$. In [6], Bravo and Luca, and independently in [12], Marques, have found all positive integers which occur in two different generalized Fibonacci sequences, which had been a problem proposed by Noe and Von Post [14]. Additionally, in [4], the authors have determined all the solutions of Diophantine equation

$$P_n^{(k)} = F_m^{(l)}$$

in positive integers $n, k, m, l \geq 2$.

In this paper, we solve the problem treated independently in [6], and [12] for the k-generalized Pell sequences. Here is our theorem.

Theorem 1.1. *Let n, m, k, l be positive integers such that $2 \leq l < k$ and*

$$P_n^{(k)} = P_m^{(l)}. \tag{1}$$

Then $n = m \leq l + 1$.

Kılıç [11] proved that

$$P_n^{(k)} = F_{2n-1} \tag{2}$$

for all $1 \leq n \leq k+1$. In Equation (1), if one of $n \leq k+1$ or $m \leq l+1$ holds, then $P_n^{(k)} = P_m^{(l)}$ is a Fibonacci number. In this case, by the Theorem 1 given in [4], we can see that $n = m \leq \min\{k+1, l+1\}$. Thus, in order to prove our theorem, it suffices to show that there are no solutions with $n \geq k+2$ and $m \geq l+2$.

2 Preliminaries

The characteristic polynomial of the sequence $(P_n^{(k)})_{n\geq 2-k}$ is

$$\Psi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1. \tag{3}$$

From Lemma 1 of [17] we know that this polynomial has exactly one positive real root located between 2 and 3. We denote the roots of the polynomial in (3) by $\alpha_1, \alpha_2, \ldots, \alpha_k$. Particularly, let $\alpha := \alpha(k) = \alpha_1$ denote the positive real root of $\Psi_k(x)$. This is called the dominant root of $\Psi_k(x)$.

The other roots are strictly inside the unit circle. The Binet's formula for the k-generalized Pell numbers appears in [5] and is given by

$$P_n^{(k)} = \sum_{j=1}^{k} \frac{\alpha_j - 1}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n. \tag{4}$$

It was also shown in [5] that the contribution of the roots inside the unit circle to the right-hand side of formula (4) is very small, more precisely the inequality

$$\left| P_n^{(k)} - g_k(\alpha)\alpha^n \right| < \frac{1}{2} \tag{5}$$

holds for all $n \geq 2 - k$, where

$$g_k(z) := \frac{z - 1}{(k+1)z^2 - 3kz + k - 1}. \tag{6}$$

From [3], we can derive the inequality,

$$|g_k(\alpha_j)| < \frac{2}{k-2} \tag{7}$$

valid for all $k \geq 3$ and $2 \leq j \leq k$, and

$$|g_k(\alpha_j)| < 1 \tag{8}$$

valid for $1 \leq j \leq k$. The following relation between α and $P_n^{(k)}$ given in [5]

$$\alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1} \tag{9}$$

holds for all $n \geq 1$.

The following result from the proof of the Lemma 9 given in [15]. In what follows, $\varphi := (1 + \sqrt{5})/2$ is the Golden section.

Lemma 2.1. *Let $n < \varphi^{k/2-2}$ and let α be the dominant root of the polynomial $\Psi_k(x)$. Then*

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi + 2} + \frac{\delta}{\varphi + 2} + \eta\varphi^{2n} + \eta\delta, \tag{10}$$

where δ and η are real numbers such that

$$|\delta| < \frac{\varphi^{2n}}{\varphi^{k/2}} \text{ and } |\eta| < \frac{4k}{\varphi^k}. \tag{11}$$

Furthermore,

$$\left| \alpha^n - \varphi^{2n} \right| < \frac{\varphi^{2n}}{\varphi^{k/2}}. \tag{12}$$

Lemma 2.2 ([5], Lemma 3.2)**.** *Let $k,\ l \geq 2$ be integers. Then*

(a) *If $k > l$, then $\alpha(k) > \alpha(l)$, where $\alpha(k)$ and $\alpha(l)$ are the values of α relative to k and l, respectively.*

(b) $\varphi^2(1 - \varphi^{-k}) < \alpha < \varphi^2$.

(c) $g_k(\varphi^2) = \frac{1}{\varphi+2}$.

(d) $0.276 < g_k(\alpha) < 0.5$.

To solve our equation, we use linear forms in logarithms and Baker's theory. For this, we will give some notations, lemmas and a theorem.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are integers with $\gcd(a_0, \ldots, a_n) = 1$ and $a_0 > 0$ and $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{13}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b \geq 1$, then $h(\eta) = \log\left(\max\left\{|a|, b\right\}\right)$.

We give some properties of the logarithmic height whose proofs can be found in [7]:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^m) &= |m| h(\eta). \end{aligned} \tag{14}$$

Now, from Lemma 6 given in [4], we can deduce the estimate

$$h(g_k(\alpha)) < 5 \log k \text{ for } k \geq 2, \tag{15}$$

which will be used in the proof of Theorem 1.1.

We next give a theorem deduced from Corollary 2.3 of Matveev [13], which provides a large upper bound for the subscript n in Equation (1) (also see Theorem 9.4 in [8]).

Theorem 2.1. *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2 \cdots A_t\right),$$

where $B \geq \max\{|b_1|, \ldots, |b_t|\}$ and $A_i \geq \max\{Dh(\gamma_i),\ |\log \gamma_i|,\ 0.16\}$ for all $i = 1, \ldots, t$.

In [9], Dujella and Pethő proposed a reduction method based on an argument of Baker and Davenport [1]. Later the authors of [2], proved the following lemma, which is an immediate variation of the result due to Dujella and Pethő from [9]. This lemma is based on the theory of continued fractions and will be used to lower the upper bound obtained by Theorem 2.1 for the subscript n in Equation (1).

Lemma 2.3. *Let M be a positive integer, let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from x to the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemma can be found in [16].

Lemma 2.4. *Let $a,\ x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

Finally, we give the following lemma, which can be found in [10].

Lemma 2.5. *If $m \geq 1$, $T \geq (4m^2)^m$ and $\frac{x}{(\log x)^m} < T$, then $x < 2^m \cdot T \cdot (\log T)^m$.*

2.1 The proof of Theorem 1.1

Assume $P_n^{(k)} = P_m^{(l)}$ holds with positive integers m, n, k, l such that $2 \leq l < k$. If $1 \leq n \leq k+1$, then we have $P_m^{(l)} = P_n^{(k)} = F_{2n-1}$ by (2). The equation $P_m^{(l)} = F_{2n-1}$ has only the solution $(m, l, n) = (m, l, m)$ for $1 \leq m \leq l+1$ by Theorem 1 given in [4]. Then we suppose that $n \geq k+2$, which implies that $n \geq 5$. If $1 \leq m \leq l+1$, then we have $P_n^{(k)} = P_m^{(l)} = F_{2m-1}$ by (2). The equation $P_n^{(k)} = F_{2m-1}$ has no solutions by Theorem 1 given in [4] since $n \geq k+2$. Then we suppose that $m \geq l+2$, which implies that $m \geq 4$. Let $\alpha = \alpha(k)$ and $\beta = \alpha(l)$, respectively, be positive real roots of $\Psi_k(x)$ and $\Psi_l(x)$ given in (3). Then $2 < \alpha < \varphi^2 < 3$ and $2 < \beta < \varphi^2 < 3$ by Lemma 2.2 (b). Using (9), we get

$$\alpha^{n-2} \leq P_n^{(k)} = P_m^{(l)} \leq \beta^{m-1},$$

and

$$\beta^{m-2} \leq P_m^{(l)} = P_n^{(k)} \leq \alpha^{n-1}.$$

Performing some calculations, we obtain

$$n \leq 2 + (m-1)\frac{\log \beta}{\log \alpha} \leq 1.8m, \tag{16}$$

and, similarly,

$$m \leq 1.8n. \tag{17}$$

We now rearrange Equation (1) using inequality (5). Thus, we have

$$\begin{aligned} |g_k(\alpha)\alpha^n - g_l(\beta)\beta^m| &= \left|g_k(\alpha)\alpha^n - P_n^{(k)} + P_m^{(l)} - g_l(\beta)\beta^m\right| \\ &\leq \left|P_n^{(k)} - g_k(\alpha)\alpha^n\right| + \left|P_m^{(l)} - g_l(\beta)\beta^m\right| \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned} \tag{18}$$

If we divide both sides of inequality (18) by $g_l(\beta)\beta^m$, from Lemma 2.2, we get

$$\left|g_k(\alpha)(g_l(\beta))^{-1}\alpha^n\beta^{-m} - 1\right| < \frac{1}{g_l(\beta)\beta^m} < \frac{1}{0.276 \cdot \beta^m} < \frac{3.63}{\beta^m}. \tag{19}$$

In order to use Theorem 2.1, we take $t := 3$ and

$$(\gamma_1, b_1) := (\alpha, n), \ (\gamma_2, b_2) := (\beta, -m), \ (\gamma_3, b_3) := \left(g_k(\alpha)(g_l(\beta))^{-1}, 1\right).$$

The number field containing γ_1, γ_2, and γ_3 is $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$, which has degree $D = [\mathbb{K} : \mathbb{Q}] \leq k \cdot l \leq k^2$. We show that

$$\Lambda_1 := g_k(\alpha)(g_l(\beta))^{-1}\alpha^n\beta^{-m} - 1$$

is nonzero. Assume that $\Lambda_1 = 0$. Then

$$\alpha^n g_k(\alpha) = g_l(\beta)\beta^m,$$

that is,

$$\left|\frac{\alpha - 1}{(k+1)\alpha^2 - 3k\alpha + k - 1}\alpha^n\right| = \left|\frac{\beta - 1}{(l+1)\beta^2 - 3l\beta + l - 1}\beta^n\right|. \tag{20}$$

Conjugating the above equality by some automorphism belonging to the Galois group of the splitting field of $\Psi_k(x)$ and $\Psi_l(x)$ over $\mathbb{Q}$ and taking absolute values, we get

$$\left|\frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1}\alpha_i^n\right| = \left|\frac{\beta_j - 1}{(l+1)\beta_j^2 - 3l\beta_j + l - 1}\beta_j^n\right| \tag{21}$$

for $1 \leq i \leq k$ and $1 \leq j \leq l$. Let $\mathbb{L} = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_l)$ be the normal closure of $\mathbb{K}$ and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be elements of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ such that $\sigma_i(\alpha) = \alpha_i$. Then $\sigma_1, \sigma_2, \ldots, \sigma_k$ map the elements from the list in the left of (21) to the same list, as well as to the elements from the list in the right of (21). Since $k > l$, there exist $i \neq j$ in $\{1, 2, \ldots, k\}$ such that $\sigma_i(\beta) = \sigma_j(\beta)$. Now appliying $\sigma_j^{-1}\sigma_i$ to the Equation (20), we get that if we put $\sigma_j^{-1}(\alpha_i) = \alpha_s$, then $s \neq 1$. If it were not, then $\alpha_i = \sigma_j(\alpha_1) = \alpha_j$, which is not possible for $i \neq j$. Thus, $s \neq 1$. Furthermore, since $\sigma_j^{-1}(\sigma_i(\beta)) = \beta$, it follows that

$$\left|\frac{\alpha_s - 1}{(k+1)\alpha_s^2 - 3k\alpha_s + k - 1}\alpha_s^n\right| = \left|\frac{\beta - 1}{(l+1)\beta^2 - 3l\beta + l - 1}\beta^m\right|. \tag{22}$$

Besides, according to (5), we can see that

$$g_l(\beta)\beta^m > P_m^{(l)} - \frac{1}{2} \geq \beta^{m-2} - \frac{1}{2} \geq 2^{m-2} - \frac{1}{2} > 2^2 - \frac{1}{2} > \frac{7}{2},$$

that is, the right side of (22) is greater than $7/2$. But, the left side of (22) is less than 1 since $|\alpha_s| < 1$ for $s \neq 1$ and $|g_k(\alpha_s)| < 1$ by (8). This is impossible. Therefore, $\Lambda_1 \neq 0$.

Moreover, since

$$h(\alpha) = \frac{\log \alpha}{k} < \frac{\log 3}{k}, \quad \text{and} \quad h(\beta) = \frac{\log \beta}{l} < \frac{\log 3}{l}$$

by (13) and

$$h(g_k(\alpha)(g_l(\beta))^{-1}) \leq h(g_k(\alpha)) + h((g_l(\beta)) < 5 \log k + 5 \log l < 10 \log k$$

by (15), we can take $A_1 := k \log 3,\ A_2 := k \log 3$, and $A_3 := 10k^2 \log k$. Also, since $n \leq 1.8m$, it follows that $B := 1.8m$. Thus, taking into account inequality (19) and using Theorem 2.1, we obtain

$$\begin{aligned} \frac{3.63}{\beta^m} &> |\Lambda_1| \\ &> \exp\left(-C \cdot D^2(1 + \log D)(1 + \log 1.8m)\,(k \log 3)\,(k \log 3)\left(10k^2 \log k\right)\right) \end{aligned}$$

and so

$$m \log \beta - \log(3.63) < C \cdot k^4 \cdot 3 \log k \cdot 3 \log m \cdot (k \log 3)\,(k \log 3)\left(10k^2 \log k\right),$$

where $C := 1.4 \cdot 30^6 \cdot 3^{4.5}$ and we have used the fact that $D^2 \leq k^2 l^2 < k^4$, $1 + \log D < 1 + \log k^2 < 3 \log k$ for $k \geq 3$ and $1 + \log 1.8m < 3 \log m$ for $m \geq 4$. From the last inequality, a quick computation with *Mathematica* yields

$$m \log \beta < 1.56 \cdot 10^{13} \cdot k^8 \cdot (\log k)^2 \cdot \log m,$$

or

$$\frac{m}{\log m} < 2.251 \cdot 10^{13} \cdot k^8 \cdot (\log k)^2. \tag{23}$$

By Lemma 2.5, inequality (23) yields that

$$m < 2 \cdot T \cdot \log(T),$$

where $T := 2.251 \cdot 10^{13} \cdot k^8 \cdot (\log k)^2$. Performing the necessary calculations, we get

$$m < 2.39 \cdot 10^{15} \cdot k^8 \cdot (\log k)^3 \tag{24}$$

valid for $k \geq 3$.

Let $k \in [3, 2600]$. Then, we obtain $n < 1.8m < 4.37 \cdot 10^{45}$ from (24). We now reduce this upper bound on n by applying Lemma 2.3. Let

$$z_1 := n \log \alpha - m \log \beta + \log\left[g_k(\alpha)(g_l(\beta))^{-1}\right]$$

and $x := e^{z_1} - 1$. Then, from (19), it is seen that

$$|x| = |e^{z_1} - 1| < \frac{3.63}{\beta^m} < 0.25$$

for $m \geq 4$. Choosing $a := 0.25$, we get the inequality

$$|z_1| = |\log(x+1)| < \frac{\log(100/75)}{(0.25)} \cdot \frac{3.63}{\beta^m} < \frac{4.18}{\beta^m}$$

by Lemma 2.4. Thus, it follows that

$$0 < \left| n\log\alpha - m\log\beta + \log\left[g_k(\alpha)(g_l(\beta))^{-1}\right]\right| < \frac{4.18}{\beta^m}.$$

Dividing this inequality by $\log\beta$, we get

$$0 < |n\gamma - m + \mu| < A \cdot B^{-w}, \tag{25}$$

where

$$\gamma := \frac{\log\alpha}{\log\beta},\ \mu := \frac{\log\left[g_k(\alpha)(g_l(\beta))^{-1}\right]}{\log\beta},\ A := 6.04,\ B := \beta, \text{ and } w := m.$$

It can be easily seen that $\frac{\log\alpha}{\log\beta}$ is irrational. If it were not, then we could write $\frac{\log\alpha}{\log\beta} = \frac{a}{b}$ for some positive integers a and b. This implies that $\alpha^b = \beta^a$, which is wrong since α^b has k conjugates and β^a has $l < k$ conjugates. Now, put

$$M := 4.37 \cdot 10^{45},$$

which is an upper bound on n since $n < 1.8m < 4.37 \cdot 10^{45}$. We find that q_{109}, the denominator of the 109-th convergent of γ exceeds $6M$. We also obtained $\epsilon > 8.02 \cdot 10^{-6}$. Furthermore, a computation with *Mathematica* gives us the inequality

$$m < \frac{\log(Aq_{109}/\epsilon)}{\log B} < 3014.$$

This gives that $n < 1.8m = 5426$ for all $k \in [3, 2600]$. A computation with *Mathematica* gives us that the equation $P_n^{(k)} = P_m^{(l)}$ has no solution in the range $n \geq k+2$ and $m \geq l+2$. This completes the analysis in the case $k \in [3, 2600]$.

From now on, we can assume that $k > 2600$. Then we can see from (24) that the inequality

$$n < 1.8m < 4.302 \cdot 10^{15} \cdot k^8 \cdot (\log k)^3 < \varphi^{k/2-2} < \varphi^{k/2} \tag{26}$$

holds for $k > 2600$. Thus, by Lemma 2.1, we have

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi+2} + \frac{\delta}{\varphi+2} + \eta\varphi^{2n} + \eta\delta, \tag{27}$$

where δ and η are real numbers such that

$$|\delta| < \frac{\varphi^{2n}}{\varphi^{k/2}} \text{ and } |\eta| < \frac{4k}{\varphi^k}. \tag{28}$$

<u>The case $m < \varphi^{l/2-2}$.</u>

In this case, we have

$$g_l(\beta)\beta^m = \frac{\varphi^{2m}}{\varphi+2} + \frac{\delta_1}{\varphi+2} + \eta_1\varphi^{2m} + \eta_1\delta_1, \tag{29}$$

where δ and η are real numbers such that

$$|\delta_1| < \frac{\varphi^{2m}}{\varphi^{l/2}} \quad\text{and}\quad |\eta_1| < \frac{4l}{\varphi^l} \tag{30}$$

by Lemma 2.1.

So, from (27) and (29), we obtain

$$\begin{aligned}\left|g_k(\alpha)\alpha^n-\frac{\varphi^{2n}}{\varphi+2}\right| &\le \frac{|\delta|}{\varphi+2}+|\eta|\,\varphi^{2n}+|\eta|\,|\delta| \\ &\le \frac{\varphi^{2n}}{\varphi^{k/2}}\left(\frac{1}{\varphi+2}+\frac{4k}{\varphi^{k/2}}+\frac{4k}{\varphi^{k}}\right) \\ &< \frac{\varphi^{2n}}{\varphi^{k/2}},\end{aligned}\tag{31}$$

and

$$\begin{aligned}\left|g_l(\beta)\beta^m-\frac{\varphi^{2m}}{\varphi+2}\right| &\le \frac{|\delta_1|}{\varphi+2}+|\eta_1|\,\varphi^{2m}+|\eta_1|\,|\delta_1| \\ &\le \frac{\varphi^{2m}}{\varphi^{l/2}}\left(\frac{1}{\varphi+2}+\frac{4l}{\varphi^{l/2}}+\frac{4l}{\varphi^{l}}\right) \\ &< \frac{9\cdot\varphi^{2m}}{\varphi^{l/2}},\end{aligned}\tag{32}$$

where we have used the fact that

$$\frac{1}{\varphi+2}+\frac{4k}{\varphi^{k/2}}+\frac{4k}{\varphi^{k}}<1 \qquad \text{for } k>2600$$

and

$$\frac{1}{\varphi+2}+\frac{4l}{\varphi^{l/2}}+\frac{4l}{\varphi^{l}}<9 \qquad \text{for } l\ge 2.$$

Using the inequalities (18), (31), and (32), we can see that

$$\begin{aligned}\left|\frac{\varphi^{2n}}{\varphi+2}-\frac{\varphi^{2m}}{\varphi+2}\right| &= \left|-g_k(\alpha)\alpha^n+g_k(\alpha)\alpha^n+\frac{\varphi^{2n}}{\varphi+2}-g_l(\beta)\beta^m+g_l(\beta)\beta^m-\frac{\varphi^{2m}}{\varphi+2}\right| \\ &\le \left|g_k(\alpha)\alpha^n-\frac{\varphi^{2n}}{\varphi+2}\right|+\left|g_l(\beta)\beta^m-\frac{\varphi^{2m}}{\varphi+2}\right|+|g_k(\alpha)\alpha^n-g_l(\beta)\beta^m| \\ &< \frac{\varphi^{2n}}{\varphi^{k/2}}+\frac{9\cdot\varphi^{2m}}{\varphi^{l/2}}+1 \\ &< \varphi^{2m}\left(\frac{1}{\varphi^{2m-2n+k/2}}+\frac{9}{\varphi^{l/2}}+\frac{1}{\varphi^{2m}}\right).\end{aligned}\tag{33}$$

Since $m\ge l+2$, it follows that $2m\ge 2l+4>l/2$. Also, $k>l$ and $P_n^{(k)}=P_m^{(l)}$ implies that $n<m$. Therefore, $\varphi^{2m-2n+k/2}>\varphi^{2+k/2}>\varphi^{2+l/2}$. Thus, from (33), we get

$$\begin{aligned}\left|\frac{\varphi^{2n}}{\varphi+2}-\frac{\varphi^{2m}}{\varphi+2}\right| &< \varphi^{2m}\left(\frac{1}{\varphi^{2+l/2}}+\frac{9}{\varphi^{l/2}}+\frac{1}{\varphi^{l/2}}\right) \\ &\le \varphi^{2m}\left(\frac{\varphi^{-2}+9+1}{\varphi^{l/2}}\right) \\ &< \frac{10.4\cdot\varphi^{2m}}{\varphi^{l/2}}.\end{aligned}\tag{34}$$

Dividing both sides of the above inequality by $\dfrac{\varphi^{2m}}{\varphi+2}$, we get

$$\left|1-\varphi^{2n-2m}\right|<\frac{10.4\cdot(\varphi+2)}{\varphi^{l/2}}<\frac{37.63}{\varphi^{l/2}}.\tag{35}$$

On the other hand, since

$$1-\varphi^{2n-2m}=1-\frac{1}{\varphi^{2m-2n}}>1-\frac{1}{\varphi^2}>0.618>\frac{1}{2},$$

the inequality (35) gives

$$\frac{1}{2}<\frac{37.63}{\varphi^{l/2}},\qquad \text{and so}\qquad l<18.$$

Thus, we obtain $m<\varphi^{l/2-2}<\varphi^7<29.1$, which contradicts the fact that $2600<k\leq n-2<m$.

The case $m>\varphi^{l/2-2}$.

In this case, we have

$$\varphi^{l/2}<6.26\cdot 10^{15}\cdot k^8\cdot(\log k)^3$$

from (24). Since the inequality $6.26\cdot 10^{15}\cdot k^8\cdot(\log k)^3<k^{14}$ holds for $k>2600$, it follows that $\varphi^{l/2}<k^{14}$, which implies that

$$l<41\log k. \tag{36}$$

On the other hand, using (18) and (31), we can see that

$$\begin{aligned}\left|g_l(\beta)\beta^m-\frac{\varphi^{2n}}{\varphi+2}\right| &= \left|g_l(\beta)\beta^m-\frac{\varphi^{2n}}{\varphi+2}+g_k(\alpha)\alpha^n-g_k(\alpha)\alpha^n\right|\\ &\leq \left|g_k(\alpha)\alpha^n-\frac{\varphi^{2n}}{\varphi+2}\right|+|g_l(\beta)\beta^m-g_k(\alpha)\alpha^n|\\ &< \frac{\varphi^{2n}}{\varphi^{k/2}}+1.\end{aligned}$$

Dividing both sides of the last inequality by $\dfrac{\varphi^{2n}}{\varphi+2}$, we get

$$\begin{aligned}\left|\sqrt{5}g_l(\beta)\beta^m\varphi^{-2n+1}-1\right| &< \frac{\varphi+2}{\varphi^{k/2}}+\frac{\varphi+2}{\varphi^{2n}}\\ &< \frac{2\varphi+4}{\varphi^{k/2}}<\frac{7.24}{\varphi^{k/2}},\end{aligned} \tag{37}$$

where we have used the fact that $n\geq k+2$ and so $2n>k/2$. Now, we show that the number $\Lambda_2:=\sqrt{5}g_l(\beta)\beta^m\varphi^{-2n+1}-1$ is nonzero. Assume that $\Lambda_2=0$. Then, we get $5g_l^2(\beta)=\varphi^{4n-2}\beta^{-2m}\in O_{\mathbb{K}}$, where $O_{\mathbb{K}}$ the ring of integers of the field $\mathbb{K}=\mathbb{Q}(\sqrt{5},\beta)$. It is clear that $5g_l^2(\beta)<5/4$ by Lemma 2.2 (d). Also, by (7), we have

$$|g_l(\beta_i)|<\frac{2}{l-2}$$

for $2\leq i\leq l$ and $l\geq 4$. It follows that $5\left|g_l^2(\beta_i)\right|<\frac{20}{(l-2)^2}$. Let $l\geq 7$. Then, we can see that

$$\left|N_{\mathbb{Q}(\beta)/\mathbb{Q}}\left(5g_l^2(\beta)\right)\right|=\prod_{i=1}^{l}5\left|g_l^2(\beta_i)\right|<\frac{5}{4}\cdot\left(\frac{20}{(l-2)^2}\right)^{l-1}<\frac{5}{4}\cdot\left(\frac{4}{5}\right)^6<1.$$

So, $5g_l^2(\beta)$ is not an algebraic integer. This contradicts $5g_l^2(\beta) \in O_{\mathbb{K}}$. Also, it can be checked with *Mathematica* that $5g_l^2(\beta)$ is not an algebraic integer for $l \in \{2,3,4,5,6\}$. Therefore, $\Lambda_2 \neq 0$. Now, we can apply Theorem 2.1. In order to use Theorem 2.1, we take $t := 3$, and

$$(\gamma_1, b_1) := (\beta, m), \ (\gamma_2, b_2) := (\varphi, -2n+1), \ (\gamma_3, b_3) := \left(\sqrt{5}g_l(\beta), 1\right).$$

The number field containing $\gamma_1, \gamma_2, \gamma_3$ is $\mathbb{K}$, which has degree $D = [\mathbb{K} : \mathbb{Q}] \leq 2l$. Moreover, since

$$h(\gamma_1) = h(\beta) = \frac{\log \beta}{l} < \frac{\log 3}{l}, \qquad h(\gamma_2) = h(\varphi) = \frac{\log \varphi}{2},$$

and

$$h(\gamma_3) \leq h(\sqrt{5}) + h(g_l(\beta)) \leq \log \sqrt{5} + 5 \log l < 6.5 \cdot \log l$$

by (14) and (15), we can take $A_1 := \log 9$, $A_2 := l \log \varphi$, and $A_3 := 13 l \log l$. Also, since $m < 1.8n$, we can take $B := 2n - 1$. Thus, taking into account inequality (37) and using Theorem 2.1, we obtain

$$\begin{aligned} 7.24 \cdot \varphi^{-k/2} &> |\Lambda_1| \\ &> \exp\left(-C \cdot (2l)^2 (1 + \log(2n-1))(1 + \log 2l) (\log 9) (l \log \varphi) (13 l \log l)\right), \end{aligned}$$

where $C := 1.4 \cdot 30^6 \cdot 3^{4.5}$. This implies that

$$k < 2.62 \cdot 10^{14} \cdot l^4 \cdot (\log l)^2 \cdot \log n, \tag{38}$$

where we have used the fact that $(1 + \log(2n-1)) < 2 \log n$ for $n \geq k + 2 > 2602$ and $1 + \log 2l < 4 \log l$ for $l \geq 2$.

Now, let

$$z_2 := m \log \beta - (2n-1) \log \varphi + \log(\sqrt{5} g_l(\beta))$$

and $x := 1 - e^{z_2}$. Then

$$|x| = |1 - e^{z_2}| < \frac{7.24}{\varphi^{k/2}} < 0.01,$$

by (37) since $k > 2600$. Choosing $a := 0.01$, we obtain the inequality

$$|z_2| = |\log(x+1)| < \frac{\log(100/99)}{0.01} \cdot \frac{7.24}{\varphi^{k/2}} < \frac{7.28}{\varphi^{k/2}},$$

by Lemma 2.4. That is,

$$0 < \left| m \log \beta - (2n-1) \log \varphi + \log\left(\sqrt{5} g_l(\beta)\right) \right| < \frac{7.28}{\varphi^{k/2}}.$$

Dividing both sides of the above inequality by $\log \varphi$, it is seen that

$$0 < |m\gamma - (2n-1) + \mu| < A \cdot B^{-w}, \tag{39}$$

where

$$\gamma := \frac{\log \beta}{\log \varphi}, \ \mu := \frac{\log\left(\sqrt{5} g_l(\beta)\right)}{\log \varphi}, \ A := 15.13, \ B := \varphi, \text{ and } w := k/2.$$

It is clear that $\frac{\log \beta}{\log \varphi}$ is irrational. If it were not, then $\frac{\log \beta}{\log \varphi} = \frac{a}{b}$ for some positive integers a and b.

Thus, we get that $\beta^b = \varphi^a$, which is false. That is, for $l \geq 3$, β^b has more conjugates than φ^a, whereas for $l = 2$, we have that $\beta = 1 + \sqrt{2}$ is a quadratic unit living in a different quadratic field than φ. Therefore, $\frac{\log \beta}{\log \varphi}$ is irrational. Besides, since $l < 63 \log k$ by (36) and $n < m$, it follows from (36) and (38) that

$$k < 2.62 \cdot 10^{14} \cdot (41 \log k)^4 \cdot (\log (41 \log k))^2 \cdot \log \left(4.302 \cdot 10^{15} \cdot k^8 \cdot (\log k)^3\right),$$

which implies that

$$k < 1.06 \cdot 10^{33}.$$

Substituting this bound of k into (24) and (36), we get

$$m < 1.68 \cdot 10^{285}$$

and

$$l \leq 3118.$$

If we take $M := 1.68 \cdot 10^{285}$, which is an upper bound on m, we find that q_{584}, the denominator of the 584-th convergent of γ exceeds $6M$. Furthermore, a quick computation with *Mathematica* gives us that the value

$$\frac{\log (Aq_{584}/\epsilon)}{\log B}$$

is less than 11320. So, if (39) has a solution, then

$$\frac{k}{2} < \frac{\log (Aq_{585}/\epsilon)}{\log B} < 7736.85,$$

that is, $k \leq 15473$. Hence, from (24), we get $m < 2.39 \cdot 10^{15} \cdot k^8 \cdot (\log k)^3$, which implies that $m < 7.05 \cdot 10^{51}$. Also, $l < 41 \log k < 396$ by (36). If we apply again Lemma 2.3 to inequality (39) with $M := 7.05 \cdot 10^{51}$, we find that q_{114}, the denominator of the 114-th convergent of γ exceeds $6M$. After doing this, then a quick computation with *Mathematica* shows that inequality (39) has solutions only for $k \leq 2180$. This contradicts the fact that $k > 2600$. Thus, the proof is complete.

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