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Bivariate Leonardo polynomials and Riordan arrays

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Abstract: In this paper, bivariate Leonardo polynomials are defined, which are closely related to bivariate Fibonacci polynomials. Bivariate Leonardo polynomials are generalizations of the Leonardo polynomials and Leonardo numbers. Some properties and identities (Cassini, Catalan, Honsberger, d'Ocagne) for the bivariate Leonardo polynomials are obtained. Then, the Riordan arrays are defined by using bivariate Leonardo polynomials.
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1 Introduction

There are many remarkable polynomial sequences. Fibonacci polynomials are among the most commonly studied polynomial sequences. Eugene Charles Catalan and E. Jacobsthal investigated Fibonacci polynomials for the first time in 1883. Then Bicknell researched Lucas polynomials in 1970 [11]. More information on Fibonacci and Lucas polynomials can be found in [11, 13].

In recent years, various generalizations of the Fibonacci and Lucas polynomials, which are bivariate Fibonacci and Lucas polynomials, have been studied. Bivariate Fibonacci and Lucas polynomials are defined as follows:

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y), \quad F_0(x,y) = 0,\ F_1(x,y) = 1$$

and

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y), \quad L_0(x,y) = 2,\ L_1(x,y) = x$$

for $n \geq 2$. Binet's formulas for bivariate Fibonacci and Lucas polynomials are

$$F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n(x,y) = \alpha^n + \beta^n$$

where α and β are the roots of the characteristic equation, [10]. In [20], the explicit forms of bivariate Fibonacci and Lucas polynomials are provided as

$$F_n(x,y) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1} y^j$$

and

$$L_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} x^{n-2j} y^j.$$

The bivariate Fibonacci and Lucas polynomials have the following relations:

$$L_n(x,y) = F_{n+1}(x,y) + yF_{n-1}(x,y) \tag{1}$$

and

$$L_{r+s}(x,y) - (-y)^s L_{r-s}(x,y) = \left(x^2 + 4y\right) F_r(x,y) F_s(x,y) \tag{2}$$

for positive integers r and s with $r \geq s$, [5]. For other studies for the bivariate Fibonacci and Lucas polynomials refer to [3, 9, 22, 23]. The generalizations of bivariate Fibonacci and Lucas polynomials have been extensively studied in [4, 7, 8, 21].

Leonardo numbers are introduced, and some properties are given by Catarino and Borges in [6]. The Leonardo sequence is defined by the following recurrence relation for $n \geq 2$,

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \tag{3}$$

with the initial conditions $Le_0 = Le_1 = 1$. Also, another recurrence relation of Leonardo numbers for $n \geq 2$ is given as

$$Le_{n+1} = 2Le_n - Le_{n-2}. \tag{4}$$

Binet's formula for the Leonardo numbers sequence is

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta}, \tag{5}$$

where α and β are roots of the characteristic equation. It is clear that

$$Le_n = 2F_{n+1} - 1, \tag{6}$$

where F_n is the n-th Fibonacci number [6]. Cassini, Catalan, and d'Ocagne's identities for Leonardo numbers are given in [6]. In [1], the authors gave a matrix representation of Leonardo numbers and obtained for them new identities. Leonardo Pisano polynomials are defined and several properties of Leonardo Pisano hybrinomials are investigated using these polynomials in [12]. It is important to note that the Leonardo polynomials discussed in this article are distinct from the Leonardo Pisano polynomials defined in [12].

Riordan arrays have various applications in combinatorics. Consider the following formal power series:

$$g(t) = g_0 + g_1 t + g_2 t^2 + \cdots$$

and

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots$$

with $g_0 \neq 0,\ f_0 = 0$ and $f_1 \neq 0$. A Riordan array is represented as a pair of formal power series with $D = (g(t), f(t))$. The generating function of the k-th column of this array is $g(t)\left(f(t)\right)^k$, for $k \geq 0$. The entries of Riordan matrix $D = (d_{n,k})$ are

$$d_{n,k} = [t^n]\, g(t) \left(f(t)\right)^k, \tag{7}$$

where $[t^n]$ is the coefficient operator. The multiplication of two Riordan arrays is defined by

$$(g(t), f(t)) \left(h(t)\,,\ l(t)\right) = \left(g(t) h\left(f(t)\right),\ l\left(f(t)\right)\right). \tag{8}$$

The set of Riordan arrays is a group with the multiplication in (8). This group is called the Riordan group and is denoted by $\mathcal{R}$. The identity element of the Riordan group is $I = (1, t)$ and the inverse of $(g(t), f(t))$ is

$$(g(t), f(t))^{-1} = \left(\frac{1}{g\left(\overline{f}(t)\right)}, \overline{f}(t)\right), \tag{9}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$, [18]. Note that, the coefficients of the power series can be chosen from any ring where these operations are valid. Each of these rings corresponds to a Riordan group in [2]. Let $h(t)$ be the generating function of the sequence $\{h_k\}_{k\in\mathbb{N}}$ and $D = (g(t), f(t))$ be the Riordan array. Then,

$$\sum_{k=0}^{n} d_{n,k} h_k = [t^n]\, g(t) h\left(f(t)\right), \tag{10}$$

see [19]. Also, coefficient differentiation can be performed by following rule

$$[t^n]\, g'(t) = (n+1) \left[t^{n+1}\right] g(t), \tag{11}$$

where $g'(t)$ is the first derivative of $g(t)$ in [16].

There are some properties of the Riordan arrays, two of which are given in [15, 17]. In these studies, it is seen that a Riordan array can be characterized by two sequences. This is called sequence characterization of the Riordan arrays. These sequence characterizations are given in the following theorem.

Theorem 1.1. *Let $D = (d_{n,k})_{n,k\geq 0}$ be an infinite lower triangular matrix. D is a Riordan array if and only if there are two sequences $A = \{a_0, a_1, a_2, \dots\}$ and $Z = \{z_0, z_1, z_2, \dots\}$ with $a_0 \neq 0$ such that*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots \tag{12}$$

and

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots . \tag{13}$$

$A = \{a_0, a_1, a_2, \dots\}$ and $Z = \{z_0, z_1, z_2, \dots\}$ are denoted by the $A-$sequence and the $Z-$sequence of the Riordan array $D = (g(t), f(t))$, respectively.

Let $A(t)$ and $Z(t)$ be the generating functions of the $A-$ and $Z-$sequences of the Riordan array $D = (g(t), f(t))$, respectively. Then, the following identities hold (see [15]):

$$f(t) = tA(f(t)) \tag{14}$$

and

$$g(t) = \frac{g_0}{1 - tZ(f(t))}. \tag{15}$$

Many researchers have studied the subgroup of the Riordan groups. Marshall and Nkwanta have considered the stochastic subgroup, which is the set of Riordan matrices whose row sums are equal to 1. They have defined stochastic Lucas arrays and Fibonacci arrays, [14].

Motivated by the above papers, we introduce bivariate Leonardo polynomials and give some properties of these polynomials. In addition, the Riordan arrays are defined by using the bivariate Leonardo polynomials.

2 Bivariate Leonardo polynomials

Bivariate Leonardo polynomials are introduced. Following that, we give Binet's formula, the generating function, sum formulas, some interesting identities, and explicit formulas of these polynomials.

Definition 2.1. *For any integer $n \geq 2$, the n-th bivariate Leonardo polynomial is defined by the following recurrence relation:*

$$\mathcal{L}_n(x, y) = x\mathcal{L}_{n-1}(x, y) + y\mathcal{L}_{n-2}(x, y) + x + y - 1 \tag{16}$$

with the initial conditions $\mathcal{L}_0(x, y) = 1$ and $\mathcal{L}_1(x, y) = 2x - 1$.

Taking $y = 1$ in Equation (16), we obtain $\mathcal{L}_n(x, 1) = \mathcal{L}_n(x)$ Leonardo polynomials. Similarly, if we take $x = y = 1$ replaced in Equation (16), we get $\mathcal{L}_n(1, 1) = Le_n$ Leonardo numbers defined in [6]. The bivariate Leonardo polynomials with negative subscripts are defined as follows:

$$\mathcal{L}_{-n}(x,y) = -(-y)^{1-n}\left(\mathcal{L}_{n-2}(x,y)+1\right)-1 \tag{17}$$

for $n \geq 2$.

Considering recurrence relation (16), we find the new form of bivariate Leonardo polynomials as follows:

$$\mathcal{L}_{n+1}(x,y) = (x+1)\,\mathcal{L}_n(x,y) + (y-x)\,\mathcal{L}_{n-1}(x,y) - y\mathcal{L}_{n-2}(x,y). \tag{18}$$

The Binet's formula for the bivariate Leonardo polynomials is given as follows:

$$\mathcal{L}_n(x,y) = 2\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)-1, \tag{19}$$

where α and β roots of the characteristic equation. Then, the relation between bivariate Leonardo and bivariate Fibonacci polynomials is given as follows:

$$\mathcal{L}_n(x,y) = 2F_{n+1}(x,y)-1, \tag{20}$$

where $F_n(x,y)$ is the n-th bivariate Fibonacci polynomials. Using (20) and (1), the relation among bivariate Leonardo, Fibonacci and Lucas polynomials is given by

$$\mathcal{L}_n(x,y) = xF_n(x,y) + L_n(x,y) - 1, \tag{21}$$

where $F_n(x,y)$ and $L_n(x,y)$ are the n-th bivariate Fibonacci and Lucas polynomials, respectively. The generating function of the bivariate Leonardo polynomials sequence is given as follows:

$$\sum_{n=0}^{\infty}\mathcal{L}_n(x,y)t^n = \frac{1+t\,(x-2)+t^2y}{1-t\,(x+1)-t^2\,(y-x)+t^3y}. \tag{22}$$

Now, let us give special cases of Equation (22). Taking $y=1$ in (22), we have the generating function of Leonardo poynomials sequence as follows:

$$\sum_{n=0}^{\infty}\mathcal{L}_n(x)t^n = \frac{1+t\,(x-2)+t^2}{1-t\,(x+1)-t^2\,(1-x)+t^3}.$$

If we take $x=y=1$ in Equation (22), the generating function of Leonardo numbers sequence is obtained, as follows:

$$\sum_{n=0}^{\infty}Le_nt^n = \frac{1-t+t^2}{1-2t+t^3}$$

(see Proposition 5.1 in [6]).

Theorem 2.1. *The sum formulas for bivariate Leonardo polynomials are given as follows:*

$$\begin{aligned}
\sum_{k=0}^{n}\mathcal{L}_k(x,y) &= \frac{\mathcal{L}_{n+1}(x,y)+y\mathcal{L}_n(x,y)-x}{x+y-1}-n,\\
\sum_{k=0}^{n}\mathcal{L}_{2k}(x,y) &= \frac{\mathcal{L}_{2n+2}(x,y)-y^2\mathcal{L}_{2n}(x,y)-x^2}{x^2-(y-1)^2}-n,\\
\sum_{k=0}^{n}\mathcal{L}_{2k+1}(x,y) &= \frac{\mathcal{L}_{2n+3}(x,y)-y^2\mathcal{L}_{2n+1}(x,y)-(y^2+2x-1)}{x^2-(y-1)^2}-(n+1),
\end{aligned}$$

for $n \geq 0$ *and* $x+y \neq 1$.

Proof. Let us prove the first given sum formula. Using the recurrence relation of the bivariate Leonardo polynomials, we have

$$\begin{aligned}\mathcal{L}_2(x,y) &= x\mathcal{L}_1(x,y)+y\mathcal{L}_0(x,y)+x+y-1,\\ \mathcal{L}_3(x,y) &= x\mathcal{L}_2(x,y)+y\mathcal{L}_1(x,y)+x+y-1,\\ &\vdots\\ \mathcal{L}_{n+2}(x,y) &= x\mathcal{L}_{n+1}(x,y)+y\mathcal{L}_n(x,y)+x+y-1.\end{aligned}$$

From these equations, we get

$$\begin{aligned}(1-x-y)\sum_{k=0}^{n-2}\mathcal{L}_{k+2}(x,y) &= (x+y)\,\mathcal{L}_1\,(x,y)+y\mathcal{L}_0\,(x,y)\\ &\quad-\mathcal{L}_{n+1}\,(x,y)-\mathcal{L}_{n+2}\,(x,y)\\ &\quad+x\mathcal{L}_{n+1}\,(x,y)+(n+1)\,(x+y-1)\,.\end{aligned}$$

From Equation (16), the result is obtained. Using Binet's formula, we can derive the other sum formulas. □

Now, let us give some special cases of sum formulas. If we take $y=1$, sum formulas for Leonardo polynomials are obtained, as follows:

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{L}_k(x) &= \frac{1}{x}\left(\mathcal{L}_{n+1}(x)+\mathcal{L}_n(x)\right)-(n+1)\,,\\ \sum_{k=0}^{n}\mathcal{L}_{2k}(x) &= \frac{1}{x}\left(\mathcal{L}_{2n+1}(x)+1\right)-(n+1)\,,\\ \sum_{k=0}^{n}\mathcal{L}_{2k+1}(x) &= \frac{1}{x}\left(\mathcal{L}_{2n+2}(x)-1\right)-(n+1)\,.\end{aligned}$$

Taking $x=y=1$, we have the following sum formulas for the Leonardo numbers:

$$\sum_{k=0}^{n}Le_k=Le_{n+2}-(n+2)\,,\quad \sum_{k=0}^{n}Le_{2k}=Le_{2n+1}-n,\quad \sum_{k=0}^{n}Le_{2k+1}=Le_{2n+2}-(n+2)$$

(see Proposition 3.1 in [6]).

Theorem 2.2. *For a nonnegative integer n with $n\geq 1$, the alternating sum of bivariate Leonardo polynomials is given as follows:*

$$\sum_{k=0}^{n}(-1)^{k+1}\mathcal{L}_k(x,y)=\begin{cases}\dfrac{(y-x)\left(\mathcal{L}_n(x,y)+1\right)-y\left(\mathcal{L}_{n-1}(x,y)+1\right)-2}{x-y+1}+1, & n \text{ is even}\\[2ex] \dfrac{y\left(\mathcal{L}_{n-1}(x,y)+1\right)-(y-x)\left(\mathcal{L}_n(x,y)+1\right)-2}{x-y+1}, & n \text{ is odd}\end{cases},\quad (23)$$

where $x-y+1\neq 0$.

Proof. Using Equations (16) and (19), the result is clear. □

Taking $y=1$ and $x=y=1$ respectively in Equation (23), we obtain the following special cases:

$$\sum_{k=0}^{n}(-1)^{k+1}\mathcal{L}_k(x)=\begin{cases}\frac{1}{x}\left(\mathcal{L}_n(x)-\mathcal{L}_{n+1}(x)+x-2\right), & n \text{ is even}\\ \frac{1}{x}\left(\mathcal{L}_{n+1}(x)-\mathcal{L}_n(x)-2\right), & n \text{ is odd}\end{cases}$$

and

$$\sum_{k=0}^{n}(-1)^{k+1}Le_k=\begin{cases}-\left(Le_{n-1}+2\right), & n \text{ is even}\\ \left(Le_{n-1}-1\right), & n \text{ is odd}\end{cases},$$

where $\mathcal{L}_n(x)$ and Le_n are the n-th Leonardo polynomial and Leonardo number, respectively.

Theorem 2.3. *For positive integers n, r and s with $r \geq s$, we have*

$$\begin{aligned}\mathcal{L}_n(x,y)\mathcal{L}_{n+r+s}(x,y)-\mathcal{L}_{n+r}(x,y)\mathcal{L}_{n+s}(x,y) &= \left(\mathcal{L}_{n+r}(x,y)+\mathcal{L}_{n+s}(x,y)\right)\\ &\quad-\left(\mathcal{L}_n(x,y)+\mathcal{L}_{n+r+s}(x,y)\right)\\ &\quad-4\left(-y\right)^{n+1}F_r\left(x,y\right)F_s\left(x,y\right),\end{aligned} \tag{24}$$

where $F_n\left(x,y\right)$ is the n-th bivariate Fibonacci polynomial.

Proof. Using Binet's formula in (19) to left-hand side (LHS), we obtain

$$\begin{aligned}LHS &= \left(\frac{2\alpha^{n+1}-2\beta^{n+1}}{\alpha-\beta}-1\right)\left(\frac{2\alpha^{n+r+s+1}-2\beta^{n+r+s+1}}{\alpha-\beta}-1\right)\\ &\quad-\left(\frac{2\alpha^{n+r+1}-2\beta^{n+r+1}}{\alpha-\beta}-1\right)\left(\frac{2\alpha^{n+s+1}-2\beta^{n+s+1}}{\alpha-\beta}-1\right).\end{aligned}$$

From Binet's formulas of bivariate Fibonacci and bivariate Lucas polynomials, we get

$$\begin{aligned}LHS &= \frac{-4\left(-y\right)^{n+1}\left(L_{r+s}\left(x,y\right)-\left(-y\right)^{s}L_{r-s}\left(x,y\right)\right)}{x^2+4y}\\ &\quad-2F_{n+r+s+1}(x,y)-2F_{n+1}(x,y)\\ &\quad+2F_{n+r+1}(x,y)+2F_{n+s+1}(x,y).\end{aligned}$$

Considering Equations (2) and (20), the result is clear. □

From some special cases of Equation (24), we have Catalan and Cassini identities for bivariate Leonardo polynomials.

Taking $s=-r$ in Equation (24), we get Catalan's identity for the bivariate Leonardo polynomials as follows:

$$\begin{aligned}\mathcal{L}_n^2(x,y)-\mathcal{L}_{n+r}(x,y)\mathcal{L}_{n-r}(x,y) &= \mathcal{L}_{n+r}(x,y)+\mathcal{L}_{n-r}(x,y)-2\mathcal{L}_n(x,y)\\ &\quad+\left(-y\right)^{n-r+1}\left(\mathcal{L}_{r-1}(x,y)+1\right)^2\end{aligned} \tag{25}$$

for $n \geq r$.

If we take $r=1$ and $s=-1$ in (24), Cassini's identity for the bivariate Leonardo polynomials is obtained as

$$\begin{aligned}\mathcal{L}_n^2(x,y)-\mathcal{L}_{n+1}(x,y)\mathcal{L}_{n-1}(x,y) &= \left(y+1\right)\mathcal{L}_{n-1}(x,y)+\left(x-2\right)\mathcal{L}_n(x,y)\\ &\quad+4\left(-y\right)^{n}+x+y-1\end{aligned} \tag{26}$$

for $n \geq 1$.

From Equation (26), we have Cassini's identity for Leonardo polynomials as follows:

$$\mathcal{L}_n^2(x) - \mathcal{L}_{n+1}(x)\mathcal{L}_{n-1}(x) = 2\mathcal{L}_{n-1}(x) + (x-2)\,\mathcal{L}_n(x) + x + 4\,(-1)^n\,.$$

Similarly, Cassini's identity for Leonardo numbers is given as,

$$Le_n^2 - Le_{n+1}Le_{n-1} = Le_{n-1} - Le_{n-2} + 4\,(-1)^n$$

(see Proposition 4.2 in [6]).

Theorem 2.4 (d'Ocagne's Identity). *For positive integers n and m with $m \geq n$, we have*

$$\begin{aligned} \mathcal{L}_m(x,y)\mathcal{L}_{n+1}(x,y) - \mathcal{L}_{m+1}(x,y)\mathcal{L}_n(x,y) &= (\mathcal{L}_n(x,y) - \mathcal{L}_{n+1}(x,y)) \\ &\quad + (\mathcal{L}_{m+1}(x,y) - \mathcal{L}_m(x,y)) \\ &\quad + 2\,(-y)^{n+1}\,(\mathcal{L}_{m-n-1}(x,y) + 1)\,. \end{aligned} \tag{27}$$

Proof. Applying Binet's formula for bivariate Leonardo polynomials to left hand-side (LHS), we have

$$\begin{aligned} LHS &= \left(\frac{2\alpha^{m+1} - 2\beta^{m+1}}{\alpha-\beta} - 1\right)\left(\frac{2\alpha^{n+2} - 2\beta^{n+2}}{\alpha-\beta} - 1\right) \\ &\quad - \left(\frac{2\alpha^{m+2} - 2\beta^{m+2}}{\alpha-\beta} - 1\right)\left(\frac{2\alpha^{n+1} - 2\beta^{n+1}}{\alpha-\beta} - 1\right). \end{aligned}$$

Considering Binet's formulas of bivariate Fibonacci and Lucas polynomials, we get

$$\begin{aligned} LHS &= \frac{4\,(-y)^{n+1}\,(L_{m-n+1}\,(x,y) + yL_{m-n-1}\,(x,y))}{x^2+4y} \\ &\quad + 2F_{m+2}\,(x,y) + 2F_{n+1}\,(x,y) \\ &\quad - 2F_{m+1}\,(x,y) - 2F_{n+2}\,(x,y)\,. \end{aligned}$$

Form Equations (2) and (20), the result is clear. □

Taking $x = y = 1$ in Equation (27), we have the d'Ocagne's identity for Leonardo numbers as follows:

$$Le_mLe_{n+1} - Le_{m+1}Le_n = Le_{m-1} - Le_{n-1} + 2\,(-1)^{n+1}\,(Le_{m-n-1} + 1)$$

(see Proposition 4.3 in [6]).

Theorem 2.5 (Honsberger's Identity). *For positive integers m and n with $m \geq 1$ and $m \geq n$, the following equality holds:*

$$\begin{aligned} \mathcal{L}_m(x,y)\mathcal{L}_{n+1}(x,y) + y\mathcal{L}_{m-1}(x,y)\mathcal{L}_n(x,y) &= 2\mathcal{L}_{m+n+1}(x,y) - \mathcal{L}_m(x,y) + 1 - y \\ &\quad - y\mathcal{L}_{m-1}(x,y) - y\mathcal{L}_n(x,y) - \mathcal{L}_{n+1}(x,y), \end{aligned} \tag{28}$$

where $\mathcal{L}_n(x,y)$ is the n-th bivariate Leonardo polynomial.

Proof. Using the Binet's formula in (19) to left-hand side (LHS), we get

$$\begin{aligned} LHS &= \left(\frac{2\alpha^{m+1}-2\beta^{m+1}}{\alpha-\beta}-1\right)\left(\frac{2\alpha^{n+2}-2\beta^{n+2}}{\alpha-\beta}-1\right) \\ &\quad + y\left(\frac{2\alpha^{m}-2\beta^{m}}{\alpha-\beta}-1\right)\left(\frac{2\alpha^{n+1}-2\beta^{n+1}}{\alpha-\beta}-1\right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} LHS &= \frac{4\left(L_{m+n+3}(x,y)+yL_{m+n+1}(x,y)\right)}{x^2+4y} \\ &\quad -2F_{m+1}(x,y)-2F_{n+2}(x,y)+1 \\ &\quad -2yF_{m}(x,y)-2yF_{n+1}(x,y)+y \end{aligned}$$

where $F_n(x,y)$ and $L_n(x,y)$ are the n-th bivariate Fibonacci and Lucas polynomials, respectively. From Equations (2) and (20), the result is clear. □

We give special cases of the Honsberger identity by (28) in the following corollary.

Corollary 2.1. *Let $\mathcal{L}_n(x,y)$ the denote n-th bivariate Leonardo polynomial, then the following equalities hold:*

$$2\mathcal{L}_{2n}(x,y)=\mathcal{L}_n^2(x,y)+y\mathcal{L}_{n-1}^2(x,y)+2\left(\mathcal{L}_n(x,y)+y\mathcal{L}_{n-1}(x,y)\right)+y-1$$

and

$$2\mathcal{L}_{2n+1}(x,y)=\left(\mathcal{L}_{n+1}(x,y)+y\mathcal{L}_{n-1}(x,y)+y+1\right)\left(\mathcal{L}_n(x,y)+1\right)-2,$$

where n is a positive integer.

Theorem 2.6. *The explicit formula of bivariate Leonardo polynomials is given as follows:*

$$\mathcal{L}_n(x,y)=2\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\left(\binom{n-j}{j}x^{n-2j}y^j\right)-1 \tag{29}$$

where n is a positive integer with $n\geq 1$.

Proof. The formula is valid for $n=1$ and $n=2$. Assume that it is correct for $n=k$. We prove for $n=k+1$. Then, we have

$$\begin{aligned} \mathcal{L}_{k+1}(x,y) &= x\mathcal{L}_k(x,y)+y\mathcal{L}_{k-1}(x,y)+x+y-1 \\ &= x\left(2\sum_{j=0}^{\lfloor \frac{k}{2}\rfloor}\left(\binom{k-j}{j}x^{k-2j}y^j\right)-1\right) \\ &\quad + y\left(2\sum_{j=0}^{\lfloor \frac{k-1}{2}\rfloor}\left(\binom{k-1-j}{j}x^{k-1-2j}y^j\right)-1\right) \\ &\quad + x+y-1. \end{aligned}$$

If $k=2m+1$, we get

$$\begin{aligned}\mathcal{L}_{k+1}(x,y) &= 2\sum_{j=0}^{m}\left(\binom{2m+1-j}{j}x^{2m+2-2j}y^{j}\right)+2\sum_{j=0}^{m}\left(\binom{2m-j}{j}x^{2m-2j}y^{j+1}\right)-1\\ &= 2x^{2m+2}+2y^{m+1}-1+2\sum_{j=1}^{m}x^{2m+2-2j}y^{j}\left(\binom{2m+1-j}{j}+\binom{2m+1-j}{j-1}\right).\end{aligned}$$

Using Pascal identity for binomial coefficients, we have

$$\mathcal{L}_{k+1}(x,y)=2\sum_{j=0}^{m+1}\left(x^{2m+2-2j}y^{j}\binom{2m+2-j}{j}\right)-1.$$

Similarly, if $k=2m$, the formula is valid and the proof is completed. □

Theorem 2.7. *The relation among the partial derivative of the bivariate Leonardo, Fibonacci and Lucas polynomials are given as follows:*

$$\frac{\partial\mathcal{L}_n(x,y)}{\partial x}=\frac{\partial\mathcal{L}_{n+1}(x,y)}{\partial y} \tag{30}$$

and

$$\frac{\partial\mathcal{L}_n(x,y)}{\partial y}-\frac{\partial L_n(x,y)}{\partial y}=x\frac{\partial F_n(x,y)}{\partial y}, \tag{31}$$

where $\mathcal{L}_n(x,y)$, $F_n(x,y)$ and $L_n(x,y)$ are the n-th bivariate Leonardo, Fibonacci and Lucas polynomials, respectively.

Proof. Let us start with Equation (31). By applying partial differentitation to the explicit formula of bivariate Leonardo and bivariate Lucas polynomials to left-hand side (LHS), we have

$$\begin{aligned}LHS &= \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\left(2j\binom{n-j}{j}x^{n-2j}y^{j-1}-\frac{n}{n-j}j\binom{n-j}{j}x^{n-2j}y^{j-1}\right)\\ &= \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\left(x^{n-2j}y^{j-1}j\binom{n-j}{j}\left(2-\frac{n}{n-j}\right)\right)\\ &= \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\left(x^{n-2j}y^{j-1}j\binom{n-j-1}{j}\right).\end{aligned}$$

Taking $n=2m$, we obtain

$$\begin{aligned}LHS &= \sum_{j=0}^{m}\left(x^{2m-2j}y^{j-1}j\binom{2m-j-1}{j}\right)\\ &= x\sum_{j=0}^{m-1}\left(x^{2m-2j-1}y^{j-1}j\binom{2m-j-1}{j}\right)\\ &= x\frac{\partial F_n(x,y)}{\partial y}.\end{aligned}$$

Similarly, if we take $n=2m+1$, the formula holds and the proof is completed. □

3 Riordan arrays with bivariate Leonardo polynomials

In this section, the Riordan arrays with bivariate Leonardo polynomials are considered. We compute the $A-$ and $Z-$sequences of these Riordan arrays.

Let us consider an infinite lower triangular matrix which is defined by bivariate Leonardo polynomials as follows:

$$\mathcal{L}=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2x-1 & 1 & 0 & 0 & \dots \\ 2x^2+2y-1 & 2x-1 & 1 & 0 & \dots \\ 2x^3+4xy-1 & 2x^2+2y-1 & 2x-1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{32}$$

The Riordan represantation of this matrix, with a pair of formal power series, is obtained, as follows:

$$\mathcal{L}=\left(\frac{1+t\,(x-2)+t^2y}{1-t\,(x+1)-t^2\,(y-x)+t^3y},t\right). \tag{33}$$

Proposition 3.1. *The entries of Riordan matrix $\mathcal{L}$ are defined as follows:*

$$l_{n,k}=\begin{cases} \mathcal{L}_{n-k}\,(x,y)\,, & n-k\geq 0 \\ 0, & \textit{otherwise} \end{cases},$$

where $n,k\geq 0$.

Proof. From Equation (7), we have

$$\begin{aligned} l_{n,k} &= \left[t^{n-k}\right]\frac{1+t\,(x-2)+t^2y}{1-t\,(x+1)-t^2\,(y-x)+t^3y} \\ &= \left[t^{n-k}\right]\sum_{k=0}^{\infty}\mathcal{L}_k\,(x,y)\,t^k=\mathcal{L}_{n-k}\,(x,y)\,. \end{aligned}$$

□

Using Equations (14) and (15), the generating functions of the $A-$sequence and $Z-$sequence for the Riordan matrix $\mathcal{L}$ are obtained

$$A(t)=1$$

and

$$Z(t)=\frac{2x-1-t\,(x-2y)-t^2y}{1+t\,(x-2)+t^2y},$$

respectively. The first few elements of the $Z-$sequence are obtained, as follows:

$$2x-1,-2x^2+4x+2y-2,2\,(x-1)\left(x^2-3x-2y+2\right),\dots\,.$$

Now, we give the inverse of the matrix $\mathcal{L}$. From Equations (9) and (33), the Riordan representation of the inverse matrix $\mathcal{L}^{-1}$ is obtained, as follows:

$$\mathcal{L}^{-1}=\left(\frac{1-t\,(x+1)-t^2\,(y-x)+t^3y}{1+t\,(x-2)+t^2y},t\right). \tag{34}$$

Also, the inverse matrix $\mathcal{L}^{-1}$ is obtained as

$$\mathcal{L}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1-2x & 1 & 0 & 0 & \dots \\ 2x^2-4x-2y+2 & 1-2x & 1 & 0 & \dots \\ 2(x-1)(-x^2+3x+2y-2) & 2x^2-4x-2y+2 & 1-2x & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using Equations (14) and (15), the generating functions of $A-$ sequence and $Z-$sequence of the Riordan matrix $\mathcal{L}^{-1}$ are obtained, as follows:

$$A(t) = 1$$

and

$$Z(t) = \frac{1-2x+t(x-2y)+t^2y}{1-t(x+1)-t^2(y-x)+t^3y}.$$

The terms of the $Z-$sequence are obtained, as follows:

$$-\mathcal{L}_1(x,y), -\mathcal{L}_2(x,y), -\mathcal{L}_3(x,y), -\mathcal{L}_4(x,y), -\mathcal{L}_5(x,y), \dots .$$

Proposition 3.2. *Let us consider the Riordan array $H = (h_{n,k})_{n,k\geq 0} = (g'(t), t)$, where $g(t)$ is the generating function of the bivariate Leonardo polynomials sequence in* (22). *The terms of the matrix H are given as*

$$h_{n,k} = \begin{cases} (n-k+1)\mathcal{L}_{n-k+1}(x,y), & n-k \geq 0 \\ 0, & \textit{otherwise} \end{cases}. \tag{35}$$

Proof. Using Equation (7), we obtain $h_{n,k} = [t^n] g'(t)t^k = [t^{n-k}] g'(t)$. From Equation (11), we have

$$\begin{aligned} h_{n,k} &= [t^{n-k+1}](n-k+1)\sum_{k=0}^{\infty}\mathcal{L}_k(x,y)t^k \\ &= (n-k+1)\mathcal{L}_{n-k+1}(x,y). \end{aligned}$$

□

The matrix H in (35) is given as follows:

$$H = \begin{pmatrix} 2x-1 & 0 & 0 & 0 & \dots \\ 4x^2+4y-2 & 2x-1 & 0 & 0 & \dots \\ 6x^3+12xy-3 & 4x^2+4y-2 & 2x-1 & 0 & \dots \\ 8x^4+24x^2y+8y^2-4 & 6x^3+12xy-3 & 4x^2+4y-2 & 2x-1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us consider the generating function of the bivariate Leonardo polynomial sequence in (22). Taking $h(t) = \frac{1}{1-t}$ in Equation (10) to obtain the stochastic bivariate Leonardo polynomial array, we have the following.

$$\begin{aligned}
\frac{1+t(x-2)+t^2y}{1-t(x+1)-t^2(y-x)+t^3y}h(f(t)) &= \frac{1}{1-t},\\
h(f(t)) &= \frac{1-t(x+1)-t^2(y-x)+t^3y}{(1-t)(1+t(x-2)+t^2y)},\\
f(t) &= \frac{2t(1-x+t(x-y-1)+t^2y)}{1-t(x+1)-t^2(y-x)+t^3y}.
\end{aligned}$$

Then, the stochastic bivariate Leonardo polynomial array is obtained as

$$\left(\frac{1+t(x-2)+t^2y}{1-t(x+1)-t^2(y-x)+t^3y}, \frac{2t(1-x+t(x-y-1)+t^2y)}{1-t(x+1)-t^2(y-x)+t^3y}\right). \tag{36}$$

The matrix in (36) is given as follows:

$$\begin{pmatrix}
1 & 0 & 0 & \dots \\
2x-1 & 2-2x & 0 & \dots \\
2x^2+2y-1 & -6x^2+8x-2y-2 & 4x^2-8x+4 & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{37}$$

It is clear that the row sums of (37) is equal to 1. It should be noted that the matrix in (37) does not have an inverse. However, we can regulate this matrix so that it belongs to the Riordan group.

Proposition 3.3. *Let us consider the following generating function:*

$$l(t) = \frac{1+t(x-2)+t^2y}{1-tx+t^2(2x-y-2)+2t^3y}.$$

The row sum of $B=(l(t), tl(t))$ is the bivariate Leonardo polynomial.

Proof. Taking $h(t)=\frac{1}{1-t}$ in Equation (10), we obtain

$$\begin{aligned}
\sum_{k=0}^{n} b_{n,k} &= [t^n]\, l(t)\frac{1}{1-tl(t)}\\
&= [t^n]\frac{1+t(x-2)+t^2y}{1-t(x+1)-t^2(y-x)+t^3y}\\
&= [t^n]\sum_{k=0}^{\infty}\mathcal{L}_k(x,y)\,t^k = \mathcal{L}_n(x,y).
\end{aligned}$$

□

The Riordan matrix $B=(l(t), tl(t))$ is given as follows:

$$B = \begin{pmatrix}
1 & 0 & 0 & 0 & \dots \\
2x-2 & 1 & 0 & 0 & \dots \\
2x^2-4x+2y+2 & 4x-4 & 1 & 0 & \dots \\
2(x-1)(x^2-3x+2y+2) & 8x^2-16x+4y+8 & 6x-6 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

The row sums of the Riordan matrix B are given as follows:

$$1,\ 2x-1,\ 2x^2+2y-1,\ 2x^3+4xy-1,\dots,$$

which are bivariate Leonardo polynomials.

References

[1] Alp, Y., & Koçer, E. G. (2021). Some properties of Leonardo numbers. *Konuralp Journal of Mathematics*, 9(1), 183–189.

[2] Barry, P. (2019). On the halves of a Riordan array and their antecedents. *Linear Algebra and its Applications*, 582, 114–137.

[3] Belbachir, H., & Bencherif, F. (2008). On some properties of bivariate Fibonacci and Lucas polynomials. *Journal of Integer Sequences*, 11(2), Article 08.2.6.

[4] Catalani, M. (2002). *Generalized bivariate Fibonacci polynomials*. Preprint. arXiv: `math/0211366`.

[5] Catalani, M. (2004). *Some formulae for bivariate Fibonacci and Lucas polynomials*. Preprint. arXiv: `math/0406323`.

[6] Catarino, P. M. M. C., & Borges, A. (2020). On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, 89(1), 75–86.

[7] Djordjević, G. B. (1997). Some properties of a class of polynomials. *Matematički Vesnik*, 49(3–4), 265–271.

[8] Djordjević, G. B. (2001). Some properties of partial derivatives of generalized Fibonacci and Lucas polynomials. *The Fibonacci Quarterly*, 39(2), 138–141.

[9] Frei, G. (1980). Binary Lucas and Fibonacci polynomials, I. *Mathematische Nachrichten*, 96, 83–112.

[10] Hoggatt, V. E. Jr., & Long, C. T. (1974). Divisibility properties of generalized Fibonacci polynomials. *The Fibonacci Quarterly*, 12(2), 113–120.

[11] Koshy, T. (2018). *Fibonacci and Lucas Numbers with Applications*. Wiley Interscience, Hoboken, NJ.

[12] Kürüz, F., Dağdeviren, A., & Catarino, P. (2021). On Leonardo Pisano hybrinomials. *Mathematics*, 9(22), Article 2923.

[13] Lupaș, A. (1999). A guide of Fibonacci and Lucas polynomials. *Octogon Mathematics Magazine*, 7(1), 2–12.

[14] Marshall, C., & Nkwanta, A. (2021). Fibonacci and Lucas Riordan arrays and construction of pseudo-involutions. *Applicable Analysis*, 104(1), 82–93.

[15] Merlini, D., Rogers, D. G., Sprugnoli, R., & Verri, M. C. (1997). On some alternative characterizations of Riordan arrays. *Canadian Journal of Mathematics*, 49(2), 301–320.

[16] Merlini, D., Sprugnoli, R., & Verri, M. C. (2006). Lagrange inversion: When and how. *Acta Applicandae Mathematicae*, 94(3), 233–249.

[17] Rogers, D. G. (1978). Pascal triangles, Catalan numbers and renewal arrays. *Discrete Mathematics*, 22(3), 301–310.

[18] Shapiro, L. W., Getu, S., Woan, W.-J., & Woodson, L. C. (1991). The Riordan group. *Discrete Applied Mathematics*, 34(1–3), 229–239.

[19] Sprugnoli, R. (1994). Riordan arrays and combinatorial sums. *Discrete Mathematics*, 132(1–3), 267–290.

[20] Swamy, M. N. S. (1999). Generalized Fibonacci and Lucas polynomials and their associated diagonal polynomials. *The Fibonacci Quarterly*, 37(3), 213–222.

[21] Swamy, M. N. S. (1999). Network properties of a pair of generalized polynomials. *The Fibonacci Quarterly*, 37(4), 350–360.

[22] Tan, M., & Zhang, Y. (2005). A note on bivariate and trivariate Fibonacci polynomials. *Southeast Asian Bulletin of Mathematics*, 29(5), 975–990.

[23] Yu, H., & Liang, C. (1997). Identities involving partial derivatives of bivariate Fibonacci and Lucas polynomials. *The Fibonacci Quarterly*, 35(1), 19–23.