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A new approach to tridiagonal matrices related to the Sylvester–Kac matrix

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Abstract: The Sylvester–Kac matrix, a well-known tridiagonal matrix, has been extensively studied for over a century, with various generalizations explored in the literature. This paper introduces a new type of tridiagonal matrix, where the matrix entries are defined by an integer sequence, setting it apart from the classical Sylvester–Kac matrix. The aim of this paper is to investigate several fundamental properties of this generalized matrix, including its algebraic structure, determinant, inverse, LU decomposition, characteristic polynomial, and various norms.
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1 Introduction

Tridiagonal matrices play a significant role in various fields, including numerical analysis and physics, because of their efficient computational properties. They have been widely studied for their ability to solve structured linear systems and to model physical applications. As an example of a tridiagonal matrix, consider the following Sylvester–Kac (or Clement) matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ n-1 & 0 & 2 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

This matrix, introduced by Sylvester [26], is notable for its remarkable eigenvalue properties. The determinant formula for Sylvester's matrix was first proposed by Sylvester [26], though without a proof. Subsequent attempts to justify and extend it can be found in papers such as [4, 24, 25]. Kac [20] provided a systematic proof, while also deriving a general expression for the characteristic polynomial, thus deepening the understanding of the matrix's spectral properties.

Numerous papers in the literature examine various types of Sylvester–Kac matrices, focusing on properties such as the determinant, inverse, characteristic polynomial, and spectrum [1, 2, 5–16, 21, 23, 27]. These contributions reflect the ongoing interest in expanding both the applications and the theoretical significance of Sylvester–Kac matrices. In particular, Du and da Fonseca [9] presented important results on Sylvester–Kac matrices, focusing on their eigenvalues and spectra with quadratic forms. They introduced a unified approach using a lower triangular matrix based on Pascal's triangle and provided a simple proof for Sylvester's determinant claim. The paper [9] also includes a summary of important historical developments and recent contributions. Dyachenko and Tyaglov [8] studied the spectral properties of tridiagonal matrices with a two-periodic main diagonal. Their paper builds upon earlier results on Sylvester–Kac matrices, extending them to the class of irreducible complex tridiagonal matrices. Using a method based on right-eigenvectors, they analysed these matrices through a related matrix that shares the same subdiagonal and superdiagonal but has a zero main diagonal, deriving explicit expressions for the spectrum and eigenvectors. Du and da Fonseca [10] provided a combinatorial proof for determining the eigenvalues of a biperiodic extension of the Sylvester–Kac matrix, obtained by adding a constant periodically to the non-zero off-diagonal entries. They also discussed a possible biperiodic extension. A tridiagonal extension of Sylvester's matrix was investigated by deriving left and right eigenvectors, establishing their orthogonality, and computing the determinants of the corresponding eigenvector matrices in closed form by Chu and Kılıç [6].

Using number sequences in matrices helps create new types of structured matrices, whose behaviour can be studied by looking at the properties of the sequence. This method has been useful in expanding classical matrix families and finding new applications. Hu et al. [18] introduced a type of the Sylvester–Kac matrix with the Fermat numbers and obtained determinant, inverse, characteristic polynomial and eigenvalues of this matrix. Also, Jiang *et al.* [19] studied on the Sylvester–Kac matrix with the Fibonacci numbers which are known by the recurrence relation [22]

$$F_{k+1} = F_k + F_{k-1} \quad \text{with} \quad F_0 = 0 \quad \text{and} \quad F_1 = 1.$$

The Fibonacci–Sylvester–Kac matrix is defined as

$$S_{F_k,n} = \begin{pmatrix} 0 & F_1 & 0 & \cdots & 0 & 0 \\ F_{n-1} & 0 & F_2 & \cdots & 0 & 0 \\ 0 & F_{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & F_{n-1} \\ 0 & 0 & 0 & \cdots & F_1 & 0 \end{pmatrix},$$

and the entries of the n-dimensional matrix $S_{F_k,n} = [f_{ij}]$ is generated by the rule [19]

$$f_{ij} = \begin{cases} F_i, & \text{if} \quad j = i+1, \quad 1 \leq i \leq n-1 \\ F_{n+1-i}, & \text{if} \quad j = i-1, \quad 2 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

The determinant, inverse and characteristic polynomial of this matrix were examined depending on whether the dimension of the matrix is odd or even [19].

Motivated by [18] and [19], we introduce a new tridiagonal matrix related to the Sylvester–Kac matrix, which is defined as follows

$$S_{a_k,n} = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 0 & a_2 & \cdots & 0 & 0 \\ 0 & a_{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} \\ 0 & 0 & 0 & \cdots & a_1 & 0 \end{pmatrix}, \tag{1.1}$$

where $\{a_k\}_{k=1}^{n-1}$ is a real sequence. The entries of the n-dimensional matrix $S_{a_k,n} = [s_{ij}]$ are generated as

$$s_{ij} = \begin{cases} a_i, & \text{if } j = i+1, \ 1 \leq i \leq n-1 \\ a_{n+1-i}, & \text{if } j = i-1, \ 2 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

In the present paper, we aim to examine basic properties of the matrix $S_{a_k,n}$, such as algebraic structure, determinant, inverse, LU decomposition, characteristic polynomial, and some norms. Firstly, we give the following preliminaries.

Definition 1.1. [23] *The n-dimensional symmetric elementary matrix $\widehat{I}_n = [\iota_{ij}]$ is defined by*

$$\iota_{ij} = \delta_{i,n+1-j}, \quad i,j = 1,2,\ldots,n,$$

where $\delta_{i,j}$ is the Kronecker delta.

Definition 1.2. [23] *The n-dimensional matrix $\Psi = [\psi_{ij}]$ is called centrosymmetric if its entries satisfy the following condition*

$$\psi_{ij} = \psi_{n+1-i,n+1-j} \quad i,j = 1,2,\ldots,n.$$

For the n-dimensional centrosymmetric matrix Ψ, there also holds the equality

$$\Psi = \widehat{I}_n \Psi \widehat{I}_n,$$

where $\widehat{I}_n$ is the n-dimensional symmetric elementary matrix.

Definition 1.3. [23] *Let v, w be $\frac{n}{2}$- and $\frac{n-1}{2}$-dimensional vectors, respectively. Then, the following definitions hold:*

(i) *An n-dimensional vector Φ is called symmetric if $\Phi = \widehat{I}_n \Phi$, and it can be expressed as*

$$\Phi = \begin{cases} \left(v, \widehat{I}_{\frac{n}{2}} v\right)^T, & \text{if } n \text{ is even}, \\ \left(w, \Phi_{\frac{n+1}{2}}, \widehat{I}_{\frac{n-1}{2}} w\right)^T, & \text{if } n \text{ is odd}. \end{cases}$$

(ii) *An n-dimensional vector Φ is called skew symmetric if $\Phi = -\widehat{I}_n \Phi$, and it can be expressed as*

$$\Phi = \begin{cases} \left(v, -\widehat{I}_{\frac{n}{2}} v\right)^T, & \text{if } n \text{ is even}, \\ \left(w, 0, -\widehat{I}_{\frac{n-1}{2}} w\right)^T, & \text{if } n \text{ is odd}, \end{cases}$$

where $\widehat{I}_{\frac{n}{2}}$ and $\widehat{I}_{\frac{n-1}{2}}$ denote the $\frac{n}{2}$- and $\frac{n-1}{2}$-dimensional symmetric elementary matrices, respectively (see Definition 1.1).

Next, we present the following lemma, which recalls the specific block structures of real centrosymmetric matrices, depending on whether the matrix order is even or odd.

Lemma 1.1. [23, Lemma 1, pp. 214] *Let Ψ be an n-dimensional real centrosymmetric matrix. Then, for even n, the matrix Ψ can be partitioned as follows*

$$\Psi = \begin{pmatrix} A & JBJ \\ B & JAJ \end{pmatrix}, \tag{1.2}$$

where A, B and J are $\frac{n}{2}$-dimensional square matrices. For odd n, the matrix Ψ can be expressed as

$$\Psi = \begin{pmatrix} A^* & X_2 & J^* B^* J^* \\ X_1 & q & X_1 J^* \\ B^* & J^* X_2 & J^* A^* J^* \end{pmatrix}, \tag{1.3}$$

where A^*, B^* and J^* are $\frac{n-1}{2}$-dimensional square matrices, X_1, X_2^T are $\frac{n-1}{2}$-dimensional row vectors and q is the central entry of Ψ.

The block structures given in Lemma 1.1 play a fundamental role in spectral analysis of centrosymmetric matrices, particularly in deriving the eigenvalues and corresponding orthonormal eigenvectors, as formulated in the following theorem.

Theorem 1.1. [23, Theorems 1a,1b, pp. 215–216]

(i) *Let n be even and Ψ be the matrix which is partitioned as in Equation (1.2). Then, $\frac{n}{2}$ skew symmetric orthonormal eigenvectors $v_i \in \Psi$ and the corresponding eigenvalues γ_i are obtained from the solution of the equation*

$$(A - JB)\, u_i = \gamma_i u_i,$$

where $i = 1, 2, \ldots, \frac{n}{2}$, $v_i = \frac{1}{\sqrt{2}} (u_i, -Ju_i)^T$, and u_i is the form of an orthonormal set. The $\frac{n}{2}$ symmetric orthonormal eigenvectors $w_j \in \Psi$ and the corresponding eigenvalues σ_j are determined from the solution of the equation

$$(A + JB)\, y_j = \sigma_j y_j,$$

where $j = 1, 2, \ldots, \frac{n}{2}$, $w_j = \frac{1}{\sqrt{2}} (y_j, Jy_j)^T$, and y_j is the form of an orthonormal set. The set $\left(v_1, v_2, \ldots, v_{\frac{n}{2}}, w_1, w_2, \ldots, w_{\frac{n}{2}}\right)$ is an orthonormal set of n eigenvectors of Ψ.

(ii) *Let n be odd and Ψ be the matrix which is partitioned as in Equation* (1.3). *Then, $\frac{n-1}{2}$ skew symmetric orthonormal eigenvectors $v_i \in \Psi$ and the corresponding eigenvalues γ_i are obtained from the solution of the equation*

$$(A^* - J^* B^*)\, u_i = \gamma_i u_i,$$

where $i = 1, 2, \ldots, \frac{n-1}{2}$, $v_i = \frac{1}{\sqrt{2}} (u_i, 0, -J^* u_i)^T$, and u_i is the form of an orthonormal set. The $\frac{n+1}{2}$ symmetric orthonormal eigenvectors $w_j \in \Psi$ and the corresponding eigenvalues σ_j are determined from the solution of the equation

$$\begin{pmatrix} A^* + J^* B^* & \sqrt{2} X_2 \\ \sqrt{2} X_1 & q \end{pmatrix} \begin{pmatrix} y_j \\ \alpha_j \end{pmatrix} = \sigma_j \begin{pmatrix} y_j \\ \alpha_j \end{pmatrix},$$

where $j = 1, 2, \ldots, \frac{n+1}{2}$, $w_j = \frac{1}{\sqrt{2}} (y_j, 2\alpha_j, J^* y_j)^T$ and $(y_j, \alpha_j)^T$ is the form of an orthonormal set. Also, the set $\left(v_1, v_2, \ldots, v_{\frac{n-1}{2}}, w_1, w_2, \ldots, w_{\frac{n+1}{2}}\right)$ is an orthonormal set of n eigenvectors of Ψ.

2 Main results

Let $V_{S_{a_k,n}}$ denote the set of all n-dimensional matrices $S_{a_k,n}$ as defined in Equation (1.1). We begin by investigating the algebraic structure of $V_{S_{a_k,n}}$.

Theorem 2.1. *$V_{S_{a_k,n}}$ is an $(n-1)$-dimensional vector space.*

Proof. Let $S_{b_k,n} = [p_{ij}]$ and $S_{c_k,n} = [r_{ij}]$ be two matrices in $V_{S_{a_k,n}}$, associated with any real sequences $\{b_k\}_{k=1}^{n-1}$ and $\{c_k\}_{k=1}^{n-1}$, respectively. Also, let x and y be any real numbers. If

$$S_{d_k,n} = x S_{b_k,n} + y S_{c_k,n} = [q_{ij}]$$

for a real sequence $\{d_k\}_{k=1}^{n-1}$, then

$$q_{ij} = x p_{ij} + y r_{ij} = \begin{cases} x b_i + y c_i, & \text{if} \quad j = i+1, \quad 1 \le i \le n-1 \\ x b_{n+1-i} + y c_{n+1-i}, & \text{if} \quad j = i-1, \quad 2 \le i \le n \\ 0, & \text{otherwise.} \end{cases}$$

Moreover,

$$xb + yc = (x b_1 + y c_1, x b_2 + y c_2, \ldots, x b_{n-1} + y c_{n-1}) = (e_1, e_2, \ldots, e_{n-1}) = \{e_k\}_{k=1}^{n-1}.$$

If $S_{e_k,n} = [t_{ij}] \in V_{S_{a_k,n}}$ corresponds to the sequence $\{e_k\}_{k=1}^{n-1}$, then

$$\begin{aligned} t_{ij} &= \begin{cases} e_i, & \text{if } j = i+1, \ 1 \le i \le n-1 \\ e_{n+1-i}, & \text{if } j = i-1, \ 2 \le i \le n \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} xb_i + yc_i, & \text{if } j = i+1, \ 1 \le i \le n-1 \\ xb_{n+1-i} + yc_{n+1-i}, & \text{if } j = i-1, \ 2 \le i \le n \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence,

$$S_{e_k,n} = S_{xb_k+yc_k,n} = xS_{b_k,n} + yS_{c_k,n} \in V_{S_{a_k,n}}.$$

Thus, $V_{S_{a_k,n}}$ is a subspace of the vector space of all n-dimensional matrices.

Let $\{o_m\}$ be an $(n-1)$-tuple whose m-th component is one and the others are zero. For instance, $o_2 = (0, 1, 0, \ldots, 0)$. Then, every matrix $S_{a_k,n} \in V_{S_{a_k,n}}$ can be expressed as

$$S_{a_k,n} = \sum_{m=1}^{n-1} a_m S_{o_m,n},$$

where a_m's represent the terms of the real sequence $\{a_k\}_{k=1}^{n-1}$. Moreover, the matrices $S_{o_m,n}$ are linearly independent for $m = 1, 2, \ldots, n-1$. Then, the $n-1$ matrices $S_{o_m,n}$ form a basis for $V_{S_{a_k,n}}$. That is, the dimension of $V_{S_{a_k,n}}$ is $n-1$. □

Theorem 2.2. *The determinant of the matrix $S_{a_k,n}$ is*

$$\det(S_{a_k,n}) = \begin{cases} 0, & \textit{if } n \textit{ is odd} \\ (-1)^{\frac{n}{2}} \prod_{i=1}^{\frac{n}{2}} a_{2i-1}^2, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Since the odd case can be verified easily and the even case is a well-known result (see, e.g., [11, 12, 24]), we omit the proof. □

Theorem 2.3. *Let the inverse of the matrix $S_{a_k,n}$ be $S_{a_k,n}^{-1} = [q_{ij}]$ for even n. Then,*

$$q_{ij} = \begin{cases} (-1)^{\frac{i+j-1}{2}-1} \dfrac{1}{a_{i-1}} \displaystyle\prod_{t=\frac{i-1}{2}}^{\frac{j}{2}-1} \dfrac{a_{2t}}{a_{n-1-2t}}, & \textit{if } i = 1, 3, \ldots, n-1, \quad j = i+1, i+3, \ldots, n \\ (-1)^{\frac{i+j-1}{2}-1} \dfrac{1}{a_{n+1-j}} \displaystyle\prod_{t=\frac{j-1}{2}}^{\frac{i}{2}-1} \dfrac{a_{n-2t}}{a_{2t+1}}, & \textit{if } j = 1, 3, \ldots, n-1, \quad i = j+1, j+3, \ldots, n \\ 0, & \textit{otherwise,} \end{cases}$$

where $\{a_k\}_{k=1}^{n-1}$ is the real sequence, used in the matrix $S_{a_k,n}$. We note that for $i = 1$, the term a_0 appears. Here, $a_0 \neq 0$ serves as an auxiliary parameter and is not part of the sequence.

Proof. According to the determinant given in Theorem 2.2, we obtain that the matrix $S_{a_k,n}$ is singular when n is odd. Then, we consider the inverse of the matrix $S_{a_k,n}$ only for even n. Let $S_{a_k,n}^{-1} = [q_{ij}]$. Then, the equality

$$S_{a_k,n}S_{a_k,n}^{-1} = \begin{pmatrix} a_1q_{21} & a_1q_{22} & \cdots & a_1q_{2,n-1} & a_1q_{2n} \\ a_{n-1}q_{11}+a_2q_{31} & a_{n-1}q_{12}+a_2q_{32} & \cdots & a_{n-1}q_{1,n-1}+a_2q_{3,n-1} & a_{n-1}q_{1n}+a_2q_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2q_{n-2,1}+a_{n-1}q_{n1} & a_2q_{n-2,2}+a_{n-1}q_{n2} & \cdots & a_2q_{n-2,n-1}+a_{n-1}q_{n,n-1} & a_2q_{n-2,n}+a_{n-1}q_{nn} \\ a_1q_{n-1,1} & a_1q_{n-1,2} & \cdots & a_1q_{n-1,n-1} & a_1q_{n-1,n} \end{pmatrix}$$
$$= I_n,$$

where I_n is the n-dimensional identity matrix, describes the entries of $S_{a_k,n}^{-1}$ as desired. □

Remark 2.1. *We note that Theorem 2.3 is a consequence of the general inversion formula provided in* [28]. *Additionally, the matrix $S_{a_k,n}^{-1}$ is centrosymmetric, as $S_{a_k,n}$ itself is centrosymmetric, as discussed in* [3].

Next, we present the LU decomposition for the matrix $S_{a_k,n}$.

Theorem 2.4. *The LU decomposition of the matrix $S_{a_k,n}$ exists for all n, and the entries of the n-dimensional matrices L and U are as follows:*

(i) *Let n be even. Then, the entries of $L = [l_{ij}]$ and $U = [u_{ij}]$ are*

$$l_{ij} = \begin{cases} 1, & \text{if } i=j \\ \frac{a_{n-i+2}}{a_{j-1}}, & \text{if } i=j+2,\ i=2m,\ m=2,3,\ldots,\frac{n}{2} \\ 0, & \text{otherwise} \end{cases}$$

and

$$u_{ij} = \begin{cases} a_{n-i}, & \text{if } i=j,\ i=2m+1,\ m=0,1,\ldots,\frac{n}{2}-1 \\ a_{i-1}, & \text{if } i=j,\ i=2m,\ m=1,2,\ldots,\frac{n}{2} \\ a_{i+1}, & \text{if } i=j-2,\ i=2m+1,\ m=0,1,\ldots,\frac{n-1}{2}-2 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *Let n be odd. Then, the entries of $L = \left[l_{ij}^*\right]$ and $U = \left[u_{ij}^*\right]$ are*

$$l_{ij}^* = \begin{cases} 1, & \text{if } i=j \\ \frac{a_1}{a_{n-2}}, & \text{if } i=n,\ j=n-1 \\ \frac{a_{n-i+2}}{a_{j-1}}, & \text{if } i=j+2,\ \ i=2m,\ \ m=2,3,\ldots,\frac{n-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

and

$$u_{ij}^* = \begin{cases} a_{n-i}, & \text{if } i=j,\ \ i=2m+1,\ \ m=0,1,\ldots,\frac{n-1}{2}-1 \\ a_{i-1}, & \text{if } i=j,\ \ i=2m,\ \ m=1,2,\ldots,\frac{n-1}{2} \\ a_{i+1}, & \text{if } i=j-2,\ \ i=2m+1,\ \ m=0,1,\ldots,\frac{n}{2}-2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The matrix multiplication $S_{a_k,n} = LU$ yields the desired results. □

The characteristic polynomial of a matrix provides important information about the matrix in question. Now, we examine the characteristic polynomial and eigenvalues of the matrix $S_{a_k,n}$ considering Theorem 1.1.

Let n be even, then the matrix $S_{a_k,n}$ can be partitioned as

$$S_{a_k,n} = \left(\begin{array}{cc} A & \widehat{I}_{\frac{n}{2}} B \widehat{I}_{\frac{n}{2}} \\ B & \widehat{I}_{\frac{n}{2}} A \widehat{I}_{\frac{n}{2}} \end{array} \right),$$

where $\widehat{I}_{\frac{n}{2}}$ is the $\frac{n}{2}$-dimensional symmetric elementary matrix, and $A = [a_{ij}]$ and $B = [b_{ij}]$ are the $\frac{n}{2}$-dimensional matrices with entries characterized by

$$a_{ij} = \begin{cases} a_i, & \text{if } j = i+1, \ 1 \le i \le \frac{n}{2} - 1, \\ a_{n+1-i}, & \text{if } j = i-1, \ 2 \le i \le \frac{n}{2}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

and

$$b_{ij} = \begin{cases} a_{\frac{n}{2}}, & \text{if } i = 1, j = \frac{n}{2}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

respectively. By considering $S_{a_k,n}$ is a centrosymmetric matrix, we have the following lemma.

Lemma 2.1. *For even n, the matrix $S_{a_k,n}$ has the $\frac{n}{2}$ skew symmetric orthonormal eigenvectors τ_i with corresponding eigenvalues α_i, determined by solutions of the equation*

$$\left(A - \widehat{I}_{\frac{n}{2}} B\right) \xi_i = \alpha_i \xi_i,$$

where $i = 1, 2, \ldots, \frac{n}{2}$, A and B are the matrices given by Equations (2.1) *and* (2.2)*, respectively, $\widehat{I}_{\frac{n}{2}}$ is the $\frac{n}{2}$-dimensional symmetric elementary matrix and $\tau_i = \frac{1}{\sqrt{2}} \left(\xi_i, -\widehat{I}_{\frac{n}{2}} \xi_i\right)^T$'s are the n-dimensional vectors. Also, ξ_i form an orthonormal set. The $\frac{n}{2}$ symmetric orthonormal eigenvectors κ_j of $S_{a_k,n}$ and the corresponding eigenvalues β_j are obtained by the solutions of the equation*

$$\left(A + \widehat{I}_{\frac{n}{2}} B\right) \eta_j = \beta_j \eta_j,$$

where $j = 1, 2, \ldots, \frac{n}{2}$, A, B are the matrices given by Equations (2.1) *and* (2.2)*, respectively, $\widehat{I}_{\frac{n}{2}}$ is the $\frac{n}{2}$-dimensional symmetric elementary matrix and $\kappa_j = \frac{1}{\sqrt{2}} \left(\eta_j, \widehat{I}_{\frac{n}{2}} \eta_j\right)^T$'s are the n-dimensional vectors. Also, η_j form an orthonormal set. The set $\left(\tau_1, \tau_2, \ldots, \tau_{\frac{n}{2}}, \kappa_1, \kappa_2, \ldots, \kappa_{\frac{n}{2}}\right)$ is an orthonormal set of n eigenvectors of $S_{a_k,n}$.*

As a conclusion of Lemma 2.1, the eigenvalues of the matrix $S_{a_k,n}$ are the same as those of the matrices $A - \widehat{I}_{\frac{n}{2}} B$ and $A + \widehat{I}_{\frac{n}{2}} B$ in the even case of n. The next theorem is useful to compute eigenvalues of the matrix $S_{a_k,n}$ for even n.

Theorem 2.5. *Let n be even and let $Q_{\mathcal{A}_-,\frac{n}{2}}(\lambda)$ and $Q_{\mathcal{A}_+,\frac{n}{2}}(\lambda)$ be the characteristic polynomials of tridiagonal matrices $\mathcal{A}_- = A - \widehat{I}_{\frac{n}{2}} B$ and $\mathcal{A}_+ = A + \widehat{I}_{\frac{n}{2}} B$, respectively. Then,*

$$Q_{\mathcal{A}_-,\frac{n}{2}}(\lambda) = \left(\lambda + a_{\frac{n}{2}}\right) d'_{\frac{n}{2}-1} - a_{\frac{n}{2}-1} a_{\frac{n}{2}+1} d'_{\frac{n}{2}-2}$$

and

$$Q_{\mathcal{A}_+,\frac{n}{2},}(\lambda) = \left(\lambda - a_{\frac{n}{2}}\right) d'_{\frac{n}{2}-1} - a_{\frac{n}{2}-1} a_{\frac{n}{2}+1} d'_{\frac{n}{2}-2},$$

where $d_i^{'}\left(i=3,4,\ldots,\frac{n}{2}-1\right)$ *represents the determinant of the* i*-th leading principle submatrix of the matrix* $\lambda I_{\frac{n}{2}}-\mathcal{A}_{-}$ *or* $\lambda I_{\frac{n}{2}}-\mathcal{A}_{+}$ *and satisfies the following recurrence relation*

$$d_i^{'}=\lambda d_{i-1}^{'}-a_{i-1}a_{n+1-i}d_{i-2}^{'}$$

with initial conditions $d_0^{'}=1$, $d_1^{'}=\lambda$ *and* $d_2^{'}=\lambda^2-a_1a_{n-1}$.

Proof. By doing row expansion followed by the last column expansion for the matrices $\lambda I_{\frac{n}{2}}-\mathcal{A}_{-}$ and $\lambda I_{\frac{n}{2}}-\mathcal{A}_{+}$, respectively, the desired results are obtained. □

We note that for every real number $\alpha\neq 0$, if $Q_{\mathcal{A}_{-},\frac{n}{2}}(\alpha)=0$, then it is clear that $Q_{\mathcal{A}_{+},\frac{n}{2}}(-\alpha)=0$, as well. The next Corollary gives the characteristic polynomial of the matrix $S_{a_k,n}$, with the help of the characteristic polynomials of the matrices $\mathcal{A}_{-}$ and $\mathcal{A}_{+}$.

Corollary 2.1. *Let* n *be even, and let* $Q_{\mathcal{A}_{-},\frac{n}{2}}(\lambda)$ *and* $Q_{\mathcal{A}_{+},\frac{n}{2}}(\lambda)$ *denote the characteristic polynomials of tridiagonal matrices* $\mathcal{A}_{-}$ *and* $\mathcal{A}_{+}$*, respectively. Then, we have the equality*

$$Q_{S_{a_k},n}(\lambda)=Q_{\mathcal{A}_{-},\frac{n}{2}}(\lambda)\,Q_{\mathcal{A}_{+},\frac{n}{2}}(\lambda)$$

for the characteristic polynomial of $S_{a_k,n}$.

Proof. For even n, considering Lemma 2.1 and Theorem 2.5, the roots of the polynomials $Q_{\mathcal{A}_{+},\frac{n}{2}}(\lambda)$ and $Q_{\mathcal{A}_{-},\frac{n}{2}}(\lambda)$ together form the roots of the polynomial $Q_{S_{a_k},n}(\lambda)$. Therefore, by the factoring principle, we obtain the equality

$$Q_{S_{a_k},n}(\lambda)=Q_{\mathcal{A}_{-},\frac{n}{2}}(\lambda)\,Q_{\mathcal{A}_{+},\frac{n}{2}}(\lambda)$$

for even n. □

For an odd n, the matrix $S_{a_k,n}$ can be partitioned as

$$S_{a_k,n}=\begin{pmatrix} A^{*} & X_2 & 0_{\frac{n-1}{2}} \\ X_1 & 0 & X_1\widehat{I}_{\frac{n-1}{2}} \\ 0_{\frac{n-1}{2}} & \widehat{I}_{\frac{n-1}{2}}X_2 & \widehat{I}_{\frac{n-1}{2}}A^{*}\widehat{I}_{\frac{n-1}{2}} \end{pmatrix},$$

where $A^{*}=\left[a_{ij}^{*}\right]$ is an $\frac{n-1}{2}$-dimensional matrix with entries characterized by

$$a_{ij}^{*}=\begin{cases} a_i, & \text{if } j=i+1,\ 1\leq i\leq \frac{n-1}{2}-1, \\ a_{n+1-i}, & \text{if } j=i-1,\ 2\leq i\leq \frac{n-1}{2}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

where $0_{\frac{n-1}{2}}$ is the $\frac{n-1}{2}$-dimensional zero matrix, $\widehat{I}_{\frac{n-1}{2}}$ is the $\frac{n-1}{2}$-dimensional symmetric elementary matrix, $X_1=\left(0,0,\ldots,a_{\frac{n+1}{2}}\right)$, $X_2^T=\left(0,0,\ldots,a_{\frac{n-1}{2}}\right)$ are the $\frac{n-1}{2}$-dimensional row vectors, and 0 is the central entry of $S_{a_k,n}$.

Lemma 2.2. *For odd n, the matrix $S_{a_k,n}$ has the $\frac{n-1}{2}$ skew symmetric orthonormal eigenvectors τ_i with corresponding eigenvalues α_i, determined by the solutions of the equation*

$$A^*\xi_i = \alpha_i \xi_i,$$

where $i = 1, 2, \ldots, \frac{n-1}{2}$, A^* is the matrix given by Equation (2.3)*, $\tau_i = \left(\xi_i, 0, -\widehat{I}_{\frac{n-1}{2}}\xi_i\right)^T$'s are the n-dimensional vectors and $\widehat{I}_{\frac{n-1}{2}}$ is the $\frac{n-1}{2}$-dimensional symmetric elementary matrix. Also, ξ_i form an orthonormal set. The $\frac{n+1}{2}$ symmetric orthonormal eigenvectors κ_j of $S_{a_k,n}$ and the corresponding eigenvalues β_j are obtained by the solutions of the equation*

$$A^{**}\begin{pmatrix} \eta_j \\ q_j \end{pmatrix} = \begin{pmatrix} A^* & \sqrt{2}X_2 \\ \sqrt{2}X_1 & 0 \end{pmatrix}\begin{pmatrix} \eta_j \\ q_j \end{pmatrix} = \beta_j \begin{pmatrix} \eta_j \\ q_j \end{pmatrix},$$

where $j = 1, 2, \ldots, \frac{n+1}{2}$, $\kappa_j = \frac{1}{\sqrt{2}}\left(\eta_j, 2q_j, \widehat{I}_{\frac{n+1}{2}}\eta_j\right)^T$'s are the n-dimensional vectors, A^* is the matrix given by Equation (2.3)*, $X_1 = \left(0, 0, \ldots, a_{\frac{n+1}{2}}\right)$, $X_2^T = \left(0, 0, \ldots, a_{\frac{n-1}{2}}\right)$ are $\frac{n+1}{2}$-dimensional row vectors. Also, η_j form an orthonormal set. The set $\left(\tau_1, \tau_2, \ldots, \tau_{\frac{n-1}{2}}, \kappa_1, \kappa_2, \ldots, \kappa_{\frac{n+1}{2}}\right)$ is an orthonormal set of n eigenvectors of $S_{a_k,n}$.*

Lemma 2.2 demonstrates that the eigenvalues of $S_{a_k,n}$ coincide with those of the matrices A^* and A^{**} when n is odd. Building on this result, we present the following theorem, which serves as a useful tool for calculating the eigenvalues of the matrix $S_{a_k,n}$ in the case of odd n.

Theorem 2.6. *Let n be odd and let $Q_{A^*,\frac{n-1}{2}}(\lambda)$ and $Q_{A^{**},\frac{n+1}{2}}(\lambda)$ be the characteristic polynomials of the tridiagonal matrices A^* and A^{**}, respectively. Then,*

$$Q_{A^*,\frac{n-1}{2}}(\lambda) = \lambda d''_{\frac{n-1}{2}-1} - a_{\frac{n-1}{2}-1}a_{\frac{n+1}{2}+1}d''_{\frac{n-1}{2}-2}$$

and

$$Q_{A^{**},\frac{n+1}{2}}(\lambda) = \lambda d''_{\frac{n+1}{2}-1} - 2a_{\frac{n-1}{2}}a_{\frac{n+1}{2}}d''_{\frac{n+1}{2}-2},$$

where d''_i $\left(i = 3, 4, \ldots, \frac{n-1}{2} - 1\right)$ represents the determinant of the i-th leading principle submatrix of the matrix A^* or A^{**} and satisfies the following recurrence relation

$$d''_i = \lambda d''_{i-1} - a_{i-1}a_{n+1-i}d''_{i-2}$$

with initial conditions $d''_0 = 1$, $d''_1 = \lambda$ and $d''_2 = \lambda^2 - a_1a_{n-1}$.

Proof. Performing a row expansion followed by an expansion along the last column for the matrices A^* and A^{**}, respectively, yields the desired results. □

We note that, for every real number $\alpha \neq 0$, if $Q_{A^*,\frac{n-1}{2}}(\alpha) = 0$, then $Q_{A^*,\frac{n-1}{2}}(-\alpha) = 0$. Similarly, for every real number $\alpha \neq 0$, if $Q_{A^{**},\frac{n+1}{2}}(\alpha) = 0$, then $Q_{A^{**},\frac{n+1}{2}}(-\alpha) = 0$. The next Corollary gives the characteristic polynomial of the matrix $S_{a_k,n}$ with the help of the characteristic polynomials of the matrices A^* and A^{**}.

Corollary 2.2. *Let n be odd and let $Q_{A^*,\frac{n-1}{2}}(\lambda)$ and $Q_{A^{**},\frac{n+1}{2}}(\lambda)$ denote the characteristic polynomials of tridiagonal matrices A^* and A^{**}, respectively. Then, we have the equality*

$$Q_{S_{a_k},n}(\lambda) = Q_{A^*,\frac{n-1}{2}}(\lambda)\, Q_{A^{**},\frac{n+1}{2}}(\lambda)$$

for the characteristic polynomial of $S_{a_k,n}$.

Proof. For odd n, considering Lemma 2.2 and Theorem 2.6, the roots of the polynomials $Q_{A^*,\frac{n-1}{2}}(\lambda)$ and $Q_{A^{**},\frac{n+1}{2}}(\lambda)$ together form the roots of the polynomial $Q_{S_{a_k},n}(\lambda)$. Therefore, by the factoring principle, we obtain the equality

$$Q_{S_{a_k},n}(\lambda) = Q_{A^*,\frac{n-1}{2}}(\lambda)\, Q_{A^{**},\frac{n+1}{2}}(\lambda),$$

where n is odd. □

Remark 2.2. *The eigenvalues of the matrix $S_{a_k,n}$ are distinct and symmetric around zero.*

The results on the Euclidean and spectral norms of the matrix $S_{a_k,n}$ are presented here. Before proceeding, it is important to recall some essential concepts that will be necessary for the discussion.

The Euclidean (Frobenius) norm and spectral norm of $m \times n$ matrix $X = [x_{ij}]$ are defined as

$$\|X\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} |x_{ij}|^2\right]^{\frac{1}{2}} \quad \text{and} \quad \|X\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i \left(X^H X\right)},$$

respectively, where X^H is the conjugate transpose of the matrix X and $\lambda_i\left(X^H X\right)$'s denote eigenvalues of $X^H X$, [17]. Let $X = [x_{ij}]$, $Y = [y_{ij}]$ and $Z = [z_{ij}]$ be the $m \times n$ matrices and $Y \circ Z = [y_{ij} z_{ij}] = X$. Then,

$$\|X\|_2 \le r_1(Y)\, c_1(Z),$$

where $r_1(Y)$ is the maximum row length norm of the matrix Y and $c_1(Z)$ is the maximum column length norm of the matrix Z defined by

$$r_1(Y) = \max_i \sqrt{\sum_j |y_{ij}|^2} \quad \text{and} \quad c_1(Z) = \max_j \sqrt{\sum_i |z_{ij}|^2},$$

respectively, [17].

Theorem 2.7. *The following results hold for the Euclidean and spectral norms of the matrix $S_{a_k,n}$.*

(i) *The Euclidean norm of the matrix $S_{a_k,n}$ is*

$$\|S_{a_k,n}\|_F = \sqrt{2\sum_{i=1}^{n-1} a_i^2},$$

(ii) *The upper bound for the spectral norm of the matrix $S_{a_k,n}$ is as follows:*

- *Let the sequence* $\{a_k\}_{k=1}^{n-1}$ *be positive increasing, where* $a_m - a_{m-1} \geq 1$ $(m = 2, 3, \ldots, n)$. *Then,*

$$\|S_{a_k,n}\|_2 \leq a_{n-1}^2 + 1.$$

- *Let the sequence* $\{a_k\}_{k=1}^{n-1}$ *be positive decreasing, where* $a_1 \geq 1$. *Then,*

$$\|S_{a_k,n}\|_2 \leq a_1\sqrt{a_1^2 + n - 1}.$$

Proof. (i) The result is obtained easily considering the definition of the Euclidean norm.

(ii) By using the Hadamard product, the matrix $S_{a_k,n}$ can be expressed as

$$S_{a_k,n} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ a_{n-1} & 0 & 1 & \cdots & 0 & 0 \\ 0 & a_{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & a_1 & 0 \end{pmatrix}}_{Y} \circ \underbrace{\begin{pmatrix} 1 & a_1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & a_2 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & a_{n-1} \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}}_{Z}.$$

- For the positive increasing sequence $\{a_k\}_{k=1}^{n-1}$, where $a_m - a_{m-1} \geq 1$ $(m = 2, 3, \ldots, n)$, the maximum row length norm of the matrix $Y = [y_{ij}]$ and maximum column length norm of the matrix $Z = [z_{ij}]$ are

$$r_1(Y) = \max_i \sqrt{\sum_j |y_{ij}|^2} = \sqrt{a_{n-1}^2 + 1}$$

and

$$c_1(Z) = \max_j \sqrt{\sum_i |z_{ij}|^2} = \sqrt{a_{n-1}^2 + 1},$$

respectively. Hence, we have

$$\|S_{a_k,n}\|_2 \leq r_1(Y)\, c_1(Z) = a_{n-1}^2 + 1.$$

- For the positive decreasing sequence $\{a_k\}_{k=1}^{n-1}$, where $a_1 \geq 1$, we have

$$r_1(Y) = \max_i \sqrt{\sum_j |y_{ij}|^2} = \sqrt{a_1^2} = a_1$$

and

$$c_1(Z) = \max_j \sqrt{\sum_i |z_{ij}|^2} = \sqrt{a_1^2 + n - 1}.$$

Hence, we obtain

$$\|S_{a_k,n}\|_2 \leq r_1(Y)\, c_1(Z) = a_1\sqrt{a_1^2 + n - 1}.$$

as desired. □

The Fibonacci-sequence Sylvester–Kac matrix $S_{F_k,n}$, a specific case of $S_{a_k,n}$, is obtained by replacing the sequence $\{a_k\}_{k=1}^{n-1}$ with the Fibonacci sequence $\{F_k\}_{k=1}^{n-1}$. We proceed to derive results concerning the Euclidean and spectral norms of this matrix. In our analysis, we utilize the Lucas numbers, defined recursively by the relation

$$L_{k+1} = L_k + L_{k-1},$$

as well as the Binet formulas for the Fibonacci and Lucas numbers, which are given by

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \quad \text{and} \quad L_k = \alpha^k + \beta^k,$$

respectively, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, [22].

Corollary 2.3. (i) *The Euclidean norm of the Fibonacci–Sylvester–Kac matrix is*

$$\|S_{F_k,n}\|_F = \sqrt{\frac{2}{5}\left(L_{2n-1} - L_1 + (-1)^n + 1\right)},$$

where L_n is the n-th Lucas number.

(ii) *The upper bound for the spectral norm of the Fibonacci–Sylvester–Kac matrix is*

$$\|S_{F_k,n}\|_2 \le F_{n-1}^2 + 1.$$

Proof. (i)

$$\begin{aligned}
\|S_{F_k,n}\|_F^2 &= 2\sum_{i=1}^{n-1} F_i^2 \\
&= 2\sum_{i=1}^{n-1}\left(\frac{\alpha^i - \beta^i}{\sqrt{5}}\right)^2 \\
&= \frac{2}{5}\sum_{i=1}^{n-1}\left(\alpha^{2i} + \beta^{2i} - 2(-1)^i\right)
\end{aligned}$$

By the means of the well-known equality $\sum_{i=1}^{n-1} p^i = \dfrac{p^n - p}{p-1}$ for an arbitrary p, ($p \neq 1$), we have

$$\begin{aligned}
\|S_{F_k,n}\|_F^2 &= \frac{2}{5}\left(\frac{\alpha^{2n} - \alpha^2}{\alpha^2 - 1} + \frac{\beta^{2n} - \beta^2}{\beta^2 - 1}\right) - \frac{4}{5}\sum_{i=1}^{n-1}(-1)^i \\
&= \frac{2}{5}\left(\frac{\alpha^{2n} - \alpha^2}{\alpha^2 - 1} + \frac{\beta^{2n} - \beta^2}{\beta^2 - 1} + (-1)^n + 1\right) \\
&= \frac{2}{5}\left(\alpha^{2n-1} + \beta^{2n-1} - \alpha - \beta + (-1)^n + 1\right) \\
&= \frac{2}{5}\left(L_{2n-1} - L_1 + (-1)^n + 1\right).
\end{aligned}$$

(ii) The proof is similar to the proof of Theorem 2.7. □

Finally, we present an example to illustrate our results. This example involves the Pell number sequence, a sequence as significant as the Fibonacci sequence, defined by the recurrence relation

$$P_k = 2P_{k-1} + P_{k-2}$$

for all $k \ge 2$ with initial conditions $P_0 = 0$ and $P_1 = 1$ [22]. The first few terms of the Pell sequence are $0, 1, 2, 5, 12, 29, 169, 408, 985, \dots$.

Example 2.1. *By substituting* $\{a_k\}_{k=1}^{n-1}$ *with* $\{P_k\}_{k=1}^{n-1}$ *in the matrix* $S_{a_k,n}$ *for* $n = 5, 6$*, we obtain the following matrices*

$$S_{P_k,5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 12 & 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \textit{and} \quad S_{P_k,6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 29 & 0 & 2 & 0 & 0 & 0 \\ 0 & 12 & 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 & 12 & 0 \\ 0 & 0 & 0 & 2 & 0 & 29 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The determinants of the matrices $S_{P_k,5}$ *and* $S_{P_k,6}$ *are*

$$\det(S_{P_k,5}) = 0$$

and

$$\det(S_{P_k,6}) = -\prod_{i=1}^{3} P_{2i-1}^2 = -21025.$$

Since the matrix $S_{P_k,5}$ *is singular, we only have the inverse of the matrix* $S_{P_k,6}$ *as*

$$S_{P_k,6}^{-1} = \begin{pmatrix} 0 & \frac{1}{29} & 0 & -\frac{2}{145} & 0 & \frac{24}{145} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 & -\frac{12}{5} \\ -\frac{12}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{24}{145} & 0 & -\frac{2}{145} & 0 & \frac{1}{29} & 1 \end{pmatrix}.$$

The LU decompositions of the matrices $S_{P_k,5}$ *and* $S_{P_k,6}$ *are*

$$S_{P_k,5} = LU = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 1 \end{pmatrix} \begin{pmatrix} 12 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 12 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$S_{P_k,6} = LU = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 12 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & 0 & 1 \end{pmatrix} \begin{pmatrix} 29 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 12 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 29 \end{pmatrix},$$

respectively.
For the auxiliary matrices

$$A^* = \begin{pmatrix} 0 & 1 \\ 12 & 0 \end{pmatrix} \quad \textit{and} \quad A^{**} = \begin{pmatrix} 0 & 1 & 0 \\ 12 & 0 & 2\sqrt{2} \\ 0 & 5\sqrt{2} & 0 \end{pmatrix},$$

the characteristic polynomial of $S_{P_k,5}$ *is*

$$
\begin{aligned}
Q_{S_{P_k},5}(\lambda) &= Q_{A^*,2}(\lambda)\, Q_{A^{**},3}(\lambda) \\
&= (\lambda^2 - 12)(\lambda^3 - 32\lambda) \\
&= \lambda^5 - 44\lambda^3 + 384\lambda,
\end{aligned}
$$

where $Q_{A^*,2}(\lambda)$ and $Q_{A^{**},3}(\lambda)$ denote the characteristic polynomials of A^* and A^{**}, respectively. The eigenvalues of $S_{P_k,5}$ are

$$
\lambda_1 = 3.464, \;\; \lambda_2 = 5.656, \;\; \lambda_3 = 0, \;\; \lambda_4 = -5.656, \;\; \lambda_5 = -3.464.
$$

For the auxiliary matrices

$$
\mathcal{A}_- = \begin{pmatrix} 0 & 1 & 0 \\ 29 & 0 & 2 \\ 0 & 12 & -5 \end{pmatrix} \quad \textit{and} \quad \mathcal{A}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 29 & 0 & 2 \\ 0 & 12 & 5 \end{pmatrix},
$$

the characteristic polynomial of $S_{P_k,6}$ is

$$
\begin{aligned}
Q_{S_{P_k},6}(\lambda) &= Q_{\mathcal{A}_-,3}(\lambda)\, Q_{\mathcal{A}_+,3}(\lambda) \\
&= (\lambda^3 + 5\lambda^2 - 53\lambda - 145)(\lambda^3 - 5\lambda^2 - 53\lambda + 145), \\
&= \lambda^6 - 131\lambda^4 + 4259\lambda^2 - 21025,
\end{aligned}
$$

where $Q_{\mathcal{A}_-,3}(\lambda)$ and $Q_{\mathcal{A}_+,3}(\lambda)$ denote the characteristic polynomials of $\mathcal{A}_-$ and $\mathcal{A}_+$, respectively. The eigenvalues of $S_{P_k,6}$ are

$$
\lambda_1 = -6.526, \;\; \lambda_2 = 2.447, \;\; \lambda_3 = 9.079, \;\; \lambda_4 = -9.079, \;\; \lambda_5 = -2.447, \;\; \lambda_6 = 6.526.
$$

The Euclidean norms of the matrices $S_{P_k,5}$ and $S_{P_k,6}$ are

$$
\|S_{P_k,5}\|_F = \sqrt{2\sum_{i=1}^{4} P_i^2} = 18.65475811
$$

and

$$
\|S_{P_k,6}\|_F = \sqrt{2\sum_{i=1}^{5} P_i^2} = 45.05552130.
$$

The upper bounds for the spectral norms of these matrices are

$$
\|S_{P_k,5}\|_2 = 12.32882801 \leq P_4^2 + 1 = 145
$$

and

$$
\|S_{P_k,6}\|_2 = 29.07142814 \leq P_5^2 + 1 = 842.
$$

3 Conclusion

This paper introduces a new Sylvester–Kac type matrix $S_{a_k,n}$, whose entries are defined by an integer sequence $\{a_k\}_{k=1}^{n-1}$. The main result is the derivation of a recursive relation for the factorization of the characteristic polynomial of the matrix $S_{a_k,n}$. This factorization expresses the characteristic polynomial as the product of the characteristic polynomials of two smaller matrices obtained through a specific block decomposition method introduced by Muthiyalu and Usha [23]. Although the block decomposition used in this paper differs structurally from that of Cantoni and Butler [3], both approaches yield consistent spectral results. Additionally, the matrix's determinant, inverse, LU decomposition, and several norms are examined, and a numerical example is provided to confirm the results.

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