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The quaternion-type cyclic-balancing sequence in groups

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Abstract: In this research, several types of definitions of the quaternion-type cyclic-balancing sequence are presented. The Cassini formula and generating function of these sequences are also obtained for all types. The quaternion-type cyclic-balancing sequences modulo m, the first step to transferring this topic to group theory, are examined. These sequences in finite groups are then defined. Eventually, the lengths of periods for these sequences of the generalized quaternion group are calculated.

Keywords: Balancing sequence, Groups, Period, Presentation.
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1 Introduction and Preliminaries

The study of number sequences has attracted considerable attention from the mathematicians since ancient times. Since then, many of them have concentrated their interest on the study of ravishing triangular numbers. In [1], Behera and Panda introduced the notion of balancing numbers $(B_n)_{n\in\mathbb{N}}$ as solutions to a certain Diophantine equation in 1999. Then, the recurrence relation of this number is $B_{n+1} = 6B_n - B_{n-1}$ for $n \geqslant 1$, where $B_0 = 0$, $B_1 = 1$. A study on the Lucas-balancing numbers $C_n = \sqrt{8B_n^2 + 1}$ was published in 2009 by Panda [11]. The recurrence relation of this number is $C_{n+1} = 6C_n - C_{n-1}$ for $n \geqslant 1$, where $C_0 = 1$, $C_1 = 3$. Also, periodicity of these numbers was examined in [12, 13].

Additionally, matrices can be used to represent the balancing numbers and can be extended to related sequences. In [14], Ray introduced balancing F-matrix as follows:

$$F = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$$

and gave a general formula for the n-th powers of this matrix:

$$F^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}.$$

Also, for a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_i = a_{i+1}, 0 \leq i \leq n-1,\ x_{n+i} = \prod_{j=1}^{n} x_{i+j-1},\ i \geq 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$ (see [2–4]).

It is well known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. The study of linear recurrence sequences in groups began with the earlier work of Wall [15], where the ordinary Fibonacci sequences in cyclic groups were investigated. Recently, many authors have studied some special linear recurrence sequences in groups; see, for example, [5–10, 16].

Firstly, we define the three different quaternion-type cyclic-balancing sequences and then present some properties, such as, the Cassini formulas, generating functions, relationships between the Balancing sequence and these quaternions in Section 2. Secondly, in Section 3, we study quaternion-type cyclic-balancing sequences modulo m, and then we give the relationships between the lengths of the periods of the quaternion-type cyclic-balancing sequences of the first, second and third kinds modulo m and the generating matrices of these sequences. Finally, in Section 4, we introduce the quaternion-type cyclic-balancing sequences in groups and calculate the quaternion balancing lengths of generalized quaternion groups.

2 The quaternion-type cyclic-balancing sequences

In this section, we introduce three different quaternion-type cyclic-balancing sequences for $n \geq 2$ any positive integer numbers. Then, we present the miscellaneous properties of such sequences.

Definition 2.1. *We define the quaternion-type cyclic-balancing sequences of the first, second and third kinds, respectively:*

$$x_n^1 = \begin{cases} 6kx_{n-1}^1 - jx_{n-2}^1, & \text{if } n \equiv 0 \pmod 3, \\ 6jx_{n-1}^1 - ix_{n-2}^1, & \text{if } n \equiv 1 \pmod 3, \\ 6ix_{n-1}^1 - kx_{n-2}^1, & \text{if } n \equiv 2 \pmod 3, \end{cases} \qquad x_n^2 = \begin{cases} 6ix_{n-1}^2 - kx_{n-2}^2, & \text{if } n \equiv 0 \pmod 3, \\ 6kx_{n-1}^2 - jx_{n-2}^2, & \text{if } n \equiv 1 \pmod 3, \\ 6jx_{n-1}^2 - ix_{n-2}^2, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

$$x_n^3 = \begin{cases} 6jx_{n-1}^3 - ix_{n-2}^3, & \text{if } n \equiv 0 \pmod 3, \\ 6ix_{n-1}^3 - kx_{n-2}^3, & \text{if } n \equiv 1 \pmod 3, \\ 6kx_{n-1}^3 - jx_{n-2}^3, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

the initial conditions for all type are $x_0^\tau = 0$ and $x_1^\tau = 1$ $(1 \leq \tau \leq 3)$.

Let the entries of the matrices A and B be the elements of the quaternion-type cyclic-balancing sequences,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

so that the following properties hold:

(i). $A \times B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$

(ii). $\det A = a_{11}a_{22} - a_{12}a_{21}.$

(iii). $\det(A \cdot B) = \det A \cdot \det B.$

(iv). $A^n = A^{n-1} \times A \quad (n \in \mathbb{Z}^+).$

Since the multiplication of quaternions is not commutative, the above properties need to consider the multiplicative order. Therefore, it is easy to see that

$$\det A \cdot \det B \neq \det B \cdot \det A$$

and

$$A^{n-1} \times A \neq A \times A^{n-1}.$$

In order to render the operation easier, we define $\epsilon(\eta)$ as follows:

$$\epsilon(\eta) = \begin{cases} j, & \text{if } n \equiv 0 \pmod 3, \\ k, & \text{if } n \equiv 1 \pmod 3, \\ i, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{1}$$

where $\eta \in \mathbb{Z}^+$. We can relate these sequences to the well-known classic Balancing sequence by

$$x_n^\tau = \begin{cases} (-1)^{\frac{n+3}{3}} B_n \epsilon(\tau+2), & \text{if } n \equiv 0 \pmod 3, \\ (-1)^{\frac{n-1}{3}} B_n, & \text{if } n \equiv 1 \pmod 3, \\ (-1)^{\frac{n-2}{3}} B_n \epsilon(\tau+1), & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where $\tau = 1, 2, 3$ and $\epsilon(\tau)$ is defined in Equation (1). We can then write the quaternion-type cyclic-balancing sequences

$$G_\tau = \begin{bmatrix} -204 & 35\epsilon(\tau+2) \\ 35\epsilon(\tau+2) & 6 \end{bmatrix} \text{ for } \tau = 1, 2, 3. \tag{2}$$

By mathematical induction on n, we find that

$$(G_\tau)^n = \begin{bmatrix} x^\tau_{3n+1} & x^\tau_{3n} \\ x^\tau_{3n} & -x^\tau_{3n-1}\epsilon(\tau+1) \end{bmatrix} \text{ for } \tau = 1, 2, 3, \tag{3}$$

where $n \geq 1$.

Now we obtain the Cassini formula for the quaternion-type cyclic-balancing sequences. By using the determinant function and Equations (2), (3), we have

$$-x^\tau_{3n+1}x^\tau_{3n-1}\epsilon(\tau+1) - (x^\tau_{3n})^2 = 1 \text{ for } \tau = 1, 2, 3. \tag{4}$$

Lemma 2.1. *The recurrence relations for the $\{x^\tau_n\}$ are as follows:*

$$x^\tau_n = -198x^\tau_{n-3} - x^\tau_{n-6},$$

where $n \geq 6$ and $\tau = 1, 2, 3$.

Proof. The proof will only be done for case $\tau = 1$, the others are done similarly.
By Definition 2.1, we get

$$\begin{cases} x^1_{3n} &= 6kx^1_{3n-1} - jx^1_{3n-2}, \\ x^1_{3n+1} &= 6jx^1_{3n} - ix^1_{3n-1}, \\ x^1_{3n+2} &= 6ix^1_{3n+1} - kx^1_{3n}. \end{cases}$$

Thus, we have

$$\begin{aligned} x^1_{3n+2} &= 6ix^1_{3n+1} - kx^1_{3n} \\ &= 35kx^1_{3n} + 6x^1_{3n-1} \\ &= 6x^1_{3n-1} + 35k\left(6kx^1_{3n-1} - jx^1_{3n-2}\right) \\ &= -204x^1_{3n-1} - k35jx^1_{3n-2}. \end{aligned}$$

Then, since $35jx^1_{3n-2} = k\left(6x^1_{3n-1} - x^1_{3n-4}\right)$, we obtain

$$x^1_{3n+2} = -198x^1_{3n-1} - x^1_{3n-4}. \tag{5}$$

Similarly, we can write

$$\begin{aligned} x^1_{3n+1} &= 6jx^1_{3n} - ix^1_{3n-1} \\ &= 35ix^1_{3n-1} + 6x^1_{3n-2} \\ &= 6x^1_{3n-2} + 35i\left(6ix^1_{3n-2} - kx^1_{3n-3}\right) \\ &= -204x^1_{3n-2} - i35kx^1_{3n-3}. \end{aligned}$$

And then, since $k35x_{3n-3}^1 = i\left(6x_{3n-2}^1 - x_{3n-5}^1\right)$, we acquire

$$x_{3n+1}^1 = -198x_{3n-2}^1 - x_{3n-5}^1. \tag{6}$$

Similarly, we have

$$\begin{aligned} x_{3n}^1 &= 6kx_{3n-1}^1 - jx_{3n-2}^1 \\ &= 35jx_{3n-2}^1 + 6x_{3n-3}^1 \\ &= 6x_{3n-3}^1 + 35j\left(6jx_{3n-3}^1 - ix_{3n-4}^1\right) \\ &= -204x_{3n-3}^1 - j35ix_{3n-4}^1. \end{aligned}$$

And then, since $35ix_{3n-4}^1 = j\left(6x_{3n-3}^1 - x_{3n-6}^1\right)$, we get

$$x_{3n}^1 = -198x_{3n-3}^1 - x_{3n-6}^1. \tag{7}$$

From Equations (5), (6) and (7), we obtain $x_n^1 = -198x_{n-3}^1 - x_{n-6}^1$, as required. □

In the following theorem, we develop the generating functions for the quaternion-type cyclic-balancing sequences.

Theorem 2.1. *The generating functions of the* $\{x_n^\tau\}$ *are*

$$\sum_{n=0}^{\infty} x_n^\tau t^n = \frac{t + 6\epsilon(\tau+1)t^2 + 35\epsilon(\tau+2)t^3 - 6t^4 - \epsilon(\tau+1)t^5}{1 + 198t^3 + t^6},$$

where $\tau = 1,\ 2,\ 3$.

Proof. Assume that $f(t)$ is the generating function of the $\{x_n^\tau\}$ for $\tau = 1,\ 2,\ 3$. Then we have

$$f\left(t\right) = \sum_{n=0}^{\infty} x_n^\tau t^n$$

From Lemma 2.1, we obtain

$$\begin{aligned} f\left(t\right) &= x_0^\tau + x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + x_4^\tau t^4 + x_5^\tau t^5 + \sum_{n=6}^{\infty}\left(-198x_{n-3}^\tau - x_{n-6}^\tau\right)t^n \\ &= x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + x_4^\tau t^4 + x_5^\tau t^5 - 198\left(f(t) - x_0^\tau - x_1^\tau t - x_2^\tau t^2\right)t^3 - f(t)t^6 \,. \end{aligned}$$

Now rearrangement of the equation implies that

$$f(t) = \frac{x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + \left(x_4^\tau + 198x_1^\tau\right)t^4 + \left(x_5^\tau + 198x_2^\tau\right)t^5}{1 + 198t^3 + t^6},$$

which is equal to the $\sum\limits_{n=0}^{\infty} x_n^\tau t^n$ in the Theorem. □

3 The quaternion-type cyclic-balancing sequences modulo m

In this section, we study quaternion-type cyclic-balancing sequences modulo m. Then, we give the relationships among the lengths of periods of the quaternion-type cyclic-balancing sequences of the first, second and third kinds modulo m and the generating matrices of these sequences.

If we reduce the quaternion-type cyclic-balancing sequences of the first, second and third kinds modulo m, taking least non-negative residues, then we obtain the following recurrence sequences:

$$\{x_n^\tau(m)\} = \{x_1^\tau(m),\ x_2^\tau(m),\ \dots\}$$

for every integer $1 \le \tau \le 3$, $x_u^\tau(m)$ is used to mean the u-th element of the τ-th quaternion-type cyclic-balancing sequence when read modulo m. We note here that the recurrence relations in the sequences $\{x_n^\tau(m)\}$ and $\{x_n^\tau\}$ are the same.

Theorem 3.1. *The sequences $\{x_n^\tau(m)\}$ are periodic, and the lengths of their periods are divisible by* 3.

Proof. Let us consider the quaternion-type cyclic-balancing sequence of the first kind $\{x_n^1\}$ as an example. Consider the set

$$\begin{aligned} Q \;=\; \{(q_1,q_2) \mid\ & q_u\text{'s are quaternions } a_u+b_ui+c_uj+d_uk,\ \text{where } a_u,b_u,c_u \text{ and } d_u \\ & \text{are integers such that } 0\le a_u,b_u,c_u,\ d_u\le m-1 \text{ and } u\in\{1,2\}\}\,. \end{aligned}$$

Suppose that the cardinality of the set Q is denoted by the notation $|Q|$. Since the set Q is finite, there are $|Q|$ distinct 2-tuples of the quaternion-type cyclic-balancing sequences of the first kind $\{x_n^1\}$ modulo m. Thus, it is clear that at least one of these 2-tuples appears twice in the sequence $\{x_n^1(m)\}$. Let $x_\alpha^1(m) \equiv x_\beta^1(m)$ and $x_{\alpha+1}^1(m) \equiv x_{\beta+1}^1(m)$. If $\beta-\alpha \equiv 0 \pmod 3$, then we get $x_{\alpha+2}^1(m) \equiv x_{\beta+2}^1(m)$, $x_{\alpha+3}^1(m) \equiv x_{\beta+3}^1(m), \dots$. So, it is easy to see that the subsequence following this 2-tuple repeats; that is, $\{x_n^1(m)\}$ is a periodic sequence and the length of its period must be divisible by 3.

The proofs for the sequences $\{x_n^2\}$ and $\{x_n^3\}$ are directly similar to the above ones and are omitted. □

We next denote the lengths of periods of the sequences $\{x_n^\tau(m)\}$ by $l_{x_n^\tau}(m)$.

Consider the matrices

$$A_1 = \begin{bmatrix} 6i & -k \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 6k & -j \\ 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 6j & -i \\ 1 & 0 \end{bmatrix}.$$

Suppose that $G_1 = A_3A_2A_1,\ G_2 = A_2A_1A_3$ and $G_3 = A_1A_3A_2$. Using the above, we define the following matrices:

$$(M_1)^n = \begin{cases} (G_1)^{\frac{n}{3}}, & \text{if } n \equiv 0 \pmod 3, \\ A_1(G_1)^{\frac{n-1}{3}}, & \text{if } n \equiv 1 \pmod 3, \\ A_2A_1(G_1)^{\frac{n-2}{3}}, & \text{if } n \equiv 2 \pmod 3, \end{cases} \quad (M_2)^n = \begin{cases} (G_2)^{\frac{n}{3}}, & \text{if } n \equiv 0 \pmod 3, \\ A_3(G_2)^{\frac{n-1}{3}}, & \text{if } n \equiv 1 \pmod 3, \\ A_1A_3(G_2)^{\frac{n-2}{3}}, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

$$(M_3)^n=\begin{cases}(G_3)^{\frac{n}{3}}, & \text{if } n\equiv 0 \pmod 3,\\ A_2\,(G_3)^{\frac{n-1}{3}}, & \text{if } n\equiv 1 \pmod 3,\\ A_3A_2\,(G_3)^{\frac{n-2}{3}}, & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

Then we get

$$(M_\tau)^n\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_{n+1}^\tau\\x_n^\tau\end{pmatrix},$$

where τ is an integer such that $1\le\tau\le 3$. We easily derive that $\det A_1=k, \det A_2=j$ and $\det A_3=i$. Therefore, we immediately deduce that $l_{x_n^\tau}(m)$ is the smallest positive integer α such that $(M_\tau)^\alpha\equiv I \pmod m$ for every positive integer m.

4 The quaternion-type cyclic-balancing sequences in groups

In this section, we define three different quaternion-type cyclic-balancing sequences in finite groups. Subsequently, we examine the quaternion-type cyclic-balancing orbits of the first, second and third kinds of the generalized quaternion group. Finally, we give a specific example for the first-type sequences of quaternion group Q_8.

Let G be a 2-generator group and let

$$X=\{(x_1,x_2)\in G\times G\mid\langle\{x_1,x_2\}\rangle=G\}.$$

The notation (x_1,x_2) is said to be a generating pair for G.

Definition 4.1. *Let G be a 2-generator group. For the generating pair (x,y), we define the quaternion-type cyclic-balancing orbits of the first, second and third kinds of G, as follows, respectively:*

$$a_n^1=\begin{cases}(a_{n-2}^1)^{-j}(a_{n-1}^1)^{6k}, & \textit{if } n\equiv 0 \pmod 3,\\ (a_{n-2}^1)^{-i}(a_{n-1}^1)^{6j}, & \textit{if } n\equiv 1 \pmod 3,\\ (a_{n-2}^1)^{-k}(a_{n-1}^1)^{6i}, & \textit{if } n\equiv 2 \pmod 3,\end{cases}\quad a_n^2=\begin{cases}(a_{n-2}^2)^{-k}(a_{n-1}^2)^{6i}, & \textit{if } n\equiv 0 \pmod 3,\\ (a_{n-2}^2)^{-j}(a_{n-1}^2)^{6k}, & \textit{if } n\equiv 1 \pmod 3,\\ (a_{n-2}^2)^{-i}(a_{n-1}^2)^{6j}, & \textit{if } n\equiv 2 \pmod 3,\end{cases}$$

$$a_n^3=\begin{cases}(a_{n-2}^3)^{-i}(a_{n-1}^3)^{6j}, & \textit{if } n\equiv 0 \pmod 3,\\ (a_{n-2}^3)^{-k}(a_{n-1}^3)^{6i}, & \textit{if } n\equiv 1 \pmod 3,\\ (a_{n-2}^3)^{-j}(a_{n-1}^3)^{6k}, & \textit{if } n\equiv 2 \pmod 3,\end{cases}$$

for $n\ge 2$, with initial conditions $a_0^\tau=x$ and $a_1^\tau=y$ $(1\le\tau\le 3)$, where the following conditions hold for every $x,y\in G$:

(i). Let $q=a+bi+cj+dk$ such that a,b,c and d are integers and let e be the identity of G, then:

- $x^q=x^{a \pmod{|x|}+b \pmod{|x|}i+c \pmod{|x|}j+d \pmod{|x|}k}$
  $=x^{a \pmod{|x|}}\,x^{b \pmod{|x|}i}\,x^{c \pmod{|x|}j}\,x^{d \pmod{|x|}k}.$
- $(x^u)^a=(x^a)^u$, *where $u\in\{i,j,k\}$ and a is an integer.*
- $e^q=e$ *and* $x^{0+0i+0j+0k}=e$.

(ii). Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ *and* $q_2 = a_2 + b_2 i + c_2 j + d_2 k,$ *such that* $a_1, b_1, c_1, d_1, a_2, b_2,$ c_2, d_2 *are integers, then* $(x^{q_1} x^{q_2})^{-1} = x^{-q_2} x^{-q_1}$.

(iii). If $xy \neq yx$, *then* $x^u y^u \neq y^u x^u$ *for* $u \in \{i, j, k\}$.

(iv). $(xy)^u = y^u x^u$ *for* $u \in \{i, j, k\}$.

(v). $(x^{u_1} y^{u_2})^{u_3} = x^{u_3 u_1} y^{u_3 u_2}$, $(xy^{u_1})^{u_2} = x^{u_2} y^{u_2 u_1}$ *and* $(x^{u_1} y)^{u_2} = x^{u_2 u_1} y^{u_2}$ *for* u_1, u_2, u_3 $\in \{i, j, k\}$, *and so* $(x^{u_1} y^{u_1})^{u_1} = x^{-1} y^{-1}$.

(vi). *For* $u_1, u_2 \in \{i, j, k\}$ *such that* $u_1 \neq u_2$, $x^{u_1} y^{u_2} = y^{u_2} x^{u_1}$, $xy^{u_1} = y^{u_1} x$, $x^{u_1} y = y x^{u_1}$, *and so* $(xy^{u_1})^{u_1} = x^{u_1} y^{-1}$ *and* $(x^{u_1} y)^{u_1} = x^{-1} y^{u_1}$.

Let the notation $B^{q,\tau}_{(x,y)}(G)$ denote the τ-th quaternion-type cyclic-balancing orbit of the group G for the generating pair (x, y). From the definition of the orbit $B^{q,\tau}_{(x,y)}(G)$ it is clear that the length of the period of this sequence in a finite group depends on the chosen generating pair and the order in which the assignments of x, y are made.

Theorem 4.1. *Let G be a 2-generator group. If G is finite, then the quaternion-type cyclic-balancing orbits of the first, second and third kinds of G are periodic and the lengths of their periods are divisible by* 3.

Proof. Let us consider the second quaternion-type cyclic-balancing orbit of the group G. We take the set

$$S = \Big\{ (s_1)^{a_1(\bmod |s_1|) + b_1(\bmod |s_1|)i + c_1(\bmod |s_1|)j + d_1(\bmod |s_1|)k}, \\ (s_2)^{a_2(\bmod |s_2|) + b_2(\bmod |s_2|)i + c_2(\bmod |s_2|)j + d_2(\bmod |s_2|)k} \\ \mid s_1, s_2 \in G \text{ and } a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{Z} \Big\}.$$

Since the group G is finite, S is a finite set. Hence, there exists $v > u$ such that $a^2_u = a^2_v$ and $a^2_{u+1} = a^2_{v+1}$ for any $u \geq 0$. If $v - u \equiv 0 \pmod 3$, then we get $a^2_{u+2} = a^2_{v+2}$, $a^2_{u+3} = a^2_{v+3}, \ldots$. Because of the repetition, for all generating pairs the sequence $B^{q,2}_{(x,y)}(G)$ is periodic and the length of its period must be divisible by 3.

The proofs for the orbits $B^{q,1}_{(x,y)}(G)$ and $B^{q,3}_{(x,y)}(G)$ are again similar to the above and are omitted. □

We next denote the lengths of the periods of the orbits $B^{q,\tau}_{(x,y)}(G)$ by $LB^{q,\tau}_{(x,y)}(G)$.

We shall now address the lengths of the periods of the orbits $B^{q,1}_{(x,y)}(Q_{2^{m+1}})$, $B^{q,2}_{(x,y)}(Q_{2^{m+1}})$ and $B^{q,3}_{(x,y)}(Q_{2^{m+1}})$ in the generalized quaternion group $Q_{2^{m+1}}$ with respect to the generating pairs (x, y).

Theorem 4.2. *Consider the generalized quaternion group $Q_{2^{m+1}}$ of order 2^m is defined by the presentation $Q_{2^{m+1}} = \langle x, y \mid x^{2^m} = y^4 = 1,\ x^{2^{m-1}} = y^2,\ y^{-1}xy = x^{-1} \rangle$. Then*

$$LB^{q,1}_{(x,y)}(Q_{2^{m+1}}) = LB^{q,2}_{(x,y)}(Q_{2^{m+1}}) = LB^{q,3}_{(x,y)}(Q_{2^{m+1}}) = 3.2^m \text{ for } m \geq 2.$$

Proof. Firstly, we calculate the lengths of the periods of the first quaternion-type cyclic-balancing orbits $B^{q,1}_{(x,y)}(Q_{2^{m+1}})$. The sequence $B^{q,1}_{(x,y)}(Q_{2^{m+1}})$ is

$$\begin{aligned}
&a^1_0 = x,\ a^1_1 = y,\ a^1_2 = y^{2i}x^{-k},\ a^1_3 = y^{3j}x^{6},\ a^1_4 = x^{35j},\ a^1_5 = y^{3i}x^{204k},\ a^1_6 = y^{2j}x^{-1189},\dots\\
&a^1_{12} = x^{-46611179},\ a^1_{13} = yx^{-271669860j},\ a^1_{14} = y^{2i}x^{-1583407981k},\ a^1_{15} = y^{3j}x^{9228778026},\dots\\
&a^1_{24} = x^{-71631910824649559},\ a^1_{25} = yx^{-417501372047787700j},\ a^1_{26} = y^{2i}x^{-2087506860238938400k},\dots\\
&\vdots\\
&a^1_{12u} = x^{-B_{12u-1}},\ a^1_{12u+1} = yx^{-B_{12u}j},\ a^1_{12u+2} = y^{2i}x^{-B_{12u+1}k},\ a^1_{12u+3} = y^{3j}x^{B_{12u+2}},\\
&a^1_{12u+4} = x^{B_{12u+3}j},\ a^1_{12u+5} = y^{3i}x^{B_{12u+4}k},\ a^1_{12u+6} = y^{2j}x^{-B_{12u+5}},\ a^1_{12u+7} = y^{3}x^{-B_{12u+6}j},\\
&a^1_{12u+8} = x^{-B_{12u+7}k},\ a^1_{12u+9} = y^{j}x^{B_{12u+8}},\ a^1_{12u+10} = y^{2}x^{B_{12u+9}j},\ a^1_{12u+11} = y^{i}x^{B_{12u+10}k}.
\end{aligned}$$

where B_n denotes the n-th member of the balancing sequence $B_0 = a$, $B_1 = b$, $B_{n+1} = 6B_n - B_{n-1}$ $(n \geq 1)$. In [12], Panda and Rout showed that the length of period of the sequence $\{B_n\} \pmod{2^m}$ is 2^m. So we get $LB^{q,1}_{(x,y)}(Q_{2^{m+1}}) = \operatorname{lcm}[12, 2^m] = 3.2^m$ for every $m \geq 2$. From the above, we easily see that $LB^{q,1}_{(x,y)}(Q_{2^{m+1}}) = 3.2^m$.

The proofs for the orbits $B^{q,2}_{(x,y)}(Q_{2^{m+1}})$ and $B^{q,3}_{(x,y)}(Q_{2^{m+1}})$ are similar to the above and are omitted. □

Now, for the generating pair (x, y), we give the first quaternion-type cyclic-balancing orbits of the quaternion group $Q_8 = \langle x, y \mid x^4 = 1,\ x^2 = y^2,\ y^{-1}xy = x^{-1}\rangle$, which is a non-Abelian group of order eight.

Example 4.1. *The sequence $B^{q,1}_{(x,y)}(Q_8)$ is*

$$x,\quad y,\quad y^{2i}x^{-k},\quad y^{-j}x^{2},\quad x^{-j},\quad y^{-i},\quad y^{2j}x^{-1},\quad y^{-1}x^{2j},\quad x^{k},\quad y^{j},\quad y^{2}x^{j},$$
$$y^{i}x^{2k},\quad x,\quad y,\quad y^{2i}x^{-k},\quad y^{-j}x^{2},\dots,$$

which implies that $LB^{q,1}_{(x,y)}(Q_8) = 12$.

References

[1] Behera, A., & Panda, G. K. (1999). On the square roots of triangular numbers. *The Fibonacci Quarterly*, 37(2), 98–105.

[2] Campbell, C. M., & Campbell, P. P. (2005). The Fibonacci length of certain centro-polyhedral groups. *Journal of Applied Mathematics and Computing*, 19(1–2), 231–240.

[3] Campbell, C. M., Campbell, P. P., Doostie, H., & Robertson, E. F. (2004). On the Fibonacci length of powers of dihedral groups. In: *Applications of Fibonacci Numbers*. Springer, Dordrecht, pp. 69–85.

[4] Campbell, C. M., Doostie, H., & Robertson, E. F. (1990). Fibonacci length of generating pairs in groups. In: *Applications of Fibonacci Numbers*. Springer, Dordrecht, pp. 27–35.

[5] Deveci, Ö., & Karaduman, E. (2012). The generalized order-k Lucas sequences in finite groups. *Journal of Applied Mathematics*, 2012, Article ID 464580.

[6] Deveci, Ö., Karaduman, E., & Campbell, C. M. (2010). The periods of k-nacci sequences in centropolyhedral groups and related groups. *Ars Combinatoria*, 97A, 193–210.

[7] Deveci, Ö., Karaduman, E., & Campbell, C. M. (2011). On the k-nacci sequences in finite binary polyhedral groups. *Algebra Colloquium*, 18(1), 945–954.

[8] Doostie, H., & Hashemi, M. (2006). Fibonacci lengths involving the Wall number $k(n)$. *Journal of Applied Mathematics and Computing*, 20, 171–180.

[9] Karaduman, E., & Deveci, Ö. (2009). K-nacci sequences in finite triangle groups. *Discrete Dynamics in Nature and Society*, 2009, Article ID 453750.

[10] Knox, S. W. (1992). Fibonacci sequences in finite groups. *The Fibonacci Quarterly*, 30(2), 116–120.

[11] Panda, G. K. (2009). Some fascinating properties of balancing numbers. In: *Proceeding of the Eleventh International Conference on Fibonacci Numbers and Their Applications*, Braunschweig, Germany, July 5–9, 2004. *Congressus Numerantium*, 194, 185–189.

[12] Panda, G. K., & Rout, S. S. (2014). Periodicity of balancing numbers. *Acta Mathematica Hungarica*, 143(2), 274–286.

[13] Patel, B. K., & Ray, P. K. (2016). The period, rank and order of the sequence of balancing numbers modulo m. *Mathematical Reports*, 18(3), 395–401.

[14] Ray, P. K. (2012). Certain matrices associated with balancing and Lucas-balancing numbers. *Matematika*, 28(1), 15–22.

[15] Wall, D. D. (1960). Fibonacci series modulo m. *The American Mathematical Monthly*, 67(6), 525–532.

[16] Wilcox, H. J. (1986). Fibonacci sequences of period n in groups. *The Fibonacci Quarterly*, 24(4), 356–361.